SMARANDACHE CURVES ACCORDING TO BISHOP FRAME IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we investigate special Smarandache curves according to Bishop frame in Euclidean 3-space and we give some differential geometric properties of Smarandache curves. Also we find the centers of the osculating spheres and curvature spheres of Smarandache curves.

1. Introduction

Special Smarandache curves have been investigated by some differential geometers [1,8]. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a *Smarandache Curve*. M. Turgut and S. Yilmaz have defined a special case of such curves and call it *Smarandache* TB_2 *Curves* in the space E_1^4 [8]. They have dealed with a special Smarandache curves which is defined by the tangent and second binormal vector fields. They have called such curves as *Smarandache* TB_2 *Curves*. Additionally, they have computed formulas of this kind curves by the method expressed in [10]. A. T. Ali has introduced some special Smarandache curves in the Euclidean space. He has studied Frenet-Serret invariants of a special case [1].

In this paper, we investigate special Smarandache curves such as *Smarandache Curves* TN_1, TN_2, N_1N_2 *and* TN_1N_2 according to Bishop frame in Euclidean 3-space. Furthermore, we find differential geometric properties of these special curves and we calculate first and second curvature (natural curvatures) of these curves. Also we find the centers of the curvature spheres and osculating spheres of Smarandache curves.

2. Preliminaries

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(N_1(s), N_2(s))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point. If the derivatives of $(N_1(s), N_2(s))$ depend only on $T(s)$ and not each other we

Key words and phrases. Smarandache curves, Bishop frame, Natural curvatures, Osculating sphere, Curvature sphere, Centers of osculating spheres.

2000 AMS Subject Classifications:53A04.

can make $N_1(s)$ and $N_2(s)$ vary smoothly throughout the path regardless of the curvature [2,6,7].

In addition, suppose the curve α is an arclength-parametrized C^2 curve. Suppose we have C^1 unit vector fields N_1 and $N_2 = T \wedge N_1$ along the curve α so that

$$\langle T, N_1 \rangle = \langle T, N_2 \rangle = \langle N_1, N_2 \rangle = 0,$$

i.e., T , N_1, N_2 will be a smoothly varying right-handed orthonormal frame as we move along the curve. (To this point, the Frenet frame would work just fine if the curve were C^3 with $\kappa \neq 0$) But now we want to impose the extra condition that $\langle N_1', N_2 \rangle = 0$. We say the unit first normal vector field N_1 is parallel along the curve α. This means that the only change of N_1 is in the direction of T. A Bishop frame can be defined even when a Frenet frame cannot (e.g., when there are points with $\kappa = 0$). Therefore, we have the alternative frame equations

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}.$$

One can show that

$$\kappa(s) = \sqrt{k_1^2 + k_2^2},\ \theta(s) = \arctan\left(\frac{k_2}{k_1}\right),\ k_1 \neq 0,\ \tau(s) = -\frac{d\theta(s)}{ds}$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(s) ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation [2,6,7].

Bishop curvatures are defined by

$$k_1 = \kappa \cos\theta, \quad k_2 = \kappa \sin\theta \tag{2.1}$$

and

$$T = T \ , \quad N_1 = N\cos\theta - B\sin\theta \ , \quad N_2 = N\sin\theta + B\cos\theta \tag{2.2}$$

[9].

3. Smarandache Curves According to Bishop Frame

In [1], author gave following definitions:

Definition 1. *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Serret-Frenet frame. Smarandache curves TN are defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}(T + N).$$

Definition 2. *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Serret-Frenet frame. Smarandache curves NB are defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}(N + B).$$

Definition 3. *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Serret-Frenet frame. Smarandache curves TNB are defined by*

$$\beta(s^*) = \frac{1}{\sqrt{3}}(T + N + B).$$

In this section, we investigate Smarandache curves according to Bishop frame in Euclidean 3-space. Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and denote by $\{T_\alpha, N_1^\alpha, N_2^\alpha\}$ the moving Bishop frame along the curve α. The following Bishop formulae is given by

$$\dot{T}_\alpha = k_1^\alpha N_1^\alpha + k_2^\alpha N_2^\alpha \ \ , \ \ \dot{N}_1^\alpha = -k_1^\alpha T_\alpha \ \ , \ \ \dot{N}_2^\alpha = -k_2^\alpha T_\alpha \tag{3.1}$$

with

$$T_\alpha = N_1^\alpha \wedge N_2^\alpha \ , \ N_1^\alpha = -T_\alpha \wedge N_2^\alpha \ , \ N_2^\alpha = T_\alpha \wedge N_1^\alpha .$$

3.1. TN_1-**Smarandache Curves.**

Definition 4. *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T_\alpha, N_1^\alpha, N_2^\alpha\}$ be its moving Bishop frame. $T_\alpha N_1^\alpha-$Smarandache curves can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}} \left(T_\alpha + N_1^\alpha\right). \tag{3.2}$$

Now, we can investigate Bishop invariants of $T_\alpha N_1^\alpha-$Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.2) with respect to s, we get

$$\dot{\beta} = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}} \left(k_1^\alpha T_\alpha - k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha\right) \tag{3.3}$$

and

$$T_\beta \frac{ds^*}{ds} = \frac{-1}{\sqrt{2}} \left(k_1^\alpha T_\alpha - k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2}{2}}. \tag{3.4}$$

The tangent vector of curve β can be written as follow,

$$T_\beta = \frac{-1}{\sqrt{2\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2}} \left(k_1^\alpha T_\alpha - k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha\right). \tag{3.5}$$

