Geometry on Non-Solvable Equations – A Survey

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Abstract: As we known, an objective thing not moves with one's volition, which implies that all contradictions, particularly, in these semiotic systems for things are artificial. In classical view, a contradictory system is meaningless, contrast to that of geometry on figures of things catched by eyes of human beings. The main objective of sciences is holding the global behavior of things, which needs one knowing both of compatible and contradictory systems on things. Usually, a mathematical system included contradictions is said to be a *Smarandache system*. Beginning from a famous fable, i.e., the 6 blind men with an elephant, this report shows the geometry on contradictory systems, including non-solvable algebraic linear or homogenous equations, non-solvable ordinary differential equations and non-solvable partial differential equations, classify such systems and characterize their global behaviors by combinatorial geometry, particularly, the global stability of non-solvable differential equations. Applications of such systems to other sciences, such as those of gravitational fields, ecologically industrial systems can be also found in this report. All of these discussions show that a non-solvable system is nothing else but a system underlying a topological structure $G \not\simeq K_n$ with a common intersection, i.e., *mathematical combinatorics* contrast to those of solvable system underlying K_n, where n is the number of equations in this system. However, if we stand on a geometrical viewpoint, they are compatible and both of them are meaningful for human beings.

Key Words: Smarandache system, non-solvable system of equations, topological graph, G^L-solution, global stability, ecologically industrial systems, gravitational field, mathematical combinatorics.

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§1. Introduction

Loosely speaking, a geometry is mainly concerned with shape, size, position, $\cdots$ etc., i.e., local or global characters of a figure in space. Its mainly objective is to hold the global behavior of things. However, things are always complex, even hybrid with other things. So it is difficult to know its global characters, or true face of a thing sometimes.

Let us consider a famous fable, i.e., the 6 blind men with an elephant following.

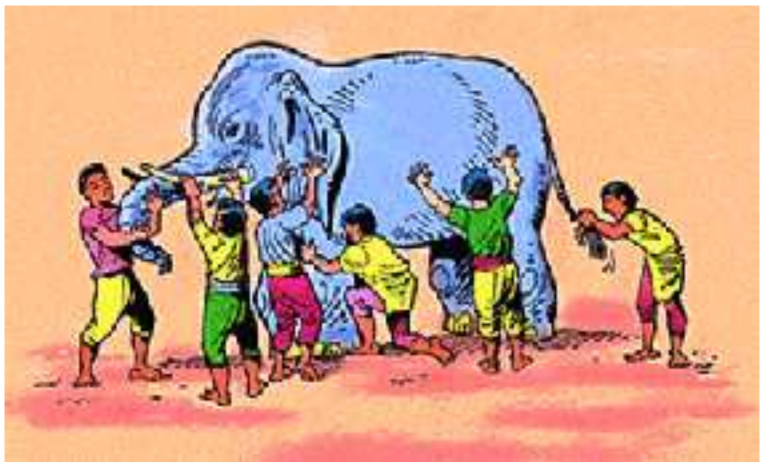

Fig.1

In this proverb, there are 6 blind men were asked to determine what an elephant looked like by feeling different parts of the elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, such as those shown in Fig.1 following. Each of them insisted on his own and not accepted others. They then entered into an endless argument. *All of you are right*! A wise man explains to them: *why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said*. Thus, the best result on an elephant for these blind men is

$$\begin{aligned}\text{An elephant} &= \{4 \text{ pillars}\}\bigcup\{1 \text{ rope}\}\bigcup\{1 \text{ tree branch}\}\\ &\bigcup \{2 \text{ hand fans}\}\bigcup\{1 \text{ wall}\}\bigcup\{1 \text{ solid pipe}\},\end{aligned}$$

i.e., a *Smarandache multi-spaces* ([23]-[25]) defined following.

Definition 1.1([12]-[13]) *Let* $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ *be* m *mathematical systems, different two by two. A Smarandache multi-system* $\widetilde{\Sigma}$ *is a union* $\bigcup\limits_{i=1}^{m}\Sigma_i$ *with rules* $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ *on* $\widetilde{\Sigma}$, *denoted by* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.

Then, what is the philosophical meaning of this fable for one understanding the world? In fact, the situation for one realizing behaviors of things is analogous to the blind men determining what an elephant looks like. Thus, this fable means the limitation or unilateral of one's knowledge, i.e., *science* because of all of those are just correspondent with the sensory cognition of human beings.

Besides, we know that contradiction exists everywhere by this fable, which comes from the limitation of unilateral sensory cognition, i.e., artificial contradiction of human beings, and all scientific conclusions are nothing else but an approximation for things. For example, let $\mu_1,\mu_2,\cdots,\mu_n$ be known and $\nu_i, i\geq 1$ unknown characters at time t for a thing T. Then, the thing T should be understood by

$$T=\left(\bigcup_{i=1}^{n}\{\mu_i\}\right)\bigcup\left(\bigcup_{k\geq 1}\{\nu_k\}\right)$$

in logic but with an approximation $T^\circ=\bigcup\limits_{i=1}^{n}\{\mu_i\}$ for T by human being at time t. Even for T°, these are maybe contradictions in characters $\mu_1,\mu_2,\cdots,\mu_n$ with endless argument between researchers, such as those implied in the fable of 6 blind men with an elephant. Consequently, if one stands still on systems without contradictions, he will never hold the real face of things in the world, particularly, the true essence of geometry for limited of his time.

However, all things are inherently related, not isolated in philosophy, i.e., underlying an invariant topological structure G ([4],[22]). Thus, one needs to characterize those things on contradictory systems, particularly, by geometry. The main objective of this report is to discuss the geometry on contradictory systems, including non-solvable algebraic equations, non-solvable ordinary or partial differential equations, classify such systems and characterize their global behaviors by combinatorial geometry, particularly, the global stability of non-solvable differential equations. For terminologies and notations not mentioned here, we follow references [11], [13] for topological graphs, [3]-[4] for topology, [12],[23]-[25] for Smarandache multi-spaces and [2],[26] for partial or ordinary differential equations.

§2. Manifolds on Equation Systems

Let us beginning with two systems of linear equations in 2 variables:

$$(LES_4^S)\ \begin{cases} x+2y &= 4 \\ 2x+y &= 5 \\ x-2y &= 0 \\ 2x-y &= 3 \end{cases} \qquad\qquad (LES_4^N)\ \begin{cases} x+2y &= 2 \\ x+2y &= -2 \\ 2x-y &= -2 \\ 2x-y &= 2 \end{cases}$$

Clearly, (LES_4^S) is solvable with a solution $x=2$ and $y=1$, but (LES_4^N) is not because $x+2y=-2$ is contradictious to $x+2y=2$, and so that for equations $2x-y=-2$ and $2x-y=2$. Thus, (LES_4^N) is a contradiction system, i.e., a Smarandache system defined following.

