DOI: 10.2478/auom-2014-0050
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 22(3),2014, 59–72

An introduction to the theory of algebraic multi-hyperring spaces

Kostaq Hila and Bijan Davvaz

Abstract

A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer $n \geq 2$ which can be used both for discrete or connected spaces, particularly for geometries and spacetimes in theoretical physics. In this paper, applying the Smarandaches notion and combining this with hyperrings in hyperring theory, we introduce the notion of multi-hyperring space and initiate a study of multi-hyperring theory. Some characterizations and properties of multi-hyperring spaces are investigated and obtained. Some open problems are suggested for further study and investigation.

1 Introduction and preliminaries

The applications of mathematics in other disciplines, for example in informatics, play a key role and they represent, in the last decades, one of the purpose of the study of the experts of hyperstructures theory all over the world. Hyperstructures, as a natural extension of classical algebraic structures, in particular hypergroups, were introduced in 1934 by the French mathematician, Marty, at the 8th Congress of Scandinavian Mathematicians [16]. Since then, a lot of papers and several books have been written on this topic. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science(see [2, 19]) and they are studied in many countries of the

Key Words: Hyperrings, Multi-hyperring space, Hyperideal subspace chain, Smarandache multi-space.
2010 Mathematics Subject Classification: Primary 20N20; Secondary 16Y99.
Received: 12 April, 2013
Revised: 3 October, 2013
Accepted: 14 October, 2013

world. This theory has been subsequently developed by Corsini [2], Mittas [14], and by various authors. Basic definitions and propositions about the hyperstructures are found in [1, 2, 5, 19]. Krasner [13] has studied the notion of hyperfields, hyperrings, and then some researchers, namely, Davvaz [3, 7, 10], Vougiouklis [19, 20] and others followed him.

Hyperrings are essentially rings with approximately modified axioms. There are different notions of hyperrings $(R, +, \cdot)$. If the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a binary operation, then the hyperring is called Krasner (additive) hyperring [13]. In 2007, Davvaz and Leoreanu-Fotea [5] published a book titled Hyperring Theory and Applications.

A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer $n \geq 2$, which can be used both for discrete or connected spaces, particularly for geometries and spacetimes in theoretical physics. The notion of multi-spaces was introduced by Smarandache under his idea of hybrid mathematics: combining different fields into a unifying field [17, 18], which is more closer to our real life world. Today, this idea is widely accepted by the world of sciences (cf. [15]). In this paper, applying the Smarandaches notion and combining this with hyperrings in hyperring theory, we introduce the notion of multi-hyperring space and initate a study of multi-hyperring theory. Some characterizations and properties of multi-hyperring spaces are investigated and obtained. Some open problems are suggested for futher study and investigation.

Recall first the basic terms and definitions from the hyperstructure theory. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

An algebraic hyperstructure is a non-empty set H together with a map $\circ : H \times H \to \mathcal{P}^*(H)$ called *hyperoperation* or *join operation*, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H. A hyperstructure $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

If $x \in H$ and A, B are nonempty subsets of H then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\}, \text{ and } x \circ B = \{x\} \cdot B.$$

A non-empty subset B of a semihypergroup H is called a *sub-semihypergroup* of H if $B \circ B \subseteq B$ and H is called in this case *super-semihypergroup* of B. Let $(H, \circ)$ be a semihypergroup. Then H is called a *hypergroup* if it satisfies the reproduction axiom, for all $a \in H$, $a \circ H = H \circ a = H$. An element e in a semihypergroup H is called *identity* if

$$x \circ e = e \circ x = \{x\}, \forall x \in H.$$

An element 0 in a semihypergroup H is called *zero element* if

$$x \circ 0 = 0 \circ x = \{0\}, \forall x \in H.$$

A non-empty set H with a hyperoperation $+$ is said to be a *canonical hypergroup* if the following conditions hold:

1. for every $x, y \in H, x + y = y + x$,
2. for every $x, y, z \in H, x + (y + z) = (x + y) + z$,
3. there exists $0 \in H$, (called neutral element of H) such that $0 + x = \{x\} = x + 0$ for all $x \in H$,
4. for every $x \in H$, there exists a unique element denoted by $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
5. for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

For any subset A of a canonical hypergroup H, $-A$ denotes the set $\{-a : a \in A\}$. A non-empty subset N of a canonical hypergroup H is called a *subhypergroup* of H if N is a canonical hypergroup under the same hyperoperation as that of H. Equivalently, for every $x, y \in N, x - y \subseteq N$. In particular, for any $x \in N, x - x \subseteq N$. Since $0 \in x - x$, it follows that $0 \in N$.