Differentiating (3.5) with respect to s, we obtain

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\left(2\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2\right)^{\frac{3}{2}}} \left(\lambda_1 T_\alpha + \lambda_2 N_1^\alpha + \lambda_3 N_2^\alpha\right) \tag{3.6}$$

where

$$\begin{aligned}
\lambda_1 &= -\dot{k}_1^\alpha \left(k_2^\alpha\right)^2 - 2\left(k_1^\alpha\right)^4 - 3\left(k_1^\alpha\right)^2\left(k_2^\alpha\right)^2 - \left(k_2^\alpha\right)^4 + k_1^\alpha k_2^\alpha \dot{k}_2^\alpha \\
\lambda_2 &= -2\left(k_1^\alpha\right)^4 - \left(k_1^\alpha\right)^2\left(k_2^\alpha\right)^2 + \dot{k}_1^\alpha \left(k_2^\alpha\right)^2 - k_1^\alpha k_2^\alpha \dot{k}_2^\alpha \\
\lambda_3 &= -2\left(k_1^\alpha\right)^3 k_2^\alpha - k_1^\alpha \left(k_2^\alpha\right)^3 + 2\left(k_1^\alpha\right)^2 \dot{k}_2^\alpha - 2k_1^\alpha \dot{k}_1^\alpha k_2^\alpha .
\end{aligned}$$

Substituting (3.4) in (3.6), we get

$$T_\beta' = \frac{\sqrt{2}}{\left(2\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2\right)^2} \left(\lambda_1 T_\alpha + \lambda_2 N_1^\alpha + \lambda_3 N_2^\alpha\right).$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\kappa_\beta = \left\|T'_\beta\right\| = \frac{\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}{\left(2\left(k_1^\alpha\right)^2+\left(k_2^\alpha\right)^2\right)^2}$$

and

$$N_\beta = \frac{1}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\lambda_1 T_\alpha + \lambda_2 N_1^\alpha + \lambda_3 N_2^\alpha\right).$$

On the other hand, we express

$$B_\beta = \frac{1}{\sqrt{2\left(k_1^\alpha\right)^2+\left(k_2^\alpha\right)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\begin{vmatrix} T_\alpha & N_1^\alpha & N_2^\alpha \\ -k_1^\alpha & k_1^\alpha & k_2^\alpha \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$B_\beta = \frac{1}{\sqrt{2\left(k_1^\alpha\right)^2+\left(k_2^\alpha\right)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\sigma_1 T_\alpha + \sigma_2 N_1^\alpha + \sigma_3 N_2^\alpha\right)$$

where

$$\sigma_1 = \lambda_3 k_1^\alpha - \lambda_2 k_2^\alpha \ \ , \ \ \sigma_2 = \lambda_3 k_1^\alpha + \lambda_1 k_2^\alpha \ \ , \ \ \sigma_3 = -k_1^\alpha\left(\lambda_1+\lambda_2\right).$$

We differentiate (3.3) with respect to s in order to calculate the torsion

$$\ddot{\beta} = \frac{-1}{\sqrt{2}}\left\{(\dot{k}_1^\alpha + (k_1^\alpha)^2 + (k_2^\alpha)^2)T_\alpha - (\dot{k}_1^\alpha - (k_1^\alpha)^2)N_1^\alpha - (\dot{k}_2^\alpha - k_1^\alpha k_2^\alpha)N_2^\alpha\right\},$$

and similarly

$$\dddot{\beta} = \frac{1}{\sqrt{2}}\left(\rho_1 T_\alpha + \rho_2 N_1^\alpha + \rho_3 N_2^\alpha\right),$$

where

$$\begin{aligned}
\rho_1 &= -\ddot{k}_1^\alpha - 3k_1^\alpha \dot{k}_1^\alpha - 3k_2^\alpha \dot{k}_2^\alpha + \left(k_1^\alpha\right)^3 + k_1^\alpha\left(k_2^\alpha\right)^2 \\
\rho_2 &= -3k_1^\alpha \dot{k}_1^\alpha - \left(k_1^\alpha\right)^3 - k_1^\alpha\left(k_2^\alpha\right)^2 + \ddot{k}_1^\alpha \\
\rho_3 &= -2\dot{k}_1^\alpha k_2^\alpha - \left(k_1^\alpha\right)^2 k_2^\alpha - \left(k_2^\alpha\right)^3 - k_1^\alpha \dot{k}_2^\alpha + \ddot{k}_2^\alpha.
\end{aligned}$$

The torsion of curve β is

$$\tau_\beta = \frac{\sqrt{2}\left\{\begin{array}{c} ((k_1^\alpha)^2 - \dot{k}_1^\alpha)(\rho_3 k_1^\alpha + \rho_1 k_2^\alpha) + k_1^\alpha(\dot{k}_2^\alpha - k_1^\alpha k_2^\alpha)(\rho_1+\rho_2) \\ -(\dot{k}_1^\alpha + (k_1^\alpha)^2 + (k_2^\alpha)^2)(\rho_2 k_2^\alpha - \rho_3 k_1^\alpha) \end{array}\right\}}{(k_1^\alpha \dot{k}_2^\alpha - \dot{k}_1^\alpha k_2^\alpha)^2 + (k_1^\alpha \dot{k}_2^\alpha - 2\left(k_1^\alpha\right)^2 k_2^\alpha - \dot{k}_1^\alpha k_2^\alpha - \left(k_2^\alpha\right)^3)^2 + (2\left(k_1^\alpha\right)^3 + k_1^\alpha\left(k_2^\alpha\right)^2)^2}.$$