Definition 2.1([11]-[13]) *A rule in a mathematical system $(\Sigma;\mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma;\mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule in $\mathcal{R}$.

In geometry, we are easily finding conditions for systems of equations solvable or not. For integers $m,n\geq 1$, denote by

$$S_{f_i}=\{(x_1,x_2,\cdots,x_{n+1})|f_i(x_1,x_2,\cdots,x_{n+1})=0\}\subset\mathbb{R}^{n+1}$$

the solution-manifold in $\mathbb{R}^{n+1}$ for integers $1\leq i\leq m$, where f_i is a function hold with conditions of the implicit function theorem for $1\leq i\leq m$. Clearly, the system

$$(ES_m)\ \begin{cases} f_1(x_1,x_2,\cdots,x_{n+1})=0 \\ f_2(x_1,x_2,\cdots,x_{n+1})=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(x_1,x_2,\cdots,x_{n+1})=0 \end{cases}$$

is solvable or not dependent on

$$\bigcap_{i=1}^{m} S_{f_i}\neq\emptyset \quad\text{or}\quad =\emptyset.$$

Conversely, if $\mathscr{D}$ is a geometrical space consisting of m manifolds $\mathscr{D}_1, \mathscr{D}_2, \cdots, \mathscr{D}_m$ in $\mathbb{R}^{n+1}$, where,

$$\mathscr{D}_i = \{(x_1, x_2, \cdots, x_{n+1}) | f_k^{[i]}(x_1, x_2, \cdots, x_{n+1}) = 0, 1 \leq k \leq m_i\} = \bigcap_{k=1}^{m_i} S_{f_k^{[i]}}.$$

Then, the system

$$\left.\begin{array}{l} f_1^{[i]}(x_1, x_2, \cdots, x_{n+1}) = 0 \\ f_2^{[i]}(x_1, x_2, \cdots, x_{n+1}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{m_i}^{[i]}(x_1, x_2, \cdots, x_{n+1}) = 0 \end{array}\right\} 1 \leq i \leq m$$

is solvable or not dependent on the intersection

$$\bigcap_{i=1}^{m} \mathscr{D}_i \neq \emptyset \quad \text{or} \quad = \emptyset.$$

Thus, we obtain the following result.

Theorem 2.2 *If a geometrical space $\mathscr{D}$ consists of m parts $\mathscr{D}_1, \mathscr{D}_2, \cdots, \mathscr{D}_m$, where, $\mathscr{D}_i = \{(x_1, x_2, \cdots, x_{n+1}) | f_k^{[i]}(x_1, x_2, \cdots, x_{n+1}) = 0, 1 \leq k \leq m_i\}$, then the system (ES_m) consisting of*

$$\left.\begin{array}{l} f_1^{[i]}(x_1, x_2, \cdots, x_{n+1}) = 0 \\ f_2^{[i]}(x_1, x_2, \cdots, x_{n+1}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{m_i}^{[i]}(x_1, x_2, \cdots, x_{n+1}) = 0 \end{array}\right\} 1 \leq i \leq m$$

is non-solvable if $\bigcap_{i=1}^{m} \mathscr{D}_i = \emptyset$.

Now, *whether is it meaningless for a contradiction system in the world*? Certainly not! As we discussed in the last section, a contradiction is artificial if such a system indeed exists in the world. The objective for human beings is not just finding contradictions, but holds behaviors of such systems. For example, although the system (LES_4^N) is contradictory, but it really exists, i.e., 4 lines in $\mathbb{R}^2$, such as those shown in Fig.2.

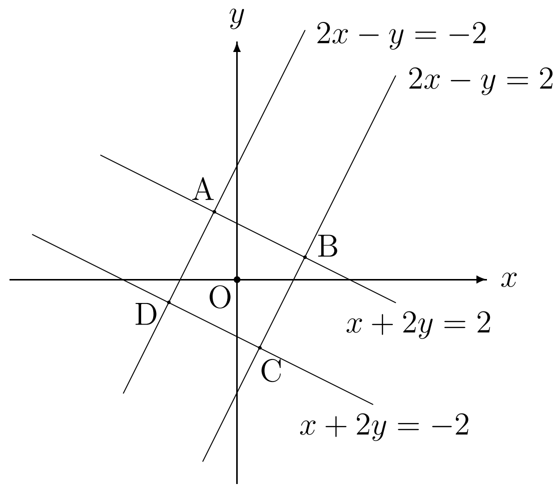

Fig.2

Generally, let

$$AX = (b_1, b_2, \cdots, b_m)^T \qquad (LEq)$$

be a linear equation system with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

for integers $m,\ n \geq 1$. Its underlying graph $\widehat{G}[LEq]$ of a system of linear equations is easily determined. For integers $1 \leq i, j \leq m,\ i \neq j$, two linear equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_i,$$
$$a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n = b_j$$

are called *parallel* if there exists a constant c such that

$$c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in} \neq b_j/b_i.$$

Otherwise, *non-parallel*. The following result is known in [16].

Theorem 2.3([16]) *Let (LEq) be a linear equation system for integers $m, n \geq 1$. Then $\widehat{G}[LEq] \simeq K_{n_1, n_2, \cdots, n_s}$ with $n_1 + n + 2 + \cdots + n_s = m$, where $\mathscr{C}_i$ is the parallel* *family by the property that all equations in a family $\mathscr{C}_i$ are parallel and there are no other equations parallel to lines in $\mathscr{C}_i$ for integers $1 \leq i \leq s$, $n_i = |\mathscr{C}_i|$ for integers $1 \leq i \leq s$ in (LEq) and (LEq) is non-solvable if $s \geq 2$.*

Particularly, for linear equation system on 2 variables, let H be a planar graph with edges straight segments on $\mathbb{R}^2$. The *c-line graph* $L_C(H)$ on H is defined by

$$\begin{aligned} V(L_C(H)) &= \{\text{straight lines } L = e_1e_2\cdots e_l, s \geq 1 \text{ in } H\};\\ E(L_C(H)) &= \{(L_1, L_2)|\ L_1 = e_1^1e_2^1\cdots e_l^1,\ L_2 = e_1^2e_2^2\cdots e_s^2,\ l, s \geq 1\\ &\qquad \text{and there adjacent edges } e_i^1,\ e_j^2 \text{ in } H, 1 \leq i \leq l, 1 \leq j \leq s\}. \end{aligned}$$

Then, a simple criterion in [16] following is interesting.