Example 1. [1, 5]

(1) Let $C(n) = \{e_0, e_1, ..., e_{k(n)}\}$, where $k(n) = n/2$ if n is an even natural number and $k(n) = (n-1)/2$ if n is an odd natural number. For all e_s, e_t of $C(n)$, define $e_s \circ e_t = \{e_p, e_v\}$,where $p = \min\{s+t, n-(s+t)\}$, $v = |s-t|$. Then $(C(n), \circ)$ is a canonical hypergroup.

(2) Let (S, T) be a projective geometry, i.e., a system involving a set S of elements called *points* and a set T of sets of points called *lines*, which satisfies the following postulates:

- Any lines contains at least three points;
- Two distinct points a, b are contained in a unique line, that we shall denote by $L(a, b)$;
- If a, b, c, d are distinct points and $L(a, b) \cap L(c, d) \neq \emptyset$, then $L(a, c) \cap L(b, d) \neq \emptyset$.

Let e be an element which does not belong to S and let $S' = S \cup \{e\}$. We define the following hyperoperation on S':

- For all different points a, b of S, we consider $a \circ b$=$L(a, b)\backslash\{a, b\}$;

• If $a \in S$ and any line contains exactly three points, let $a \circ a$=$\{e\}$, otherwise $a \circ a$=$\{a, e\}$;

• For all $a \in S'$, we have $e \circ a = a \circ e = a$.

Then $(S', \circ)$ is a canonical hypergroup.

Algebraic hyperstructure theory has many applications in other disciplines. In [6], inheritance issue based on genetic information is looked at carefully via a new hyperalgebraic approach. Several examples are provided from different biology points of view, and it is shown that the theory of hyperstructures exactly fits the inheritance issue. In [12], the authors used the concept of algebraic hyperstructures in the F_2-genotypes with cross operation.

A physical example of hyperstructures associated with the elementary particle physics is presented in [11]. The theory of algebraic hyperstructures allows us to expand the group theory to much more sets of objects. In [11], the authors have shown the Leptons set, as an important group of elementary particles, along with a hyperoperation form a hyperstructure. The hyperoperation is the interaction between the Leptons considering the conservation rules. This theory is a new overlook to the elementary particle physics and helps us to make a new arrangement to the elementary particles.

Another applications of algebraic hyperstructures is in Chemistry. In [7, 8, 9], Davvaz et al. presented examples of algebraic hyperstructures associated with chain reactions and dismutation reactions.

There are several kinds of hyperrings that can be defined on a non-empty set R. In what follows, we shall consider one of the most general types of hyperrings.

Definition 1.1. A *hyperring* is a triple $(R, +, \cdot)$, where R is a non-empty set with a hyperaddition $+$ and a hypermultiplication $\cdot$ satisfying the following axioms:

1. $(R, +)$ is a canonical hypergroup,
2. $(R, \cdot)$ is a semihypergroup such that $x \cdot 0 = 0 \cdot x = \{0\}$ for all $x \in R$, (i.e, 0 is a bilaterally absorbing element),
3. The hypermultiplication $\cdot$ is distributive with respect to the hyperoperation $+$. That is, for every $x, y, z \in R, x \cdot (y + z) = x \cdot y + x \cdot z$, and $(x + y) \cdot z = x \cdot z + y \cdot z$.

Definition 1.2. A non-empty subset R' of R is called a *subhyperring* of $(R, +, \cdot)$ if $(R', +)$ is a subhypergroup of $(R, +)$ and $\forall x, y \in R', x \cdot y \in \mathcal{P}^*(R')$.

EXAMPLE 2. Let $R = \{0, a, b\}$ be a set with two hyperoperations defined as follows.

$+$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a,b\}$	R
b	$\{b\}$	R	$\{a,b\}$

$\cdot$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	R	R
b	$\{0\}$	R	R

Clearly, $(R, +, \cdot)$ is a hyperring.

EXAMPLE 3. Let $(R, +, \cdot)$ be a hyperring. Then $(M_n(R), \oplus, \odot)$ is a hyperring, where $M_n(R)$ is the set of all $n \times n$ matrices over R for some natural number n and the hyperoperations $\oplus$ and $\odot$ are defiend as follows:

For $x = (x_{ij}), y = (y_{ij}) \in M_n(R)$,

$$x \oplus y = \{z \in M_n(R) : z = (z_{ij}), z_{ij} \in x_{ij} + y_{ij}, 1 \leq i, j \leq n\} \text{ and}$$
$$x \odot y = \{z \in M_n(R) : z = (z_{ij}), z_{ij} \in \sum_{k=1}^{n} x_{ik} \cdot y_{kj}, 1 \leq i, j \leq n\}.$$

If there exists $u \in R$ such that $x \cdot u = u \cdot x = \{x\}$ for all $x \in R$, then u is called the scalar unit of R and is denoted by 1. The element 0 is called the *zero element* of R if $0 \cdot x = x \cdot 0 = \{0\}$ for all $x \in R$.