The first and second normal field of curve β are as follow. Then, from (2.2) we obtain

$$N_1^\beta = \frac{1}{\sqrt{\mu}\eta}\left\{\begin{array}{c} \left(\sqrt{\mu}\cos\theta_\beta\lambda_1 - \sin\theta_\beta\sigma_1\right)T_\alpha \\ +\left(\sqrt{\mu}\cos\theta_\beta\lambda_2 - \sin\theta_\beta\sigma_2\right)N_1^\alpha \\ +\left(\sqrt{\mu}\cos\theta_\beta\lambda_3 - \sin\theta_\beta\sigma_3\right)N_2^\alpha \end{array}\right\},$$

and

$$N_2^\beta = \frac{1}{\sqrt{\mu}\eta}\left\{\begin{array}{c}\left(\sqrt{\mu}\sin\theta_\beta\lambda_1+\cos\theta_\beta\sigma_1\right)T_\alpha\\ +\left(\sqrt{\mu}\sin\theta_\beta\lambda_2+\cos\theta_\beta\sigma_2\right)N_1^\alpha\\ +\left(\sqrt{\mu}\sin\theta_\beta\lambda_3+\cos\theta_\beta\sigma_3\right)N_2^\alpha\end{array}\right\}$$

where $\theta_\beta = -\int \tau_\beta(s)ds$ and $\mu = 2\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2$, $\eta = \sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}$

Now, we can calculate natural curvatures of curve β, so from (2.1) we get

$$k_1^\beta = \frac{\sqrt{2(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\cos\theta_\beta}{\left(2\left(k_1^\alpha\right)^2+\left(k_2^\alpha\right)^2\right)^2}$$

and

$$k_2^\beta = \frac{\sqrt{2(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\sin\theta_\beta}{\left(2\left(k_1^\alpha\right)^2+\left(k_2^\alpha\right)^2\right)^2}.$$

3.2. TN_2-**Smarandache Curves.**

Definition 5. *Let $\alpha=\alpha(s)$ be a unit speed regular curve in E^3 and $\{T_\alpha, N_1^\alpha, N_2^\alpha\}$ be its moving Bishop frame. $T_\alpha N_2^\alpha-$Smarandache curves can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}\left(T_\alpha + N_2^\alpha\right). \tag{3.7}$$

Now, we can investigate Bishop invariants of $T_\alpha N_2^\alpha-$Smarandache curves according to $\alpha=\alpha(s)$. Differentiating (3.7) with respect to s, we get

$$\dot{\beta} = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left(k_2^\alpha T_\alpha - k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha\right) \tag{3.8}$$

and

$$T_\beta\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left(k_2^\alpha T_\alpha - k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{\left(k_1^\alpha\right)^2+2\left(k_2^\alpha\right)^2}{2}}. \tag{3.9}$$

The tangent vector of curve β can be written as follow,

$$T_\beta = \frac{-1}{\sqrt{\left(k_1^\alpha\right)^2+2\left(k_2^\alpha\right)^2}}\left(k_2^\alpha T_\alpha - k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha\right). \tag{3.10}$$

Differentiating (3.10) with respect to s, we obtain

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\left(\left(k_1^\alpha\right)^2+2\left(k_2^\alpha\right)^2\right)^{\frac{3}{2}}}\left(\lambda_1 T_\alpha + \lambda_2 N_1^\alpha + \lambda_3 N_2^\alpha\right) \tag{3.11}$$

where

$$\begin{array}{rcl}\lambda_1 &=& -\left(k_1^\alpha\right)^2\dot{k_2^\alpha} - \left(k_1^\alpha\right)^4 - 3\left(k_1^\alpha\right)^2\left(k_2^\alpha\right)^2 - 2\left(k_2^\alpha\right)^4 + k_1^\alpha\dot{k_1^\alpha}k_2^\alpha\\ \lambda_2 &=& -\left(k_1^\alpha\right)^3 k_2^\alpha - 2k_1^\alpha\left(k_2^\alpha\right)^3 + 2\dot{k_1^\alpha}\left(k_2^\alpha\right)^2 - 2k_1^\alpha k_2^\alpha\dot{k_2^\alpha}\\ \lambda_3 &=& -\left(k_1^\alpha\right)^2\left(k_2^\alpha\right)^2 - 2\left(k_2^\alpha\right)^4 + \left(k_1^\alpha\right)^2\dot{k_2^\alpha} - k_1^\alpha\dot{k_1^\alpha}k_2^\alpha.\end{array}$$

Substituting (3.9) in (3.11), we get

$$T'_\beta = \frac{\sqrt{2}}{\left(\left(k_1^\alpha\right)^2 + 2\left(k_2^\alpha\right)^2\right)^2}\left(\lambda_1 T_\alpha + \lambda_2 N_1^\alpha + \lambda_3 N_2^\alpha\right).$$

Then, the first curvature and the principal normal vector field of curve β are respectively,

$$\kappa_\beta = \left\|T'_\beta\right\| = \frac{\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}{\left(\left(k_1^\alpha\right)^2 + 2\left(k_2^\alpha\right)^2\right)^2}$$

and

$$N_\beta = \frac{1}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\lambda_1 T_\alpha + \lambda_2 N_1^\alpha + \lambda_3 N_2^\alpha\right).$$

On the other hand, we express

$$B_\beta = \frac{1}{\sqrt{\left(k_1^\alpha\right)^2 + 2\left(k_2^\alpha\right)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\begin{vmatrix} T_\alpha & N_1^\alpha & N_2^\alpha \\ -k_2^\alpha & k_1^\alpha & k_2^\alpha \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$B_\beta = \frac{1}{\sqrt{\left(k_1^\alpha\right)^2 + 2\left(k_2^\alpha\right)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\sigma_1 T_\alpha + \sigma_2 N_1^\alpha + \sigma_3 N_2^\alpha\right)$$

where

$$\sigma_1 = \lambda_3 k_1^\alpha - \lambda_2 k_2^\alpha \quad , \quad \sigma_2 = \left(\lambda_3 + \lambda_1\right) k_2^\alpha \quad , \quad \sigma_3 = -\left(\lambda_2 k_2^\alpha + \lambda_1 k_1^\alpha\right).$$

