

SMARANDACHE LATTICE AND PSEUDO COMPLEMENT

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ABSTRACT. In this paper, we have introduced smarandache - 2 - Algebraic structure of lattice namely smarandache lattice. A smarandache 2- algebraic structure on a set N means a weak algebraic structure A_0 on N such that there exists a proper subset M of N which is embedded with a stronger algebraic structure A_1 . Stronger algebraic structure means a structure which satisfies more axioms, by proper subset one can understand a subset different from the empty set, by the unit element if any, and from the whole set. We have defined smarandache lattice and obtained some of its characterization through Pseudo complemented. For the basic concept, we referred the PadilaRaul [4].

1. INTRODUCTION

New notions are introduced in algebra to study more about the congruence in number theory by Florentinsmarandache [1]. By <proper subset> of a set A , we consider a set P included in A , and different from A , different from the empty set, and from the unit element in A - if any they rank the algebraic structures using an order relationship.

The algebraic structures $S_1 \ll S_2$ if :both are defined on the same set; all S_1 laws are also S_2 laws ; all axioms of S_1 law are accomplished by the corresponding S_2 law; S_2 law strictly accomplishes more axioms than S_1 laws, or in other words S_2 laws has more laws than S_1 .

For example : Semi group \ll monoid \ll group \ll ring \ll field, or Semi group \ll commutative semi group, ring \ll unitary ring, etc. they define a General special structure to be a structure SM on a set A , different from a structure SN , such that a proper subset of A is an SN structure, where $SM \ll SN$.

2. PRELIMINARIES

Definition 2.1. Let P be a lattice with 0 . Let $x \in P$, x^* is a Pseudo complemented of x iff $x^* \in P$ and $x \wedge x^* = 0$ and for every $y \in P$: if $x \wedge y = 0$ then $y \leq x^*$.

Definition 2.2. Let P be a pseudo complemented lattice. $N_P = \{ x^* : x \in P \}$ is the set of complements in P . $N_P = \{ N_P \leq N, \neg N, 0_N, 1_N, N, N \}$ where:

- (i) \leq_N , is defined by: for every $x, y \in N_P$: $x \leq_N y$ iff $x \leq_P y$
- (ii) $\neg N$ is defined by :

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- for every $x \in N_P : \neg N(x) = x^*$
 (iii) \wedge_N is defined by:
 for every $x, y \in N_P : x \wedge_N y = x \wedge_P y$
 (iv) \vee_N is defined by :
 for every $x, y \in N_P : x \vee_N y = (x * \wedge_P y^*)^*$
 (v) $1_N = 0_P^*, 0_N = 0_P$.

Definition 2.3. Let P be a lattice with $0, I_P$ is the set of all ideals in P

$I_P = \langle I_P, \leq_1, \wedge_1, \vee_1, 0_1, 1_1 \rangle$ where:
 $\leq = \subseteq, i \wedge_1 j = I \cap J, i \vee_1 j = (I \cup J), 0_1 = 0_A, 1_1 = A$

Definition 2.4. If P is a distributive lattice with $0, I_P$ is a complete Pseudo Complemented lattice. Let P be a lattice with 0 . NI_P , the set of normal ideals in P , is given by $NI_P = \{ I * I_P : II_P \}$ Alternatively $NI_P = \{ II_P : I = I * * \}$. Thus $NI_P = \{ NI_P, \subseteq, \cap, \cup, \wedge_{NI}, \vee_{NI} \}$, NI_P is the set of Pseudo Complements in I_P .

Definition 2.5. A Pseudo complemented distributive lattice P is called a stone lattice if, for all $a \in P$, it satisfies the property $a^* \vee a^{**} = 1$.

Definition 2.6. Let P be a pseudo complemented distributive lattice. Then for any filter F of P , define the set $\delta(F)$ as follows $\delta(F) = \{ a^* \in P / a^* \in F \}$.

Definition 2.7. Let P be a pseudo complemented distributive lattice. An ideal I of P is called a δ -ideal if $I = \delta(F)$ for some filter F of P .

Now we have introduced a definition by [4]:

Definition 2.8. A lattice S is said to be a Smarandache lattice. If there exist a proper subset L of S , Which is a Boolean algebra with respect to the same induced operations of S .

3. CHARACTERIZATIONS

Theorem 3.1. Let S be a lattice. If there exist a proper subset N_P of S defined in definition 2.2. Then S is a smarandache lattice.

Proof. By hypothesis, let S be a lattice and whose proper subset $N_P = \{ x^* : x \in P \}$ is the set of all Pseudo complements in P . By definition, Let P is a pseudo complement lattice. $N_P = \{ x^* : x \in P \}$ is the set of complements in P .