A non-empty subset A of R is called a *hyperideal* of $(R, +, \cdot)$ if $(A, +)$ is a subhypergroup of $(R, +)$ and $\forall x \in R, \forall y \in A$, both $x \cdot y$ and $y \cdot x$ are elements of $\mathcal{P}^*(A)$.

A hyperring R is said to satisfy the ascending (resp. descending) chain condition if for every ascending (resp. descending) sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ (resp. $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$) of hyperideals of R, there exists a natural number n such that $A_n = A_k$ for all $n \geq k$. If R satisfies the ascending (resp. descending) chain condition, we say R is a Noetherian (resp. Artinian) hyperring.

A non-empty subset Q of R is said to be a *quasi-hyperideal* of $(R, +, \cdot)$ if $(Q, +)$ is a subhypergroup of $(R, +)$ and $Q \cdot R \cap R \cdot Q \subseteq Q$. A non-empty subset B of R is said to be a *bi-hyperideal* of $(R, +, \cdot)$ if $(B, +)$ is a subhypergroup of $(R, +)$ and $B \cdot R \cdot B \subseteq B$. A hyperring R is said to be regular if for all $x \in R$, there is a $a \in R$ such that $x \in x \cdot a \cdot x$.

2 Introduction of multi-hyperring spaces

The notion of multi-spaces is introduced by Smarandache in [18] under an idea of hybrid mathematics: combining different fields into a unifying field [17], which can be formally defined with mathematical words by the following definition. Today, this idea is widely accepted by the world of sciences.

Definition 2.1. For any integer $i, 1 \leq i \leq n$ let A_i be a set with ensemble of law L_i, denoted by $(A_i; L_i)$. Then the union of $(A_i; L_i), 1 \leq i \leq n$

$$\tilde{A} = \bigcup_{i=1}^{n} (A_i; L_i),$$

is called a *multi-space*.

The conception of multi-hypergroup space is a generalization of the algebraic hypergroup. By combining the above Smarandache multi-spaces with hypergroups in hyperstructure theory, a new kind of algebraic hyperstructure called multi-hypergroup space is found, the definition of which is given as follows.

Definition 2.2. Let $\tilde{G} = \bigcup\limits_{i=1}^{m} G_i$ be a complete multi-space with a binary hyperoperation set $O(\tilde{G}) = \{\times_i, 1 \leq i \leq m\}$. If for any integers $i, 1 \leq i \leq m, (G_i; \times_i)$ is a hypergroup and $\forall x, y, z \in \tilde{G}$ and any two binary hyperoperations $\times$ and $\circ$, $\times \neq \circ$, there is one hyperoperation for example the hyperoperation $\times$ satisfying the distribution law to the hyperoperation $\circ$ if their hyperoperation results exist, i.e.,

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

and

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\tilde{G}$ is called a *multi-hypergroup space*.

Remark 1. If $m = 1$, then $\tilde{G} = (G_1; \times_1)$ is just a hypergroup.

Example 4. Let $(S, \circ)$ be an algebraic hyperstructure, i.e., $a \circ b \subseteq S, \forall a, b \in S$. Whence, let we take $C, C \subseteq S$ being a cyclic hypergroup. Now consider a partition of S

$$S = \bigcup_{k=1}^{m} G_k$$

with $m \geq 2$ such that $G_i \cap G_j = C, \forall i, j, 1 \leq i, j, \leq m$.

For an integer $k, 1 \leq k \leq m$, assume that $G_k = \{g_{k1}, g_{k2}, \ldots, g_{kl}\}$. We define a hyperoperation $\times_k$ on G_k as follows, which enables $(G_k, \times_k)$ to be a cyclic hypergroup.

$$\begin{aligned} g_{k1} \times_k g_{k1} &= \{g_{k1}, g_{k2}\}, \\ g_{k2} \times_k g_{k1} &= \{g_{k1}, g_{k3}\}, \\ &\vdots \\ g_{k(l-1)} \times_k g_{k1} &= \{g_{k1}, g_{kl}\}, \\ g_{kl} \times_k g_{k1} &= \{g_{k1}\}. \end{aligned}$$

Then, $S = \bigcup_{k=1}^{m} G_k$ is a complete multi-hyperspace with $m+1$ hyperoperations.