We differentiate (3.8) with respect to s in order to calculate the torsion of curve β

$$\ddot{\beta} = \frac{-1}{\sqrt{2}}\left\{(\dot{k}_2^\alpha + (k_1^\alpha)^2 + (k_2^\alpha)^2)T_\alpha - (\dot{k}_1^\alpha - k_1^\alpha k_2^\alpha)N_1^\alpha - (\dot{k}_2^\alpha - (k_2^\alpha)^2)N_2^\alpha\right\},$$

and similarly

$$\dddot{\beta} = \frac{1}{\sqrt{2}}\left(\rho_1 T_\alpha + \rho_2 N_1^\alpha + \rho_3 N_2^\alpha\right),$$

where

$$\begin{aligned}
\rho_1 &= -\ddot{k}_2^\alpha - 3k_1^\alpha \dot{k}_1^\alpha - 3k_2^\alpha \dot{k}_2^\alpha + \left(k_1^\alpha\right)^2 k_2^\alpha + \left(k_2^\alpha\right)^3 \\
\rho_2 &= -2k_1^\alpha \dot{k}_2^\alpha - \left(k_1^\alpha\right)^3 - k_1^\alpha \left(k_2^\alpha\right)^2 - \dot{k}_1^\alpha k_2^\alpha + \ddot{k}_1^\alpha \\
\rho_3 &= -3k_2^\alpha \dot{k}_2^\alpha - \left(k_1^\alpha\right)^2 k_2^\alpha - \left(k_2^\alpha\right)^3 + \ddot{k}_2^\alpha.
\end{aligned}$$

The torsion of curve β is

$$\tau_\beta = \frac{\sqrt{2}\left\{\begin{array}{c} k_2^\alpha(k_1^\alpha k_2^\alpha - \dot{k}_1^\alpha)(\rho_3+\rho_1) + (\dot{k}_2^\alpha - (k_2^\alpha)^2)(\rho_2 k_2^\alpha + \rho_1 k_1^\alpha) \\ -(\dot{k}_2^\alpha + (k_1^\alpha)^2 + (k_2^\alpha)^2)(\rho_2 k_2^\alpha - \rho_3 k_1^\alpha) \end{array}\right\}}{(k_1^\alpha \dot{k}_2^\alpha - \dot{k}_1^\alpha k_2^\alpha)^2 + (-2\left(k_2^\alpha\right)^3 - \left(k_1^\alpha\right)^2 k_2^\alpha)^2 + (2k_1^\alpha\left(k_2^\alpha\right)^2 - \dot{k}_1^\alpha k_2^\alpha + k_1^\alpha \dot{k}_2^\alpha + \left(k_1^\alpha\right)^3)^2}.$$

The first and second normal field of curve β are as follow. Then, from (2.2) we obtain

$$N_1^\beta = \frac{1}{\sqrt{\mu}\eta}\left\{\begin{array}{c}\left(\sqrt{\mu}\cos\theta_\beta\lambda_1 - \sin\theta_\beta\sigma_1\right)T_\alpha\\ +\left(\sqrt{\mu}\cos\theta_\beta\lambda_2 - \sin\theta_\beta\sigma_2\right)N_1^\alpha\\ +\left(\sqrt{\mu}\cos\theta_\beta\lambda_3 - \sin\theta_\beta\sigma_3\right)N_2^\alpha\end{array}\right\},$$

and

$$N_2^\beta = \frac{1}{\sqrt{\mu}\eta}\left\{\begin{array}{c}\left(\sqrt{\mu}\sin\theta_\beta\lambda_1 + \cos\theta_\beta\sigma_1\right)T_\alpha\\ +\left(\sqrt{\mu}\sin\theta_\beta\lambda_2 + \cos\theta_\beta\sigma_2\right)N_1^\alpha\\ +\left(\sqrt{\mu}\sin\theta_\beta\lambda_3 + \cos\theta_\beta\sigma_3\right)N_2^\alpha\end{array}\right\},$$

where $\theta_\beta = -\int \tau_\beta(s)ds$ and $\mu = (k_1^\alpha)^2 + 2\,(k_2^\alpha)^2$, $\eta = \sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}$.

Now, we can calculate natural curvatures of curve β, so from (2.1) we get

$$k_1^\beta = \frac{\sqrt{2(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\cos\theta_\beta}{\left((k_1^\alpha)^2 + 2\,(k_2^\alpha)^2\right)^2}$$

and

$$k_2^\beta = \frac{\sqrt{2(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\sin\theta_\beta}{\left((k_1^\alpha)^2 + 2\,(k_2^\alpha)^2\right)^2}.$$

3.3. N_1N_2-**Smarandache Curves.**

Definition 6. *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T_\alpha, N_1^\alpha, N_2^\alpha\}$ be its moving Bishop frame. $N_1^\alpha N_2^\alpha-$Smarandache curves can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}\left(N_1^\alpha + N_2^\alpha\right). \tag{3.12}$$

Now, we can investigate Bishop invariants of $N_1^\alpha N_2^\alpha-$Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.12) with respect to s, we get

$$\dot{\beta} = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-\left(k_1^\alpha + k_2^\alpha\right)}{\sqrt{2}}T_\alpha \tag{3.13}$$

and

$$T_\beta\frac{ds^*}{ds} = \frac{-\left(k_1^\alpha + k_2^\alpha\right)}{\sqrt{2}}T_\alpha$$

where

$$\frac{ds^*}{ds} = \frac{k_1^\alpha + k_2^\alpha}{\sqrt{2}}. \tag{3.14}$$

The tangent vector of curve β can be written as follow,

$$T_\beta = -T_\alpha. \tag{3.15}$$

Differentiating (3.15) with respect to s, we obtain

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = -k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha \tag{3.16}$$

Substituting (3.14) in (3.16), we get

$$T_\beta' = \frac{-\sqrt{2}}{k_1^\alpha + k_2^\alpha}\left(k_1^\alpha N_1^\alpha + k_2^\alpha N_2^\alpha\right).$$