Theorem 2.4([16]) *A linear equation system $(LEq2)$ on 2 variables is non-solvable if and only if $\widehat{G}[LEq2] \simeq L_C(H))$, where H is a planar graph of order $|H| \geq 2$ on $\mathbb{R}^2$ with each edge a straight segment*

Denoted by $L_1 = \{(x, y)|x + 2y = 2\}$, $L_2 = \{(x, y)|x + 2y = -2\}$, $L_3 = \{(x, y)|2x - y = 2\}$ and $L_3 = \{(x, y)|2x - y = -2\}$ for the system (LES_4^N). Clearly, $L_1 \bigcap L_2 = \emptyset$, $L_1 \bigcap L_3 = \{B\}$, $L_1 \bigcap L_4 = \{A\}$, $L_2 \bigcap L_3 = \{C\}$, $L_2 \bigcap L_4 = \{D\}$ and $L_3 \bigcap L_4 = \emptyset$. Then, the system (LES_4^N) can also appears as a combinatorial system underlying a vertex-edge labeled graph C_4^l in $\mathbb{R}^2$ with labels vertex labeling $l(L_i) = L_i$ for integers $1 \leq i \leq 4$, edge labeling $l(L_1, L_3) = B$, $l(L_1, L_4) = A$, $l(L_2, L_3) = C$ and $l(L_2, L_4) = D$. Thus, the combinatorial system $C_4^l = (LES_4^N)$, such as those shown in Fig.3.

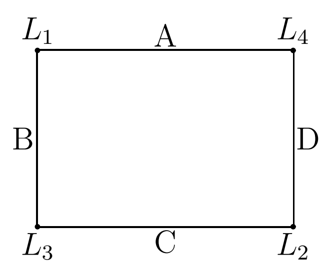

Fig.3

Generally, a Smarandache multi-system is equivalent to a combinatorial system by following, which implies the *CC Conjecture* for mathematics, i.e., *any mathematics can be reconstructed from or turned into combinatorization* (see [6] for details).

Definition 2.5([11]-[13]) *For any integer $m \geq 1$, let $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ be a Smarandache multi-system consisting of m mathematical systems $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$. An inherited topological structure $G^L\left[\widetilde{S}\right]$ of $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ is a topological vertex-edge labeled graph defined following:*

$$V\left(G^L\left[\widetilde{S}\right]\right) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E\left(G^L\left[\widetilde{S}\right]\right) = \{(\Sigma_i, \Sigma_j)\,|\Sigma_i \bigcap \Sigma_j \neq \emptyset,\ 1 \leq i \neq j \leq m\} \text{ with labeling}$$

$$L:\ \Sigma_i \to L(\Sigma_i) = \Sigma_i \quad \text{and} \quad L:\ (\Sigma_i, \Sigma_j) \to L(\Sigma_i, \Sigma_j) = \Sigma_i \bigcap \Sigma_j$$

for integers $1 \leq i \neq j \leq m$.

Thus, a Smarandache system is equivalent to a combinatorial system, i.e., $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right) \simeq G^L\left[\widetilde{S}\right]$ underlying with a graph $\widehat{G}\left[\widetilde{S}\right]$ without labels. Particularly, let these mathematical systems $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be geometrical spaces, for instance manifolds $M_1, M_2, \cdots, M_m$ with respective dimensions $n_1, n_2, \cdots, n_m$ in Definition 2.3, we get a geometrical space with $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ underlying a topological graph $G^L\left[\widetilde{M}\right]$. Such a geometrical space $G^L\left[\widetilde{M}\right]$ is called a *combinatorial manifold*, denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$. For examples, combinatorial manifolds with different dimensions in $\mathbb{R}^3$ are shown in Fig.4, in where (a) represents a combination of a 3-manifold, a torus and 1-manifold, and (b) a torus with 4 bouquets of 1-manifolds.

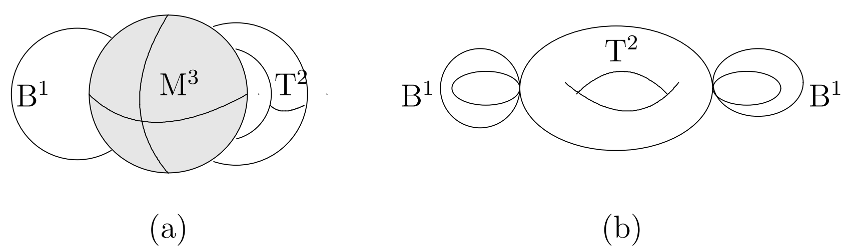

Fig.4

Particularly, if $n_i = n, 1 \leq i \leq m$, a combinatorial manifold $\widetilde{M}(n_1, \cdots, n_m)$ is nothing else but an n-manifold underlying $G^L\left[\widetilde{M}\right]$. However, this guise of G^L-systems contributes to manifolds and combinatorial manifolds (See [7]-[15] for details). For example, the fundamental groups of manifolds are characterized in [14]-[15] following.

Theorem 2.6([14]) *For any locally compact n-manifold M, there always exists an inherent graph $G^{in}_{min}[M]$ of M such that $\pi(M) \cong \pi(G^{in}_{min}[M])$.*

Particularly, for an integer $n \geq 2$ a compact n-manifold M is simply-connected if and only if $G^{in}_{min}[M]$ is a finite tree.

Theorem 2.7([15]) *Let $\widetilde{M}$ be a finitely combinatorial manifold. If for $\forall (M_1, M_2) \in E(G^L\left[\widetilde{M}\right])$, $M_1 \cap M_2$ is simply-connected, then*

$$\pi_1(\widetilde{M}) \cong \left(\bigoplus_{M \in V(G[\widetilde{M}])} \pi_1(M) \right) \bigoplus \pi_1(G\left[\widetilde{M}\right]).$$

Furthermore, it provides one with a listing of manifolds by graphs in [14].

Theorem 2.8([14]) *Let $\mathscr{A}[M] = \{ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$ be a atlas of a locally compact n-manifold M. Then the labeled graph $G^L_{|\Lambda|}$ of M is a topological invariant on $|\Lambda|$, i.e., if $H^{L_1}_{|\Lambda|}$ and $G^{L_2}_{|\Lambda|}$ are two labeled n-dimensional graphs of M, then there exists a self-homeomorphism $h : M \to M$ such that $h : H^{L_1}_{|\Lambda|} \to G^{L_2}_{|\Lambda|}$ naturally induces an isomorphism of graph.*

Why the system (ES_m) consisting of

$$\left.\begin{array}{l} f^{[i]}_1(x_1, x_2, \cdots, x_n) = 0 \\ f^{[i]}_2(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f^{[i]}_{m_i}(x_1, x_2, \cdots, x_n) = 0 \end{array}\right\} 1 \leq i \leq m$$

is non-solvable if $\bigcap\limits_{i=1}^{m} \mathscr{D}_i = \emptyset$ in Theorem 2.2? In fact, it lies in that the solution-manifold of (ES_m) is the intersection of $\mathscr{D}_i, 1 \leq i \leq m$. If it is allowed combinatorial manifolds to be solution-manifolds, then there are no contradictions once more even if $\bigcap\limits_{i=1}^{m} \mathscr{D}_i = \emptyset$. This fact implies that including combinatorial manifolds to be solution-manifolds of systems (ES_m) is a better understanding things in the world.