Now, by combining these Smarandache multi-spaces with hyperrings in hyperstructure theory, a new kind of algebraic hyperstructure called multi-hyperring spaces is found, the definition of which is given as follows.

Definition 2.3. Let $\tilde{R} = \bigcup_{i=1}^{m} R_i$ be a complete multi-space with a double binary hyperoperation set $O(\tilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$. If for any integers $i, j, i \neq j, 1 \leq i, j \leq m, (R_i; +_i, \times_i)$ is a hyperring and $\forall x, y, z \in \tilde{R}$,

$$(x +_i y) +_j z = x +_i (y +_j z), (x \times_i y) \times_j z = x \times_i (y \times_j z)$$

and

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, (y +_j z) \times_i z = y \times_i x +_j z \times_i x$$

provided all these hyperoperations result exists, then $\tilde{R}$ is called a *multi-hyperring space*. If for any integer $1 \leq i \leq m, (R; +_i, \times_i)$ is a hyperfield, then $\tilde{R}$ is called a *multi-hyperfield space*.

For a multi-hyperring space $\tilde{R} = \bigcup_{i=1}^{m} R_i$, let $\tilde{S} \subset \tilde{R}$ and $O(\tilde{S}) \subset O(\tilde{R})$. If $\tilde{S}$ is also a multi-hyperring space with a double binary hyperoperation set $O(\tilde{S})$, then $\tilde{S}$ is said a *multi-hyperring subspace* of $\tilde{R}$.

The main object of this paper is to characterize multi-hyperring spaces.

3 Characterizations of multi-hyperring spaces

The following theorem gives a criteria of being multi-hyperring subspace of a multi-hyperring space.

Theorem 3.1. *For a multi-hyperring space $\tilde{R} = \bigcup_{i=1}^{m} R_i$, a subset $\tilde{S} \subset \tilde{R}$ with a double binary hyperoperation set $O(\tilde{S}) \subset O(\tilde{R})$ is a multi-hyperring subspace of $\tilde{R}$ if and only if for any integer $k, 1 \leq k \leq m$, $(\tilde{S} \cap R_k; +_k, \times_k)$ is a subhyperring of $(R_k; +_k, \times_k)$ or $\tilde{S} \cap R_k = \emptyset$.*

Proof. For any integer $k, 1 \leq k \leq m$, if $(\tilde{S} \cap R_k; +_k, \times_k)$ is a subhyperring of $(R_k; +_k, \times_k)$ or $\tilde{S} \cap R_k = \emptyset$, then since $\tilde{S} = \bigcup_{i=1}^{m} (\tilde{S} \cap R_i)$, we know that $\tilde{S}$ is a multi-hyperring subspace by definition of multi-hyperring spaces.

Now, if $\tilde{S} = \bigcup\limits_{j=1}^{s} S_{i_j}$ is a multi-hyperring subspace of $\tilde{R}$ with a double binary hyperoperation set $O(\tilde{S}) = \{(+_{i_j}, \times_{i_j}), 1 \leq j \leq s\}$, then $(S_{i_j}; +_{i_j}, \times_{i_j})$ is a subhyperring of $(R_{i_j}; +_{i_j}, \times_{i_j})$. Therefore, for any integer $j, 1 \leq j \leq s, S_{i_j} = R_{i_j} \cap \tilde{S}$. But for other integer $l \in \{i : 1 \leq i \leq m\} \setminus \{i_j; 1 \leq j \leq s\}$, $\tilde{S} \cap S_l = \emptyset$. □

Theorem 3.2. *For a multi-hyperring space* $\tilde{R} = \bigcup\limits_{i=1}^{m} R_i$, *a subset* $\tilde{S} \subset \tilde{R}$ *with a double binary hyperoperation set* $O(\tilde{S}) \subset O(\tilde{R})$ *is a multi-hyperring subspace of* $\tilde{R}$ *if and only if for any double binary hyperoperations* $(+_j, \times_j) \in O(\tilde{S}), (\tilde{S} \cap R_j; +_j) \prec (R_j; +_j)$ *and* $(\tilde{S}; \times_j)$ *is complete.*

Proof. By Theorem 3.1, we have that $\tilde{S}$ is a multi-hyperring subspace if and only if for any integer $i, 1 \leq i \leq m$, $(\tilde{S} \cap R_i; +_i, \times_i)$ is a subhyperring of $(R_i; +_i, \times_i)$ or $\tilde{S} \cap R_i = \emptyset$. It is clear that $(\tilde{S} \cap R_i; +_i, \times_i)$ is a subhyperring of $(R_i; +_i, \times_i)$ if and only if for any double binary hyperoperation $(+_j, \times_j) \in O(\tilde{S})$, $(\tilde{S} \cap R_j; +_j) \prec (R_j; +_j)$ and $(\tilde{S}; \times_j)$ is a complete set. This completes the proof. □