Then, the first curvature and the principal normal vector field of curve β are respectively,

$$\kappa_\beta = \left\| T'_\beta \right\| = \frac{\sqrt{2}\sqrt{\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2}}{k_1^\alpha + k_2^\alpha}$$

and

$$N_\beta = \frac{-1}{\sqrt{\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2}} \left(k_1^\alpha N_1^\alpha + k_2^\alpha N_2^\alpha\right).$$

On the other hand, we express

$$B_\beta = \frac{1}{\sqrt{\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2}} \begin{vmatrix} T_\alpha & N_1^\alpha & N_2^\alpha \\ -1 & 0 & 0 \\ 0 & -k_1^\alpha & -k_2^\alpha \end{vmatrix}.$$

So, the binormal vector of curve β is

$$B_\beta = \frac{-1}{\sqrt{\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2}} \left(k_2^\alpha N_1^\alpha - k_1^\alpha N_2^\alpha\right).$$

We differentiate (3.13) with respect to s in order to calculate the torsion of curve β

$$\ddot{\beta} = \frac{-1}{\sqrt{2}} \left\{ \left(\dot{k}_1^\alpha + \dot{k}_2^\alpha \right) T_\alpha + \left(\left(k_1^\alpha\right)^2 + k_1^\alpha k_2^\alpha \right) N_1^\alpha + \left(k_1^\alpha k_2^\alpha + \left(k_2^\alpha\right)^2 \right) N_2^\alpha \right\},$$

and similarly

$$\dddot{\beta} = \frac{-1}{\sqrt{2}} \left(\rho_1 T_\alpha + \rho_2 N_1^\alpha + \rho_3 N_2^\alpha\right),$$

where

$$\begin{aligned} \rho_1 &= \ddot{k}_1^\alpha + \ddot{k}_2^\alpha - \left(k_1^\alpha\right)^3 - \left(k_1^\alpha\right)^2 k_2^\alpha - k_1^\alpha \left(k_2^\alpha\right)^2 - \left(k_2^\alpha\right)^3 \\ \rho_2 &= 3k_1^\alpha \dot{k}_1^\alpha + 2k_1^\alpha \dot{k}_2^\alpha + \dot{k}_1^\alpha k_2^\alpha \\ \rho_3 &= 2\dot{k}_1^\alpha k_2^\alpha + 3k_2^\alpha \dot{k}_2^\alpha + k_1^\alpha \dot{k}_2^\alpha. \end{aligned}$$

The torsion of curve β is

$$\tau_\beta = \frac{-\sqrt{2}\left\{ \rho_3(\left(k_1^\alpha\right)^2 + k_1^\alpha k_2^\alpha) - \rho_2(k_1^\alpha k_2^\alpha + \left(k_2^\alpha\right)^2) \right\}}{\left(k_1^\alpha + k_2^\alpha\right) \left\{ (k_1^\alpha k_2^\alpha + \left(k_2^\alpha\right)^2)^2 + (\left(k_1^\alpha\right)^2 + k_1^\alpha k_2^\alpha)^2 \right\}}.$$

The first and second normal field of curve β are as follow. Then, from (2.2) we obtain

$$N_1^\beta = \frac{1}{\sqrt{\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2}} \left\{ \left(k_2^\alpha \sin\theta_\beta - k_1^\alpha \cos\theta_\beta\right) N_1^\alpha - \left(k_2^\alpha \cos\theta_\beta + k_1^\alpha \sin\theta_\beta\right) N_2^\alpha \right\},$$

and

$$N_2^\beta = \frac{1}{\sqrt{\left(k_1^\alpha\right)^2 + \left(k_2^\alpha\right)^2}} \left\{ -\left(k_1^\alpha \sin\theta_\beta + k_2^\alpha \cos\theta_\beta\right) N_1^\alpha + \left(k_1^\alpha \cos\theta_\beta - k_2^\alpha \sin\theta_\beta\right) N_2^\alpha \right\}$$

where $\theta_\beta = -\int \tau_\beta(s) ds$.

Now, we can calculate the natural curvatures of curve β, so from (2.1) we get

$$k_1^\beta = \frac{\sqrt{2((k_1^\alpha)^2 + (k_2^\alpha)^2)}\cos\theta_\beta}{k_1^\alpha + k_2^\alpha}$$

and

$$k_2^\beta = \frac{\sqrt{2((k_1^\alpha)^2 + (k_2^\alpha)^2)}\sin\theta_\beta}{k_1^\alpha + k_2^\alpha}.$$

3.4. TN_1N_2-**Smarandache Curves.**

Definition 7. *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T_\alpha, N_1^\alpha, N_2^\alpha\}$ be its moving Bishop frame. $T_\alpha N_1^\alpha N_2^\alpha-$Smarandache curves can be defined by*

$$\beta(s^*) = \frac{1}{\sqrt{3}}\left(T_\alpha + N_1^\alpha + N_2^\alpha\right). \tag{3.17}$$

Now, we can investigate Bishop invariants of $T_\alpha N_1^\alpha N_2^\alpha-$Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.17) with respect to s, we get

$$\dot{\beta} = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{3}}\left(\left(k_1^\alpha + k_2^\alpha\right)T_\alpha - k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha\right) \tag{3.18}$$

and

$$T_\beta \frac{ds^*}{ds} = \frac{-1}{\sqrt{3}}\left(\left(k_1^\alpha + k_2^\alpha\right)T_\alpha - k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2\left((k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2\right)}{3}}. \tag{3.19}$$

The tangent vector of curve β can be written as follow,

$$T_\beta = \frac{-1}{\sqrt{2\left\{(k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2\right\}}}\left(\left(k_1^\alpha + k_2^\alpha\right)T_\alpha - k_1^\alpha N_1^\alpha - k_2^\alpha N_2^\alpha\right). \tag{3.20}$$