§3. Surfaces on Homogenous Polynomials

Let

$$P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x}) \qquad (ES^{n+1}_m)$$

be m homogeneous polynomials in variables $x_1, x_2, \cdots, x_{n+1}$ with coefficients in $\mathbb{C}$ and

$$\emptyset \neq S_{P_i} = \{(x_1, x_2, \cdots, x_{n+1}) | P_i\left(\overline{x}\right) = 0\} \subset \mathbb{P}^n\mathbb{C}$$

for integers $1 \leq i \leq m$, which are hypersurfaces, particularly, curves if $n = 2$ passing through the original of $\mathbb{C}^{n+1}$.

Similarly, parallel hypersurfaces in $\mathbb{C}^{n+1}$ are defined following.

Definition 3.1 *Let $P(\overline{x}), Q(\overline{x})$ be two complex homogenous polynomials of degree d in $n+1$ variables and $I(P,Q)$ the set of intersection points of $P(\overline{x})$ with $Q(\overline{x})$. They are said to be parallel, denoted by $P \parallel Q$ if $d \geq 1$ and there are constants $a, b, \cdots, c$ (not all zero) such that for $\forall \overline{x} \in I(P,Q)$, $ax_1 + bx_2 + \cdots + cx_{n+1} = 0$, i.e., all intersections of $P(\overline{x})$ with $Q(\overline{x})$ appear at a hyperplane on $\mathbb{P}^n\mathbb{C}$, or $d = 1$ with all intersections at the infinite $x_{n+1} = 0$. Otherwise, $P(\overline{x})$ are not parallel to $Q(\overline{x})$, denoted by $P \nparallel Q$.*

Then, these polynomials in (ES_m^{n+1}) can be classified into families $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l$ by this parallel property such that $P_i \parallel P_j$ if $P_i, P_j \in \mathscr{C}_k$ for an integer $1 \leq k \leq l$, where $1 \leq i \neq j \leq m$and it is *maximal* if each $\mathscr{C}_i$ is maximal for integers $1 \leq i \leq l$, i.e., for $\forall P \in \{P_k(\overline{x}),\ 1 \leq k \leq m\} \setminus \mathscr{C}_i$, there is a polynomial $Q(\overline{x}) \in \mathscr{C}_i$ such that $P \nparallel Q$. The following result is a generalization of Theorem 2.3.

Theorem 3.2([19]) *Let $n \geq 2$ be an integer. For a system (ES_m^{n+1}) of homogenous polynomials with a parallel maximal classification $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l$,*

$$\widehat{G}[ES_m^{n+1}] \leq K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l)$$

and with equality holds if and only if $P_i \parallel P_j$ and $P_s \nparallel P_i$ implies that $P_s \nparallel P_j$, where $K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l)$ denotes a complete l-partite graphs. Conversely, for any subgraph $G \leq K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l)$, there are systems (ES_m^{n+1}) of homogenous polynomials with a parallel maximal classification $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l$ such that

$$G \simeq \widehat{G}[ES_m^{n+1}].$$

Particularly, if all polynomials in (ES_m^{n+1}) be degree 1, i.e., hyperplanes with a parallel maximal classification $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l$, then

$$\widehat{G}[ES_m^{n+1}] = K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l).$$

The following result is immediately known by definition.

Theorem 3.3 *Let (ES_m^{n+1}) be a G^L-system consisting of homogenous polynomials $P(\overline{x}_1), P(\overline{x}_2), \cdots, P(\overline{x}_m)$ in $n+1$ variables with respectively hypersurfaces $S_{P_i}, 1 \leq i \leq m$. Then, $\widetilde{M} = \bigcup\limits_{i=1}^{m} S_{P_i}$ is an n-manifold underlying graph $\widehat{G}[ES_m^{n+1}]$ in $\mathbb{C}^{n+1}$.*

For $n = 2$, we can further determine the genus of surface $\widetilde{M}$ in $\mathbb{R}^3$ following.

Theorem 3.4([19]) *Let $\widetilde{S}$ be a combinatorial surface consisting of m orientable surfaces $S_1, S_2, \cdots, S_m$ underlying a topological graph $G^L[\widetilde{S}]$ in $\mathbb{R}^3$. Then*

$$g(\widetilde{S}) = \beta(\widehat{G}\left\langle \widetilde{S} \right\rangle) + \sum_{i=1}^{m}(-1)^{i+1} \sum_{\bigcap\limits_{l=1}^{i} S_{k_l} \neq \emptyset} \left[g\left(\bigcap_{l=1}^{i} S_{k_l}\right) - c\left(\bigcap_{l=1}^{i} S_{k_l}\right) + 1 \right],$$

where $g\left(\bigcap\limits_{l=1}^{i} S_{k_l}\right)$, $c\left(\bigcap\limits_{l=1}^{i} S_{k_l}\right)$ are respectively the genus and number of path-connected components in surface $S_{k_1} \bigcap S_{k_2} \bigcap \cdots \bigcap S_{k_i}$ and $\beta(\widehat{G}\left\langle \widetilde{S} \right\rangle)$ denotes the Betti number of topological graph $\widehat{G}\left\langle \widetilde{S} \right\rangle$.

Notice that for a curve C determined by homogenous polynomial $P(x, y, z)$ of degree d in $\mathbb{P}^2\mathbf{C}$, there is a compact connected Riemann surface S by the Noether's result such that

$$h : \; S - h^{-1}(\mathrm{Sing}(C)) \to C - \mathrm{Sing}(C)$$

is a homeomorphism with genus

$$g(S) = \frac{1}{2}(d-1)(d-2) - \sum_{p \in \mathrm{Sing}(C)} \delta(p),$$

where $\delta(p)$ is a positive integer associated with the singular point p in C. Furthermore, if $\mathrm{Sing}(C) = \emptyset$, i.e., C is non-singular then there is a compact connected Riemann surface S homeomorphism to C with genus $\frac{1}{2}(d-1)(d-2)$. By Theorem 3.4, we obtain the genus of $\widetilde{S}$ determined by homogenous polynomials following.