A hyperideal subspace $\tilde{I}$ of a multi-hyperring space $\tilde{R} = \bigcup\limits_{i=1}^{m} R_i$ with a double binary hyperoperation set $O(\tilde{R})$ is a multi-hyperring subspace of $\tilde{R}$ satisfying the following conditions:

1. $\tilde{I}$ is a multi-hypergroup subspace with a hyperoperation set $\{+ : (+, \times) \in O(\tilde{I})\}$;
2. for any $r \in \tilde{R}, a \in \tilde{I}$ and $(+, \times) \in O(\tilde{I}), r \times a \subseteq \tilde{I}$ and $a \times r \subseteq \tilde{I}$ provided these hyperoperation results exist.

Theorem 3.3. *A subset* $\tilde{I}$ *with* $O(\tilde{I}), O(\tilde{I}) \subset O(\tilde{R})$ *of a multi-hyperring space* $\tilde{R} = \bigcup\limits_{i=1}^{m} R_i$ *with a double binary hyperoperation set* $O(\tilde{R}) = \{(+_i, \times_i) | 1 \leq i \leq m\}$ *is a multi-hyperideal subspace if and only if for any integer* i, $1 \leq i \leq m, (\tilde{I} \cap R_i, +_i, \times_i)$ *is a hyperideal of the hyperring* $(R_i, +_i, \times_i)$ *or* $\tilde{I} \cap R_i = \emptyset$.

Proof. By the definition of a hyperideal subspace, the necessity condition is obvious.

For the sufficiency, denote by $\tilde{R}(+, \times)$ the set of elements in $\tilde{R}$ with binary hyperoperations "$+$" and "$\times$". If there exists an integer i such that $\tilde{I} \cap R_i \neq \emptyset$ and $(\tilde{I} \cap R_i, +_i, \times_i)$ is a hyperideal of $(R_i, +_i, \times_i)$, then for all $a \in \tilde{I} \cap R_i$, $\forall r_i \in R_i$, we know that

$$r_i \times_i a \subseteq \tilde{I} \cap R_i; a \times_i r_i \subseteq \tilde{I} \cap R_i.$$

Notice that $\tilde{R}(+_i, \times_i) = R_i$. Therefore, we get that for all $r \in \tilde{R}$,

$$r \times_i a \subseteq \tilde{I} \cap R_i; \text{ and } a \times_i r \subseteq \tilde{I} \cap R_i$$

provided these hyperoperation results exist. Whence, $\tilde{I}$ is a hyperideal subspace of $\tilde{R}$. □

A hyperideal subspace $\tilde{I}$ of a multi-hyperring space $\tilde{R}$ is maximal if for any hyperideal subspace $\tilde{I}'$, if $\tilde{R} \supseteq \tilde{I}' \supseteq \tilde{I}$, then $\tilde{I}' = \tilde{R}$ or $\tilde{I}' = \tilde{I}$. For any order of these double hyperoperations in $O(\tilde{R})$ of a multi-hyperring space $\tilde{R} = \bigcup\limits_{i=1}^{m} R_i$, not loss of generality, let us assume it being $(+_1, \times_1) \succ (+_2, \times_2) \succ ... \succ (+_m, \times_m)$, we can construct a hyperideal subspace chain of $\tilde{R}$ by the following programming.

(i) Construct a hyperideal subspace chain

$$\tilde{R} \supset \tilde{R}_{11} \supset \tilde{R}_{12} \supset ... \supset \tilde{R}_{1s_1}$$

under the double binary hyperoperation $(+_1, \times_1)$, where $\tilde{R}_{11}$ is a maximal hyperideal subspace of $\tilde{R}$ and in general, for any integer i, $1 \leq i \leq m-1$, $\tilde{R}_{1(i+1)}$ is a maximal hyperideal subspace of $\tilde{R}_{1i}$.