Differentiating (3.20) with respect to s, we obtain

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{2\sqrt{2}\left\{(k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2\right\}^{\frac{3}{2}}}\left(\lambda_1 T_\alpha + \lambda_2 N_1^\alpha + \lambda_3 N_2^\alpha\right) \tag{3.21}$$

where

$$\begin{aligned}
\lambda_1 &= -2\dot{k}_1^\alpha\left(k_2^\alpha\right)^2 - \left(k_1^\alpha\right)^2\dot{k}_2^\alpha + k_1^\alpha k_2^\alpha \dot{k}_2^\alpha - 2\left(k_1^\alpha\right)^4 - 2\left(k_1^\alpha\right)^3 k_2^\alpha \\
&\quad -4\left(k_1^\alpha\right)^2\left(k_2^\alpha\right)^2 - 2k_1^\alpha\left(k_2^\alpha\right)^3 - 2\left(k_2^\alpha\right)^4 + k_1^\alpha \dot{k}_1^\alpha k_2^\alpha + \dot{k}_1^\alpha\left(k_2^\alpha\right)^2 \\
\lambda_2 &= -2\left(k_1^\alpha\right)^4 - 4\left(k_1^\alpha\right)^3 k_2^\alpha - 4\left(k_1^\alpha\right)^2\left(k_2^\alpha\right)^2 - 2k_1^\alpha\left(k_2^\alpha\right)^3 + k_1^\alpha \dot{k}_1^\alpha k_2^\alpha \\
&\quad +2\dot{k}_1^\alpha\left(k_2^\alpha\right)^2 - \left(k_1^\alpha\right)^2\dot{k}_2^\alpha - 2k_1^\alpha k_2^\alpha \dot{k}_2^\alpha \\
\lambda_3 &= -2\left(k_1^\alpha\right)^3 k_2^\alpha - 4\left(k_1^\alpha\right)^2\left(k_2^\alpha\right)^2 - 4k_1^\alpha\left(k_2^\alpha\right)^3 - 2\left(k_2^\alpha\right)^4 \\
&\quad +2\left(k_1^\alpha\right)^2\dot{k}_2^\alpha + k_1^\alpha k_2^\alpha \dot{k}_2^\alpha - 2k_1^\alpha \dot{k}_1^\alpha k_2^\alpha - \dot{k}_1^\alpha\left(k_2^\alpha\right)^2.
\end{aligned}$$

Substituting (3.19) in (3.21), we get

$$T'_\beta = \frac{\sqrt{3}}{4\left\{(k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2\right\}^2}\left(\lambda_1 T_\alpha + \lambda_2 N_1^\alpha + \lambda_3 N_2^\alpha\right).$$

Then, the first curvature and the principal normal vector field of curve β are respectively,

$$\kappa_\beta = \left\|T'_\beta\right\| = \frac{\sqrt{3}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}{4\left\{(k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2\right\}^2}$$

and

$$N_\beta = \frac{1}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\lambda_1 T_\alpha + \lambda_2 N_1^\alpha + \lambda_3 N_2^\alpha\right). \tag{3.22}$$

On the other hand, we express

$$B_\beta = \frac{1}{\sqrt{2\left((k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2\right)}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\begin{vmatrix} T_\alpha & N_1^\alpha & N_2^\alpha \\ -\left(k_1^\alpha + k_2^\alpha\right) & k_1^\alpha & k_2^\alpha \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$B_\beta = \frac{1}{\sqrt{2\left((k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2\right)}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\sigma_1 T_\alpha + \sigma_2 N_1^\alpha + \sigma_3 N_2^\alpha\right) \tag{3.23}$$

where

$$\sigma_1 = \lambda_3 k_1^\alpha - \lambda_2 k_2^\alpha \quad , \quad \sigma_2 = \lambda_3\left(k_1^\alpha + k_2^\alpha\right) + \lambda_1 k_2^\alpha \quad , \quad \sigma_3 = -\left(\lambda_2\left(k_1^\alpha + k_2^\alpha\right) + \lambda_1 k_1^\alpha\right).$$

We differentiate (3.18) with respect to s in order to calculate the torsion of curve β,

$$\ddot{\beta} = \frac{-1}{\sqrt{3}}\left\{(\dot{k}_1^\alpha + \dot{k}_2^\alpha + (k_1^\alpha)^2 + (k_2^\alpha)^2)T_\alpha - (\dot{k}_1^\alpha - (k_1^\alpha)^2 - k_1^\alpha k_2^\alpha)N_1^\alpha - (\dot{k}_2^\alpha - k_1^\alpha k_2^\alpha - (k_2^\alpha)^2)N_2^\alpha\right\},$$

and similarly

$$\dddot{\beta} = \frac{1}{\sqrt{3}}\left(\rho_1 T_\alpha + \rho_2 N_1^\alpha + \rho_3 N_2^\alpha\right),$$

where

$$\begin{aligned}
\rho_1 &= -\ddot{k}_1^\alpha - \ddot{k}_2^\alpha - 3k_1^\alpha \dot{k}_1^\alpha - 3k_2^\alpha \dot{k}_2^\alpha + (k_1^\alpha)^3 + (k_1^\alpha)^2 k_2^\alpha + k_1^\alpha (k_2^\alpha)^2 + (k_2^\alpha)^3 \\
\rho_2 &= -3k_1^\alpha \dot{k}_1^\alpha - 2k_1^\alpha \dot{k}_2^\alpha - (k_1^\alpha)^3 - k_1^\alpha (k_2^\alpha)^2 - \dot{k}_1^\alpha k_2^\alpha + \ddot{k}_1^\alpha \\
\rho_3 &= -2\dot{k}_1^\alpha k_2^\alpha - 3k_2^\alpha \dot{k}_2^\alpha - (k_1^\alpha)^2 k_2^\alpha - (k_2^\alpha)^3 - k_1^\alpha \dot{k}_2^\alpha + \ddot{k}_2^\alpha
\end{aligned}$$