Theorem 3.5([19]) *Let $C_1, C_2, \cdots, C_m$ be complex curves determined by homogenous polynomials $P_1(x, y, z), P_2(x, y, z), \cdots, P_m(x, y, z)$ without common component, and let*

$$R_{P_i,P_j} = \prod_{k=1}^{\deg(P_i)\deg(P_j)} (c_k^{ij} z - b_k^{ij} y)^{e_k^{ij}}, \qquad \omega_{i,j} = \sum_{k=1}^{\deg(P_i)\deg(P_j)} \sum_{e_k^{ij} \neq 0} 1$$

be the resultant of $P_i(x,y,z), P_j(x,y,z)$ for $1 \leq i \neq j \leq m$. Then there is an orientable surface $\widetilde{S}$ in $\mathbb{R}^3$ of genus

$$\begin{aligned} g(\widetilde{S}) &= \beta(\widehat{G}\left\langle \widetilde{C}\right\rangle) + \sum_{i=1}^{m}\left(\frac{(\deg(P_i)-1)(\deg(P_i)-2)}{2} - \sum_{p^i\in \mathrm{Sing}(C_i)}\delta(p^i)\right) \\ &\quad + \sum_{1\leq i\neq j\leq m}(\omega_{i,j}-1) + \sum_{i\geq 3}(-1)^i \sum_{C_{k_1}\bigcap\cdots\bigcap C_{k_i}\neq\emptyset}\left[c\left(C_{k_1}\bigcap\cdots\bigcap C_{k_i}\right)-1\right] \end{aligned}$$

with a homeomorphism $\varphi: \ \widetilde{S} \to \widetilde{C} = \bigcup\limits_{i=1}^{m} C_i$. Furthermore, if $C_1, C_2, \cdots, C_m$ are non-singular, then

$$\begin{aligned} g(\widetilde{S}) &= \beta(\widehat{G}\left\langle \widetilde{C}\right\rangle) + \sum_{i=1}^{m}\frac{(\deg(P_i)-1)(\deg(P_i)-2)}{2} \\ &\quad + \sum_{1\leq i\neq j\leq m}(\omega_{i,j}-1) + \sum_{i\geq 3}(-1)^i \sum_{C_{k_1}\bigcap\cdots\bigcap C_{k_i}\neq\emptyset}\left[c\left(C_{k_1}\bigcap\cdots\bigcap C_{k_i}\right)-1\right], \end{aligned}$$

where

$$\delta(p^i) = \frac{1}{2}\left(I_{p^i}\left(P_i, \frac{\partial P_i}{\partial y}\right) - \nu_\phi(p^i) + |\pi^{-1}(p^i)|\right)$$

is a positive integer with a ramification index $\nu_\phi(p^i)$ for $p^i \in \mathrm{Sing}(C_i), 1 \leq i \leq m$.

Notice that $\widehat{G}\left[ES^3_m\right] = K_m$. We then easily get conclusions following.

Corollary 3.6 *Let $C_1, C_2, \cdots, C_m$ be complex non-singular curves determined by homogenous polynomials $P_1(x,y,z), P_2(x,y,z), \cdots, P_m(x,y,z)$ without common component, any intersection point $p \in I(P_i, P_j)$ with multiplicity 1 and*

$$\begin{cases} P_i(x,y,z) = 0 \\ P_j(x,y,z) = 0, \quad \forall i,j,k \in \{1,2,\cdots,m\} \\ P_k(x,y,z) = 0 \end{cases}$$

has zero-solution only. Then the genus of normalization $\widetilde{S}$ of curves $C_1, C_2, \cdots, C_m$ is

$$g(\widetilde{S}) = 1 + \frac{1}{2} \times \sum_{i=1}^{m}\deg(P_i)\left(\deg(P_i)-3\right) + \sum_{1\leq i\neq j\leq m}\deg(P_i)\deg(P_j).$$

Corollary 3.7 *Let $C_1, C_2, \cdots, C_m$ be complex non-singular curves determined by homogenous polynomials $P_1(x,y,z), P_2(x,y,z), \cdots, P_m(x,y,z)$ without common component and $C_i \bigcap C_j = \bigcap\limits_{i=1}^{m} C_i$ with $\left|\bigcap\limits_{i=1}^{m} C_i\right| = \kappa > 0$ for integers $1 \leq i \neq j \leq m$. Then* *the genus of normalization* $\widetilde{S}$ *of curves* $C_1, C_2, \cdots, C_m$ *is*

$$g(\widetilde{S}) = g(\widetilde{S}) = (\kappa - 1)(m-1) + \sum_{i=1}^{m} \frac{(\deg(P_i) - 1)(\deg(P_i) - 2)}{2}.$$

Particularly, if all curves in $\mathbb{C}^3$ are lines, we know an interesting result following.

Corollary 3.8 *Let* $L_1, L_2, \cdots, L_m$ *be distinct lines in* $\mathbb{P}^2\mathbf{C}$ *with respective normalizations of spheres* $S_1, S_2, \cdots, S_m$*. Then there is a normalization of surface* $\widetilde{S}$ *of* $L_1, L_2, \cdots, L_m$ *with genus* $\beta(\widehat{G}\left\langle \widetilde{L} \right\rangle)$*. Particularly, if* $\widehat{G}\left\langle \widetilde{L} \right\rangle)$ *is a tree, then* $\widetilde{S}$ *is homeomorphic to a sphere.*

§4. Geometry on Non-solvable Differential Equations

4.1 G^L-Systems of Differential Equations

Let

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) = 0 \end{cases} \qquad (PDES_m)$$

be a system of ordinary or partial differential equations of first order on a function $u(x_1, \cdots, x_n, t)$ with continuous $F_i : \mathbf{R}^n \to \mathbf{R}^n$ such that $F_i(\overline{0}) = \overline{0}$. Its *symbol* is determined by

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n) = 0, \end{cases}$$

i.e., substitutes $u_{x_1}, u_{x_2}, \cdots, u_{x_n}$ by $p_1, p_2, \cdots, p_n$ in $(PDES_m)$.

Definition 4.1 *A non-solvable* $(PDES_m)$ *is algebraically contradictory if its symbol is non-solvable. Otherwise, differentially contradictory.*

Then, we know conditions following characterizing non-solvable systems of partial differential equations.

Theorem 4.2([18],[21]) *A Cauchy problem on systems*

$$\begin{cases} F_1(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0\\ F_2(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_m(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0 \end{cases}$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} x_i|_{x_n=x_n^0}=x_i^0(s_1,s_2,\cdots,s_{n-1})\\ u|_{x_n=x_n^0}=u_0(s_1,s_2,\cdots,s_{n-1})\\ p_i|_{x_n=x_n^0}=p_i^0(s_1,s_2,\cdots,s_{n-1}), \quad i=1,2,\cdots,n \end{cases}$$

if and only if the system

$$F_k(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0,\ 1\leq k\leq m$$

is algebraically contradictory, in this case, there must be an integer $k_0,\ 1\leq k_0\leq m$ such that

$$F_{k_0}(x_1^0,x_2^0,\cdots,x_{n-1}^0,x_n^0,u_0,p_1^0,p_2^0,\cdots,p_n^0)\neq 0$$

or it is differentially contradictory itself, i.e., there is an integer $j_0,\ 1\leq j_0\leq n-1$ such that

$$\frac{\partial u_0}{\partial s_{j_0}}-\sum_{i=0}^{n-1}p_i^0\frac{\partial x_i^0}{\partial s_{j_0}}\neq 0.$$

Particularly, the following conclusion holds with quasilinear system $(LPDES_m^C)$.