(ii) If the hyperideal subspace

$$\tilde{R} \supset \tilde{R}_{11} \supset \tilde{R}_{12} \supset ... \supset \tilde{R}_{1s_1} \supset ... \supset \tilde{R}_{i1} \supset ... \supset \tilde{R}_{is_i}$$

has been constructed for $(+_1, \times_1) \succ (+_2, \times_2) \succ ... \succ (+_i, \times_i), 1 \leq i \leq m-1$, then construct a hyperideal subspace chain of $\tilde{R}_{is_i}$,

$$\tilde{R}_{is_i} \supset \tilde{R}_{(i+1)1} \supset \tilde{R}_{(i+1)2} \supset ... \supset \tilde{R}_{(i+1)s_1}$$

under the hyperoperations $(+_{i+1}, \times_{i+1})$, where $\tilde{R}_{(i+1)1}$ is a maximal hyperideal subspace of $\tilde{R}_{is_i}$ and in general, $\tilde{R}_{(i+1)(i+1)}$ is a maximal hyperideal subspace of $\tilde{R}_{(i+1)j}$ for any integer j, $1 \leq j \leq s_i - 1$. Define a hyperideal subspace chain of $\tilde{R}$ under $(+_1, \times_1) \succ (+_2, \times_2) \succ ... \succ (+_{i+1}, \times_{i+1})$ being

$$\tilde{R} \supset \tilde{R}_{11} \supset ... \supset \tilde{R}_{1s_1} \supset ... \supset \tilde{R}_{i1} \supset ... \supset \tilde{R}_{is_i} \supset \tilde{R}_{(i+1)1} \supset ... \supset \tilde{R}_{(i+1)s_{i+1}}.$$

Theorem 3.4. *For a multi-hyperring space $\tilde{R} = \bigcup\limits_{i=1}^{m} R_i$, its hyperideal subspace chain only has finite terms if and only if for any integer i, $1 \leq i \leq m$, the hyperideal chain of the hyperring $(R_i; +_i, \times_i)$ has finite terms, i.e., each hyperring $(R_i; +_i, \times_i)$ is an Artinian hyperring.*

Proof. Let the order of double hyperoperations in $\overrightarrow{O}(\tilde{R})$ be

$$(+_1, \times_1) \succ (+_2, \times_2) \succ .... \succ (+_m, \times_m)$$

and a maximal hyperideal chain in the hyperring $(R_1; +_1, \times_1)$ is

$$R_1 \succ R_{11} \succ ... \succ R_{1t_1}.$$

Calculation shows that

$$\tilde{R}_{11} = \tilde{R} \backslash \{R_1 \backslash R_{11}\} = R_{11} \bigcup (\bigcup_{i=2}^{m}) R_i,$$
$$\tilde{R}_{12} = \tilde{R}_{11} \backslash \{R_{11} \backslash R_{12}\} = R_{12} \bigcup (\bigcup_{i=2}^{m}) R_i$$
$$\vdots$$
$$\tilde{R}_{1t_1} = \tilde{R}_{1t1} \backslash \{R_{1(t_1-1)} \backslash R_{1t_1}\} = R_{1t_1} \bigcup (\bigcup_{i=2}^{m}) R_i.$$

By Theorem 3.3, we have that

$$\tilde{R} \supset \tilde{R}_{11} \supset \tilde{R}_{12} \supset ... \supset \tilde{R}_{1t_1}$$

is a maximal hyperideal subspace chain of $\tilde{R}$ under the double binary hyperoperation $(+_1, \times_1)$. In general, for any integer i, $1 \leq i \leq m-1$, let us assume

$$R_i \succ R_{i1} \succ ... \succ R_{it_i}$$

is a maximal hyperideal chain in the hyperring $(R_{(i-1)t_{i-1}}; +_i, \times_i)$. We have

$$\tilde{R}_{ik} = R_{ik} \bigcup (\bigcup_{j=i+1}^{m}) \tilde{R}_{ik} \bigcap R_i.$$

Then we know that

$$\tilde{R}_{(i-1)t_{i-1}} \supset \tilde{R}_{i1} \supset \tilde{R}_{i2} \supset ... \supset \tilde{R}_{it_i},$$

is a maximal hyperideal subspace chain of $\tilde{R}_{(i-1)t_{i-1}}$ under the double hyperoperation $(+_i, \times_i)$ by Theorem 3.3. Whence, if for any integer i, $1 \leq i \leq m$, the hyperideal chain of the hyperring $(R_i; +_i, \times_i)$ has finite terms, then the hyperideal subspace chain of the multi-hyperring space $\tilde{R}$ only has finite terms. On the other hand, if there exists one integer i_0 such that the hyperideal chain of the hyperring $(R_{i_0}; +_{i_0}, \times_{i_0})$ has infinite terms, then there must be infinite terms in the hyperideal subspace chain of the multi-hyperring space $\tilde{R}$. □

A multi-hyperring space is called an Artinian multi-hyperring space if each hyperideal subspace chain only has finite terms. The following is a consequence by Theorem 3.4.