The torsion of curve β is

$$\tau_\beta = \frac{\sqrt{3}\left\{\begin{array}{c} (\dot{k}_1^\alpha - (k_1^\alpha)^2 - k_1^\alpha k_2^\alpha)(-\rho_3 k_1^\alpha - \rho_3 k_2^\alpha - \rho_1 k_2^\alpha) \\ +(\dot{k}_2^\alpha - k_1^\alpha k_2^\alpha - (k_2^\alpha)^2)(\rho_2 k_1^\alpha + \rho_2 k_2^\alpha + \rho_1 k_1^\alpha) \\ -(\dot{k}_1^\alpha + \dot{k}_2^\alpha + (k_1^\alpha)^2 + (k_2^\alpha)^2)(-\rho_3 k_1^\alpha + \rho_2 k_2^\alpha) \end{array}\right\}}{\left\{\begin{array}{c} (k_1^\alpha \dot{k}_2^\alpha - \dot{k}_1^\alpha k_2^\alpha)^2 + (k_1^\alpha \dot{k}_2^\alpha - 2\left(k_1^\alpha\right)^2 k_2^\alpha - 2k_1^\alpha \left(k_2^\alpha\right)^2 - 2\left(k_2^\alpha\right)^3 - \dot{k}_1^\alpha k_2^\alpha)^2 \\ +(2\left(k_1^\alpha\right)^3 + 2\left(k_1^\alpha\right)^2 k_2^\alpha + 2k_1^\alpha \left(k_2^\alpha\right)^2 - \dot{k}_1^\alpha k_2^\alpha + k_1^\alpha \dot{k}_2^\alpha)^2 \end{array}\right\}}.$$

The first and second normal field of curve β are as follow. Then, from (2.2) we obtain

$$N_1^\beta = \frac{1}{\sqrt{2\mu}\eta}\left\{\begin{array}{c} \left(\sqrt{2\mu}\cos\theta_\beta\lambda_1 - \sin\theta_\beta\sigma_1\right) T_\alpha \\ +\left(\sqrt{2\mu}\cos\theta_\beta\lambda_2 - \sin\theta_\beta\sigma_2\right) N_1^\alpha \\ +\left(\sqrt{2\mu}\cos\theta_\beta\lambda_3 - \sin\theta_\beta\sigma_3\right) N_2^\alpha \end{array}\right\},$$

and

$$N_2^\beta = \frac{1}{\sqrt{2\mu}\eta}\left\{\begin{array}{c} \left(\sqrt{2\mu}\sin\theta_\beta\lambda_1 + \cos\theta_\beta\sigma_1\right) T_\alpha \\ +\left(\sqrt{2\mu}\sin\theta_\beta\lambda_2 + \cos\theta_\beta\sigma_2\right) N_1^\alpha \\ +\left(\sqrt{2\mu}\sin\theta_\beta\lambda_3 + \cos\theta_\beta\sigma_3\right) N_2^\alpha \end{array}\right\},$$

where $\theta_\beta = -\int \tau_\beta(s)ds$ and $\mu = (k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2$, $\eta = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$

Now, we can calculate the natural curvatures of curve β, so from (2.1) we get

$$k_1^\beta = \frac{\sqrt{3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}\cos\theta_\beta}{4((k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2)^2}$$

and

$$k_2^\beta = \frac{\sqrt{3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}\sin\theta_\beta}{4((k_1^\alpha)^2 + k_1^\alpha k_2^\alpha + (k_2^\alpha)^2)^2}.$$

4. Curvature Spheres and Osculating Spheres of Smarandache Curves

Definition 8. *Let $\alpha : (a,b) \to \mathbb{R}^n$ be a regular curve and let $F : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. We say that a parametrically defined curve α and an implicity defined hypersurface $F^{-1}(0)$ have contact of order k at $\alpha(t_0)$ provided we have $(Fo\alpha)(t_0) = (Fo\alpha)'(t_0) = ... = (Fo\alpha)^{(k)}(t_0) = 0$ but $(Fo\alpha)^{(k+1)}(t_0) \neq 0$ [5].*

If any circle which has a contact of first-order with a planar curve, it is called the *curvature circle*. If any sphere which has a contact of second-order with a space curve, It is called the *curvature sphere*. Furthermore, if any sphere which has a contact of third-order with a space curve, it is called the *osculating sphere*.

Theorem 1. *Let $\beta(s^*)$ be a unit speed Smarandache curve of $\alpha(s)$ with first curvature k_1^β and second curvature k_2^β, then the centres of curvature spheres of $\beta(s^*)$*

according to Bishop frame are

$$\begin{aligned} c(s^*) &= \beta(s^*)+\frac{k_1^\beta \mp k_2^\beta\sqrt{r^2\left((k_1^\beta)^2-(k_2^\beta)^2\right)-1}}{4(k_1^\beta)^2}N_1^\beta \\ &\quad+\frac{3k_1^\beta \pm k_2^\beta\sqrt{r^2((k_1^\beta)^2-(k_2^\beta)^2)-1}}{4k_1^\beta k_2^\beta}N_2^\beta. \end{aligned}$$

Proof. Assume that $c(s^*)$ is the centre of curvature sphere of $\beta(s^*)$, then radius vector of curvature sphere is

$$c-\beta=\delta_1 T_\beta+\delta_2 N_1^\beta+\delta_3 N_2^\beta.$$

Let we define the function F such that $F=\langle c(s^*)-\beta(s^*), c(s^*)-\beta(s^*)\rangle - r^2$. First and second derivatives of F are as follow.

$$F'=-2\delta_1$$

$$F''=-2(k_1^\beta\delta_2+k_2^\beta\delta_3-1).$$

The condition of contact of second-order requires $F=0$, $F'=0$ and $F''=0$. Then, coefficients $\delta_1,\delta_2,\delta_3$ are obtained

$$\begin{aligned} \delta_1 &= 0 \\ \delta_2 &= \frac{k_1^\beta \mp k_2^\beta\sqrt{r^2\left((k_1^\beta)^2-(k_2^\beta)^2\right)-1}}{4(k_1^\beta)^2} \\ \delta_3 &= \frac{3k_1^\beta \pm k_2^\beta\sqrt{r^2((k_1^\beta)^2-(k_2^\beta)^2)-1}}{4k_1^\beta k_2^\beta}. \end{aligned}$$