Corollary 4.3 *A Cauchy problem $(LPDES_m^C)$ on quasilinear, particularly, linear system of partial differential equations with initial values $u|_{x_n=x_n^0}=u_0$ is non-solvable if and only if the system $(LPDES_m)$ of partial differential equations is algebraically contradictory. Particularly, the Cauchy problem on a quasilinear partial differential equation is always solvable.*

Similarly, for integers $m,\ n\geq 1$, let

$$\dot{X}=A_1X,\cdots,\dot{X}=A_kX,\cdots,\dot{X}=A_mX \qquad (LDES_m^1)$$

be a linear ordinary differential equation system of first order and

$$\begin{cases} x^{(n)} + a_{11}^{[0]} x^{(n-1)} + \cdots + a_{1n}^{[0]} x = 0 \\ x^{(n)} + a_{21}^{[0]} x^{(n-1)} + \cdots + a_{2n}^{[0]} x = 0 \\ \cdots\cdots\cdots\cdots \\ x^{(n)} + a_{m1}^{[0]} x^{(n-1)} + \cdots + a_{mn}^{[0]} x = 0 \end{cases} \qquad (LDE_m^n)$$

a linear differential equation system of order n with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0 \le k \le m,\ 1 \le i, j \le n$. Then it is known a criterion from [16] following.

Theorem 4.4([17]) *A differential equation system $(LDES_m^1)$ is non-solvable if and only if*

$$(|A_1 - \lambda I_{n\times n}|, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|) = 1.$$

Similarly, the differential equation system (LDE_m^n) is non-solvable if and only if

$$(P_1(\lambda), P_2(\lambda), \cdots, P_m(\lambda)) = 1,$$

where $P_i(\lambda) = \lambda^n + a_{i1}^{[0]}\lambda^{n-1} + \cdots + a_{i(n-1)}^{[0]}\lambda + a_{in}^{[0]}$ for integers $1 \le i \le m$. Particularly, $(LDES_1^1)$ and $(LDE_1^n m)$ are always solvable.

According to Theorems 4.3 and 4.4, for systems $(LPDES_m^C)$, $(LDES_m^1)$ or (LDE_m^n), there are equivalent systems $G^L[LPDES_m^C]$, $G^L[LDES_m^1]$ or $G^L[LDE_m^n]$ by Definition 2.5, called $G^L[LPDES_m^C]$-*solution*, $G^L[LDES_m^1]$ -*solution* or $G^L[LDE_m^n]$-*solution* of systems $(LPDES_m^C)$, $(LDES_m^1)$ or (LDE_m^n), respectively. Then, we know the following conclusion from [17]-[18] and [21].

Theorem 4.5([17]-[18],[21]) *The Cauchy problem on system $(PDES_m)$ of partial differential equations of first order with initial values $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]},\ 1 \le i \le n$ for the kth equation in $(PDES_m)$, $1 \le k \le m$ such that*

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^{n} p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0,$$

and the linear homogeneous differential equation system $(LDES_m^1)$ *(or* (LDE_m^n)*) both are uniquely* G^L*-solvable, i.e.,* $G^L[PDES]$*,* $G^L[LDES_m^1]$ *and* $G^L[LDE_m^n]$ *are uniquely determined.*

For ordinary differential systems $(LDES_m^1)$ or (LDE_m^n), we can further replace solution-manifolds $S^{[k]}$ of the kth equation in $G^L[LDES_m^1]$ and $G^L[LDE_m^n]$ by their solution *basis*

$$\mathscr{B}^{[k]} = \{\ \overline{\beta}_i^{[k]}(t)e^{\alpha_i^{[k]}t} \mid 1 \le i \le n\ \} \quad \text{or} \quad \mathscr{C}^{[k]} = \{\ t^l e^{\lambda_i^{[k]}t} \mid 1 \le i \le s, 1 \le l \le k_i\ \}.$$

because each solution-manifold of $(LDES_m^1)$ (or (LDE_m^n)) is a linear space.

For example, let a system (LDE_m^n) be

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}$ and $\dot{x} = \dfrac{dx}{dt}$. Then the solution basis of equations $(1)-(6)$ are respectively $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ with its $G^L[LDE_m^n]$ shown in Fig.5.

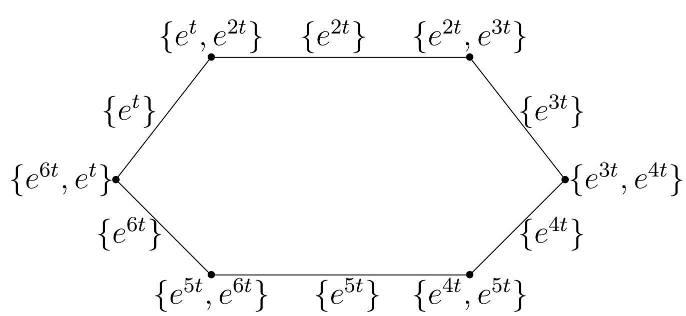

Fig.5

Such a labeling can be simplified to labeling by integers for combinatorially classifying systems $G^L[LDES_m^1]$ and $G^L[LDE_m^n]$, i.e., *integral graphs* following.

Definition 4.6 *Let G be a simple graph. A vertex-edge labeled graph $\theta: G \to \mathbb{Z}^+$ is called integral if $\theta(uv) \leq \min\{\theta(u), \theta(v)\}$ for $\forall uv \in E(G)$, denoted by G^{I_θ}.*

For two integral labeled graphs $G_1^{I_\theta}$ and $G_2^{I_\tau}$, they are called identical if $G_1 \overset{\varphi}{\simeq} G_2$ and $\theta(x) = \tau(\varphi(x))$ for any graph isomorphism φ and $\forall x \in V(G_1) \bigcup E(G_1)$, denoted by $G_1^{I_\theta} = G_2^{I_\tau}$. Otherwise, non-identical.

For example, the graphs shown in Fig.6 are all integral on $K_4 - e$, but $G_1^{I_\theta} = G_2^{I_\tau}$, $G_1^{I_\theta} \neq G_3^{I_\sigma}$.

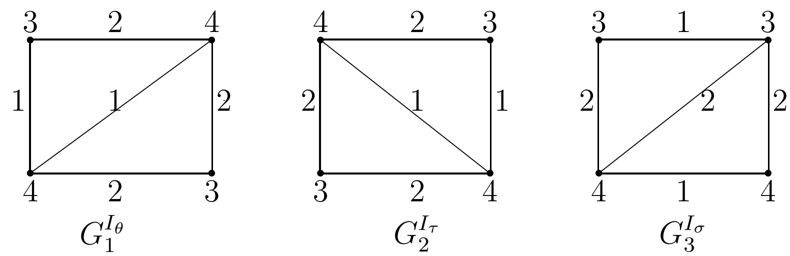

Fig.6

Applying integral graphs, the systems $(LDES_m^1)$ and (LDE_m^n) are combinatorially classified in [17] following.