Corollary 3.5. A multi-hyperring space $\tilde{R} = \bigcup\limits_{i=1}^{m} R_i$ with a double binary hyperoperation set $O(\tilde{R}) = \{(+_i, \times_i)|1 \leq i \leq m\}$ is an Artinian multi-hyperring space if and only if for any integer i, $1 \leq i \leq m$, the hyperring $(R_i; +_i, \times_i)$ is an Artinian hyperring.

For a multi-hyperring space $\tilde{R} = \bigcup\limits_{i=1}^{m} R_i$ with a double binary hyperoperation set $O(\tilde{R}) = \{(+_i, \times_i)|1 \leq i \leq m\}$, an element e is an *idempotent* element if $e \in e \times e = e_\times^2$ for a double binary hyperoperation $(+, \times) \in O(\tilde{R})$. We define the *directed sum* $\tilde{I}$ of two hyperideal subspaces $\tilde{I}_1$ and $\tilde{I}_2$ as follows:

(i) $\tilde{I} = \tilde{I}_1 \bigcup \tilde{I}_2$;

(ii) $\tilde{I}_1 \bigcap \tilde{I}_2 = \{0_+\}$, or $\tilde{I}_1 \bigcap \tilde{I}_2 = \emptyset$, where 0_+ denotes an unit element under the hyperoperation $+$.

Denote the directed sum of $\tilde{I}_1$ and $\tilde{I}_2$ by

$$\tilde{I} = \tilde{I}_1 \bigoplus \tilde{I}_2.$$

If for any $\tilde{I}_1, \tilde{I}_2, \tilde{I} = \tilde{I}_1 \oplus \tilde{I}_2$ implies that $\tilde{I}_1 = \tilde{I}$ or $\tilde{I}_2 = \tilde{I}$, then $\tilde{I}$ is said to be *non-reducible*. We get the following result for these Artinian multi-hyperring spaces.

Theorem 3.6. *Any Artinian multi-hyperring spaces $\tilde{R} = \bigcup\limits_{i=1}^{m} R_i$ with a double binary hyperoperation set $O(\tilde{R}) = \{(+_i, \times_i)|1 \leq i \leq m\}$ is a directed sum of finite non-reducible hyperideal subspaces, and if for any integer i, $1 \leq i \leq m, (R_i; , +_i, \times + i)$ has unit $1_{\times_i}$, then*

$$\tilde{R} = \bigoplus_{i=1}^{m} (\bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i)),$$

where $e_{ij}, 1 \leq j \leq s_i$ are orthogonal idempotent elements of the hyperring R_i.

Proof. Denote by $\tilde{M}$ the set of hyperideal subspaces which can not be represented by a directed sum of finite hyperideal subspaces in $\tilde{R}$. By Theorem 3.4, there is a minimal hyperideal subspace $\tilde{I}_0$ in $\tilde{M}$. It is obvious that $\tilde{I}_0$ is reducible.

Let us assume that $\tilde{I}_0 = \tilde{I}_1 + \tilde{I}_2$. Then $\tilde{I}_1 \notin \tilde{M}$ and $\tilde{I}_2 \notin \tilde{M}$. Therefore, $\tilde{I}_1$ and $\tilde{I}_2$ can be represented by directed sums of finite hyperideal subspaces. Whence, $\tilde{I}_0$ can be also represented by a directed sum of finite hyperideal subspaces. It is a contradiction, since $\tilde{I}_0 \in \tilde{M}$.

Now let $\tilde{R} = \bigoplus_{i=1}^{s} \tilde{I}_i$, where each $\tilde{I}_i, 1 \leq i \leq s$, is non-reducible. Notice that for a double hyperoperation $(+, \times)$, each non-reducible hyperideal subspace of $\tilde{R}$ has the form

$$(e \times R(\times)) \bigcup (R(\times) \times e), e \in R(\times).$$

Whence, we know that there is a set $T \subset \tilde{R}$ such that

$$\tilde{R} = \bigoplus_{e \in T, \times \in O(\tilde{R})} (e \times R(\times)) \bigcup (R(\times) \times e).$$

For any hyperoperation $\times \in O(\tilde{R})$ and a scalar unit $1_\times$, let us assume that

$$1_\times \in e_1 \oplus e_2 \oplus ... \oplus e_l, e_i \in T, 1 \leq i \leq s.$$

Then

$$e_i \times 1_\times \subseteq (e_i \times e_1) \oplus (e_i \times e_2) \oplus ... \oplus (e_i \times e_l).$$

Therefore, we get that

$$e_i \in e_i \times e_i = e_i^2 \text{ and } e_i \times e_j = 0_i \text{ for } i \neq j.$$