Thus, centres of the curvature spheres of Smarandache curve β are

$$\begin{aligned} c(s^*) &= \beta(s^*)+\frac{k_1^\beta \mp k_2^\beta\sqrt{r^2\left((k_1^\beta)^2-(k_2^\beta)^2\right)-1}}{4(k_1^\beta)^2}N_1^\beta \\ &\quad+\frac{3k_1^\beta \pm k_2^\beta\sqrt{r^2((k_1^\beta)^2-(k_2^\beta)^2)-1}}{4k_1^\beta k_2^\beta}N_2^\beta. \end{aligned}$$

□

Let $\delta_3=\lambda$, then $c(s^*)=\beta(s^*)+\delta_2 N_1^\beta+\lambda N_2^\beta$ is a line passing the point $\beta(s^*)+\delta_2 N_1^\beta$ and parallel to line N_2^β. Thus we have the following corollary.

Corollary 1. *Let $\beta(s^*)$ be a unit speed Smarandache curve of $\alpha(s)$ then each centres of the curvature spheres of β are on a line.*

Theorem 2. *Let $\beta(s^*)$ be a unit speed Smarandache curve of $\alpha(s)$ with first curvature k_1^β and second curvature k_2^β , then the centres of osculating spheres of $\beta(s^*)$ according to Bishop frame are*

$$c(s^*)=\beta(s^*)+\frac{(k_2^\beta)'}{(k_1^\beta)^2\left(\frac{k_2^\beta}{k_1^\beta}\right)'}N_1^\beta-\frac{(k_1^\beta)'}{(k_1^\beta)^2\left(\frac{k_2^\beta}{k_1^\beta}\right)'}N_2^\beta.$$

Proof. Assume that $c(s^*)$ is the centre of osculating sphere of $\beta(s^*)$, then radius vector of osculating sphere is

$$c(s^*) - \beta(s^*) = \delta_1 T_\beta + \delta_2 N_1^\beta + \delta_3 N_2^\beta.$$

Let we define the function F such that $F = \langle c(s^*) - \beta(s^*), c(s^*) - \beta(s^*)\rangle - r^2$. First, second and third derivatives of F are as follow.

$$F' = -2\delta_1$$

$$F'' = -2(k_1^\beta \delta_2 + k_2^\beta \delta_3 - 1)$$

$$F''' = -2\left((k_1^\beta)'\delta_2 - (k_1^\beta)^2\delta_1 + (k_2^\beta)'\delta_3 - (k_2^\beta)^2\delta_1\right).$$

The condition of contact of third-order requires $F = 0$, $F' = 0$, $F'' = 0$ and $F''' = 0$. Then, coefficients $\delta_1, \delta_2, \delta_3$ are obtained

$$\delta_1 = 0, \qquad \delta_2 = \frac{(k_2^\beta)'}{(k_1^\beta)^2\left(\frac{k_2^\beta}{k_1^\beta}\right)'}, \qquad \delta_3 = -\frac{(k_1^\beta)'}{(k_1^\beta)^2\left(\frac{k_2^\beta}{k_1^\beta}\right)'}.$$

Thus, centres of the osculating spheres of Smarandache curve β can be written as follow

$$c(s^*) = \beta(s^*) + \frac{(k_2^\beta)'}{(k_1^\beta)^2\left(\frac{k_2^\beta}{k_1^\beta}\right)'} N_1^\beta - \frac{(k_1^\beta)'}{(k_1^\beta)^2\left(\frac{k_2^\beta}{k_1^\beta}\right)'} N_2^\beta.$$

□

Corollary 2. *Let $\beta(s^*)$ be a unit speed Smarandache curve of $\alpha(s)$ with first curvature k_1^β and second curvature k_2^β , then the radius of the osculating sphere of $\beta(s^*)$ is*

$$r(s^*) = \frac{\sqrt{\left((k_1^\beta)'\right)^2 + \left((k_2^\beta)'\right)^2}}{(k_1^\beta)^2\left(\frac{k_2^\beta}{k_1^\beta}\right)'}.$$

Example 1. *Let $\alpha(t)$ be the Salkowski curve given by*

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \tag{4.1}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{1}{\sqrt{1+m^2}}\left(-\frac{1-n}{4(1+2n)}\sin((1+2n)t) - \frac{1+n}{4(1-2n)}\sin((1-2n)t) - \frac{1}{2}\sin t\right)\\
\alpha_2 &= \frac{1}{\sqrt{1+m^2}}\left(\frac{1-n}{4(1+2n)}\cos((1+2n)t) + \frac{1+n}{4(1-2n)}\cos((1-2n)t) + \frac{1}{2}\cos t\right)\\
\alpha_3 &= \frac{\cos(2nt)}{4m\sqrt{1+m^2}}
\end{aligned}$$

and

$$n = \frac{m}{\sqrt{1+m^2}}.$$

$\alpha(t)$ can be expressed with the arc lenght parameter. We get unit speed regular curve by using the parametrization $t = \frac{1}{n}\arcsin\left(n\sqrt{1+m^2}s\right)$. Furthermore, graphics of special Smarandache curves are as follow. Here $m = \sqrt{3}$.

Figure 1 : Smarandache Curve $T_\alpha N_1^\alpha$

Figure 2 : Smarandache Curve $T_\alpha N_2^\alpha$

Figure 3 : Smarandache Curve $N_1^\alpha N_2^\alpha$

Figure 4 : Smarandache Curve $T_\alpha N_1^\alpha N_2^\alpha$

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