Theorem 4.7([17]) *Let $(LDES_m^1)$, $(LDES_m^1)'$ (or (LDE_m^n), $(LDE_m^n)'$) be two linear homogeneous differential equation systems with integral labeled graphs H, H'. Then $(LDES_m^1) \overset{\varphi}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \overset{\varphi}{\simeq} (LDE_m^n)'$) if and only if $H = H'$.*

4.2 Differential Manifolds on G^L-Systems of Equations

By definition, the union $\widetilde{M} = \bigcup_{k=1}^{m} S^{[k]}$ is an n-manifold. The following result is immediately known.

Theorem 4.8([17]-[18],[21]) *For any simply graph G, there are differentiable solution-manifolds of $(PDES_m)$, $(LDES_m^1)$ and (LDE_m^n) such that $\widehat{G}[PDES] \simeq G$, $\widehat{G}[LDES_m^1] \simeq G$ and $\widehat{G}[LDE_m^n] \simeq G$.*

Notice that a basis on vector field $T(M)$ of a differentiable n-manifold M is

$$\left\{ \frac{\partial}{\partial x_i},\ 1 \leq i \leq n \right\}$$

and a vector field X can be viewed as a first order partial differential operator

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

where a_i is C^∞-differentiable for all integers $1 \leq i \leq n$. Combining Theorems 4.5 and 4.8 enables one to get a result on vector fields following.

Theorem 4.9([21]) *For an integer $m \geq 1$, let $U_i, 1 \leq i \leq m$ be open sets in $\mathbb{R}^n$ underlying a graph defined by $V(G) = \{U_i | 1 \leq i \leq m\}$, $E(G) = \{(U_i, U_j) | U_i \bigcap U_j \neq \emptyset, 1 \leq i, j \leq m\}$. If X_i is a vector field on U_i for integers $1 \leq i \leq m$, then there always exists a differentiable manifold $M \subset \mathbb{R}^n$ with atlas $\mathscr{A} = \{(U_i, \phi_i) | 1 \leq i \leq m\}$ underlying graph G and a function $u_G \in \Omega^0(M)$ such that $X_i(u_G) = 0, 1 \leq i \leq m$.*

§5. Applications

In philosophy, every thing is a G^L-system with contradictions embedded in our world, which implies that the geometry on non-solvable system of equations is in fact a truthful portraying of things with applications to various fields, particularly, the understanding on gravitational fields and the controlling of industrial systems.

5.1 Gravitational Fields

An immediate application of geometry on G^L-systems of non-solvable equations is that it can provides one with a visualization on things in space of dimension≥ 4 by decomposing the space into subspaces underlying a graph G^L.

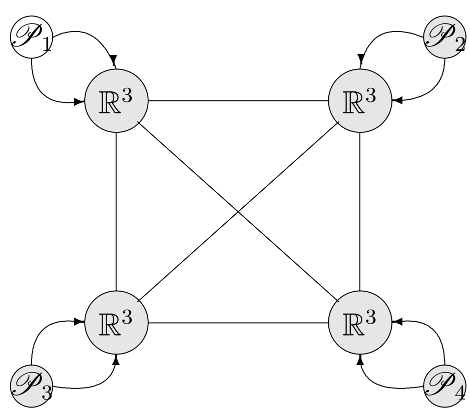

Fig.7

For example, a decomposition of a Euclidean space into $\mathbb{R}^3$ is shown in Fig.7, where $G^L \simeq K_4$, a complete graph of order 4 and $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4$ are the observations on its subspaces $\mathbb{R}^3$. Notice that $\mathbb{R}^3$ is in a general position and $\mathbb{R}^3 \bigcap \mathbb{R}^3 \not\simeq \mathbb{R}^3$ here. Generally, if $G^L \simeq K_m$, we know its dimension following.

Theorem 5.1([9],[13]) *Let $\mathscr{E}_{K_m}(3)$ be a K_m-space of $\underbrace{\mathbb{R}^3_1, \cdots, \mathbb{R}^3}_{m}$. Then its minimum dimension*

$$\dim_{min}\mathscr{E}_{K_m}(3) = \begin{cases} 3, & \text{if} \quad m = 1, \\ 4, & \text{if} \quad 2 \leq m \leq 4, \\ 5, & \text{if} \quad 5 \leq m \leq 10, \\ 2 + \lceil \sqrt{m} \rceil, & \text{if} \quad m \geq 11 \end{cases}$$

and maximum dimension

$$\dim_{max}\mathscr{E}_{K_m}(3) = 2m - 1$$

with $\mathbb{R}^3_i \bigcap \mathbb{R}^3_j = \bigcap\limits_{i=1}^{m} \mathbb{R}^3_i$ for any integers $1 \leq i, j \leq m$.

For the gravitational field, by applying the *geometrization of gravitation* in $\mathbb{R}^3$, Einstein got his gravitational equations with time ([1])

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu}$$

where $R^{\mu\nu} = R^{\mu\alpha\nu}_{\alpha} = g_{\alpha\beta}R^{\alpha\mu\beta\nu}$, $R = g_{\mu\nu}R^{\mu\nu}$ are the respective Ricci tensor, Ricci scalar curvature, $G = 6.673 \times 10^{-8} cm^3/gs^2$, $\kappa = 8\pi G/c^4 = 2.08 \times 10^{-48} cm^{-1} \cdot g^{-1} \cdot s^2$, which has a spherically symmetric solution on Riemannian metric, called *Schwarzschild spacetime*

$$ds^2 = f(t)\left(1 - \frac{r_s}{r}\right)dt^2 - \frac{1}{1 - \frac{r_s}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

for $\lambda = 0$ in vacuum, where r_g is the *Schwarzschild radius*. Thus, if the dimension of the universe≥ 4, all these observations are nothing else but a projection of the true faces on our six organs, a pseudo-truth. However, different from the string theory, we can characterize its global behavior by K^L_m-space solutions of $\mathbb{R}^3$ (See [8]-[10] for details). For example, if $m = 4$, there are 4 Einstein's gravitational equations for $\forall v \in V\left(K^L_4\right)$. We can solving it locally by spherically symmetric solutions in $\mathbb{R}^3$ and construct a K^L_4-solution $S_{f_1}, S_{f_2}, S_{f_3}$ and S_{f_4}, such as those shown in Fig.8,

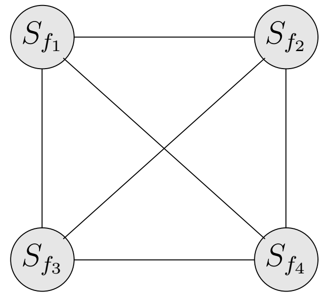

Fig.8

where, each S_{f_i} is a geometrical space determined by Schwarzschild spacetime

$$ds^2 = f(t)\left(1 - \frac{r_s}{r}\right)dt^2 - \frac{1}{1 - \frac{r_s}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

for integers $1 \leq i \leq m$. Certainly, its global behavior depends on the intersections $S_{f_i} \bigcap S_{f_j}, 1 \leq i \neq j \leq 4$.