That is, $e_i, 1 \leq i \leq l$, are orthogonal idempotent elements of $\tilde{R}(\times)$. Notice that $\tilde{R}(\times) = R_h$ for some integer h. We know that $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of the hyperring $(R_h, +_h, \times_h)$. Denote by e_{hj} for $e_j, 1 \leq j \leq l$. Consider all scalar units in $\tilde{R}$, we get that

$$\tilde{R} = \bigoplus_{i=1}^{m} (\bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i)).$$

This completes the proof. $\square$

Open problems

1. Similar to Artinian multi-hyperring spaces, one can define Notherian multi-hyperring spaces, simple multi-hyperring spaces, etc. Also, it remains to be investigated and characterized its structure similar to the above results.

2. One can define a Jacobson or Brown-McCoy radical for multi-hyperring spaces and study their properties in multi-hyperring spaces.

3. One can consider the possibility of extending and generalizing of all the results obtained for hyperring to multi-hyperring spaces.

4. One can inquire further to the study of others multi-hyperstructure spaces.

References

[1] P. Corsini, *Prolegomena of hypergroup theory*, Second edition, Aviani editori, 1993.

[2] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, vol 5 of Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, The Netherlands 2003.

[3] B. Davvaz, A. Salasi, *A realization of hyperrings*, Commun. Algebra, 34(12)(2006), 4389-4400.

[4] B. Davvaz, *Isomorphism theorems of hyperrings*, Indian J. Pure Appl. Math., 35 (2004), no. 3, 321-331.

[5] B. Davvaz and V. Leoreanu-Fotea, *Hyperring theory and applications*, International Academic Press, Palm Harbor, Fla, USA, 2007.

[6] B. Davvaz, A. Dehghan Nezhad, M.M. Heidari, *Inheritance examples of algebraic hyperstructures*, Inform. Sci., 224 (2013), 180-187.

[7] B. Davvaz, A. Dehghan Nezhad and A. Benvidi, *Chemical hyperalgebra: Dismutation reactions*, MATCH Commun. Math. Comput. Chem., 67 (2012), 55-63.

[8] B. Davvaz, A. Dehghan Nezad and A. Benvidi, *Chain reactions as experimental examples of ternary algebraic hyperstructures*, MATCH Commun. Math. Comput. Chem., 65(2) (2011), 491-499.

[9] B. Davvaz and A. Dehghan Nezhad, *Dismutation reactions as experimental verifications of ternary algebraic hyperstructures*, MATCH Commun. Math. Comput. Chem., 68 (2012), 551-559.

[10] B. Davvaz and T. Vougiouklis, *Commutative rings obtained from hyperrings (H_v-rings) with α^*-relations*, Commun. Algebra, 35 (2007), 3307-3320.

[11] A. Dehghan Nezhad, S.M. Moosavi Nejad, M. Nadjafikhah and B. Davvaz, *A physical example of algebraic hyperstructures: Leptons*, Indian J. Physics, 86(11) (2012), 1027-1032.

[12] M. Ghadiri, B. Davvaz and R. Nekouian, *H_v-Semigroup structure on F_2-offspring of a gene pool*,Int. J. Biomath., 5(4) (2012) 1250011 (13 pages).

[13] M. Krasner, *A class of hyperrings and hyperfields*, Internat. J. Math. Math. Sci. **6** (1983), 307-311.

[14] J. Mittas, *Hypergroupes canoniques*, Math. Balkanica, 2 (1972), 165-179.

[15] L.F. Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ USA 2006.

[16] F. Marty, *Sur une generalization de la notion de group*, 8th Congres Math. Scandinaves, Stockholm, (1934), 45-49.

[17] F. Smarandache, *A Unifying Field in Logics, Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[18] F. Smarandache, *Mixed noneuclidean geometries*, Stud. Cercet. Stiint. Ser. Mat. Univ. Bacau **8** (1998) 127–154.

[19] T. Vougiouklis, *Hyperstructures and their Representations*, Hadronic Press, Florida, USA, 1994.

[20] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, in Algebraic Hyperstructures and Applications, pp. 203-211, World Scientific Publishing, Teaneck, NJ, USA, 1991.

Kostaq HILA,
Department of Mathematics & Computer Science,
Faculty of Natural Sciences
University of Gjirokastra,
Albania.
Email: kostaq_hila@yahoo.com, khila@uogj.edu.al

Bijan DAVVAZ,
Department of Mathematics,
Yazd University,
Yazd, Iran.
Email: davvaz@yazd.ac.ir, bdavvaz@yahoo.com