5.2 Ecologically Industrial Systems

Determining a system, particularly, an industrial system on initial values being stable or not is an important problem because it reveals that this system is controllable or not by human beings. Usually, such a system is characterized by a system of differential equations. For example, let

$$\begin{cases} A \rightarrow X \\ 2X + Y \rightarrow 3X \\ B + X \rightarrow Y + D \\ X \rightarrow E \end{cases}$$

be the *Brusselator model* on chemical reaction, where A, B, X, Y are respectively the concentrations of 4 materials in this reaction. By the chemical dynamics if the initial concentrations for A, B are chosen sufficiently larger, then X and Y can be characterized by differential equations

$$\frac{\partial X}{\partial t} = k_1\Delta X + A + X^2Y - (B+1)X, \quad \frac{\partial Y}{\partial t} = k_2\Delta Y + BX - X^2Y.$$

As we known, the stability of a system is determined by its solutions in classical sciences. But if the system of equations is non-solvable, *what is its stability*? It should be noted that non-solvable systems of equations extensively exist in our daily life. For example, an industrial system with raw materials $M_1, M_2, \cdots, M_n$, products (including by-products) $P_1, P_2, \cdots, P_m$ but $W_1, W_2, \cdots, W_s$ wastes after a produce process, such as those shown in Fig.9 following,

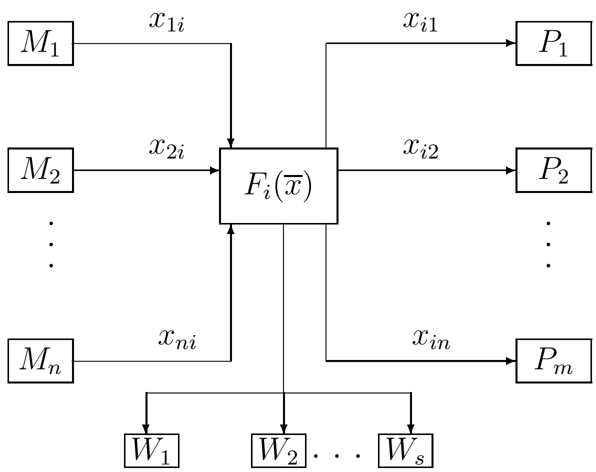

Fig.9

which is an opened system and can be transferred to a closed one by letting the environment as an additional cell, called an *ecologically industrial system*. However, such an ecologically industrial system is usually a non-solvable system of equations by the input-output model in economy, see [20] for details.

Certainly, the global stability depends on the local stabilities. Applying the G-solution of a G^L-system (DES_m) of differential equations, the global stability is defined following.

Definition 5.2 *Let $(PDES_m^C)$ be a Cauchy problem on a system of partial differential equations of first order in $\mathbb{R}^n$, $H \leq G[PDES_m^C]$ a spanning subgeraph, and $u^{[v]}$ the solution of the vth equation with initial value $u_0^{[v]}$, $v \in V(H)$. It is sum-stable on the subgraph H if for any number $\varepsilon > 0$ there exists, $\delta_v > 0$, $v \in V(H)$ such that each $G(t)$-solution with*

$$\left| u'^{[v]}_0 - u_0^{[v]} \right| < \delta_v, \quad \forall v \in V(H)$$

exists for all $t \geq 0$ and with the inequality

$$\left| \sum_{v \in V(H)} u'^{[v]} - \sum_{v \in V(H)} u^{[v]} \right| < \varepsilon$$

holds, denoted by $G[t] \overset{H}{\sim} G[0]$ *and* $G[t] \overset{\Sigma}{\sim} G[0]$ *if* $H = G[PDES_m^C]$*. Furthermore, if there exists a number* $\beta_v > 0,\ v \in V(H)$ *such that every* $G'[t]$*-solution with*

$$\left|{u'}_0^{[v]} - u_0^{[v]}\right| < \beta_v, \quad \forall v \in V(H)$$

satisfies

$$\lim_{t\to\infty}\left|\sum_{v\in V(H)} {u'}^{[v]} - \sum_{v\in V(H)} u^{[v]}\right| = 0,$$

then the $G[t]$*-solution is called asymptotically stable, denoted by* $G[t] \xrightarrow{H} G[0]$ *and* $G[t] \xrightarrow{\Sigma} G[0]$ *if* $H = G[PDES_m^C]$*.*

Let $(PDES_m^C)$ be a system

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = H_i(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ u|_{t=t_0} = u_0^{[i]}(x_1, x_2, \cdots, x_{n-1}) \end{array}\right\} \ 1 \le i \le m \qquad (APDES_m^C)$$

A point $X_0^{[i]} = (t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]})$ with $H_i(t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]}) = 0$ for an integer $1 \le i \le m$ is called an *equilibrium point* of the ith equation in $(APDES_m)$. A result on the sum-stability of $(APDES_m)$ is known in [18] and [21] following.

Theorem 5.3([18],[21]) *Let* $X_0^{[i]}$ *be an equilibrium point of the* i*th equation in* $(APDES_m)$ *for each integer* $1 \le i \le m$*. If*

$$\sum_{i=1}^{m} H_i(X) > 0 \ \ and \ \ \sum_{i=1}^{m} \frac{\partial H_i}{\partial t} \le 0$$

for $X \ne \sum\limits_{i=1}^{m} X_0^{[i]}$*, then the system* $(APDES_m)$ *is sum-stability, i.e.,* $G[t] \overset{\Sigma}{\sim} G[0]$*. Furthermore, if*

$$\sum_{i=1}^{m} \frac{\partial H_i}{\partial t} < 0$$

for $X \ne \sum\limits_{i=1}^{m} X_0^{[i]}$*, then* $G[t] \xrightarrow{\Sigma} G[0]$*.*

Particularly, if the non-solvable system is a linear homogenous differential equation systems $(LDES_m^1)$, we further get a simple criterion on its zero G^L-solution, i.e., all vertices with 0 labels in [17] following.

Theorem 5.4([17]) *The zero G-solution of linear homogenous differential equation systems* $(LDES_m^1)$ *is asymptotically sum-stable on a spanning subgraph* $H \leq G[LDES_m^1]$ *if and only if* $\mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *in* $(LDES^1)$ *hold for* $\forall v \in V(H)$.

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