$N_P = \{ N_P, \leq_N, \neg_N, 0_N, 1_N, \wedge_N, \vee_N \}$

where: (i) \leq_N , is defined by: for every $x, y \in N_P : x \leq_N y$ iff $x \leq_P y$

(ii) \neg_N is defined by :

for every $x \in N_P : \neg_N(x) = x^*$

(iii) \wedge_N is defined by:

for every $x, y \in N_P : x \wedge_N y = x \wedge_P y$

(iv) \vee_N is defined by :

for every $x, y \in N_P : x \vee_N y = (x * \wedge_P y^*)^*$

$$(v) 1_N = 0_P^*, 0_N = 0_P .$$

It is enough to prove that N_P is a Boolean algebra.

(i) For every $x, y \in N_P$,

$x \wedge_N y \in N_P$ and \wedge_N is meet under \leq_N .

If $x, y \in N_P$, then $x = x^{**}$ and $y = y^{**}$.

Since $x \wedge_P y \leq_P x$, by result if $x \leq_N y$ then $y^* \leq_N x^*$,

$x^* \leq_P (x \wedge_P y)^*$, and, with by result if $x \leq_N y$ then $y^* \leq_N x^*$,

$(x \wedge_P y)^{**} \leq_P x$. Similarly, $(x \wedge_P y)^{**} \leq_P y$.

Hence $(x \wedge_P y)^{**} \in N_P$.

By result, $x \leq_N x^{**}$, $(x \wedge_P y) \leq_P (x \wedge_P y)^{**}$,

Hence $(x \wedge_P y) \in N_P$, $(x \wedge_N y) \in N_P$.

If $a \in N_P$ and $a \leq_N x$ and $a \leq_N y$,

Then $a \leq_P x$ and $a \leq_P y$, $a \leq_P (x \wedge_P y)$,

Hence $a \leq_N (x \wedge_N y)$. So indeed N is meet in \leq_N .

(ii) For every $x, y \in N_P : x \vee_N y \in N_P$ and \vee_N is join under \leq_N .

Let $x, y \in N_P$. Then $x^*, y^* \in N_P$. Then by (i), $(x^* \wedge_P y^*) \in N_P$

Hence $(x^* \wedge_P y^*)^* \in N_P$, and hence $(x \vee_N y) \in N_P$

$(x^* \wedge_P y^*) \leq_P x^*$, hence, by result $x \leq_N x^{**}$

$x^{**} \leq_P (x^* \wedge_P y^*)^*$,

By result $N_P = \{ x \in P; x = x^{**} \}$, $x \leq_P (x^* \wedge_P y^*)^*$.

Similarly, $y \leq_P (x^* \wedge_P y^*)^*$.

If $a \in N_P$ and $x \leq_N a$ and $y \leq_N a$, then $x \leq_P a$ and $y \leq_P a$, then by result if $x \leq_N y$

Then $y^* \leq_N x^*$, $a^* \leq_P x^*$ and $a^* \leq_P y^*$, hence $a^* \leq_P (x^* \wedge_P y^*)$

Hence, by result if $x \leq_N y$ then $y^* \leq_N x^*$,

$(x^* \wedge_P y^*)^* \leq_P a^{**}$ hence, by result $N_P = \{ x \in P; x = x^{**} \}$,

$(x^* \wedge_P y^*)^* \in N_P$, hence $x \vee_N y \leq_N a$ so, indeed \vee_N is join in \leq_N .

(iii) $0_N, 1_N \in N_P$ and 0_N and 1_N are the bounds of N_P .

Obviously $1_N \in N_P$, since $1_N = 0_P^*$ since for every $a \in N_P$, $a \wedge_P 0_P = 0_P$,

For every $a \in N_P$, $a \leq_P 0_P^*$, hence $a \leq_N 1_N$.

$0_P^*, 0_P^{**} \in N_P$ Hence $0_P^* \wedge_P 0_P^{**} \in N_P$ But of course, $0_P^* \wedge_P$

$0_P^{**} = 0_P$ Hence $0_P \in N_P$, $0_N \in N_P$, Obviously, for every $a \in N_P$; $0_P \leq_P a$.

Hence for every $a \in N_P : 0_N \leq_N a$. So N_P is bounded lattice.

(iv) For every $a \in N_P : \neg_N(a) \in N_P$ and

for every $a \in N_P : a \wedge_N \neg_N(a) = 0_N$ and

For every $a \in N_P : a \vee_N \neg_N(a) = 1_N$.

Let $a \in N_P$, Obviously $\neg_N(a) \in N_P$

$a \wedge_N \neg_N(a) = a \wedge_P a^* = ((a^* \wedge_P a)^*)^* = (a^* \wedge_P a)^* = 0_P^* = 1_N$

$a \vee_N \neg_N(a) = a \vee_P a^* = 0_P = 0_N$.

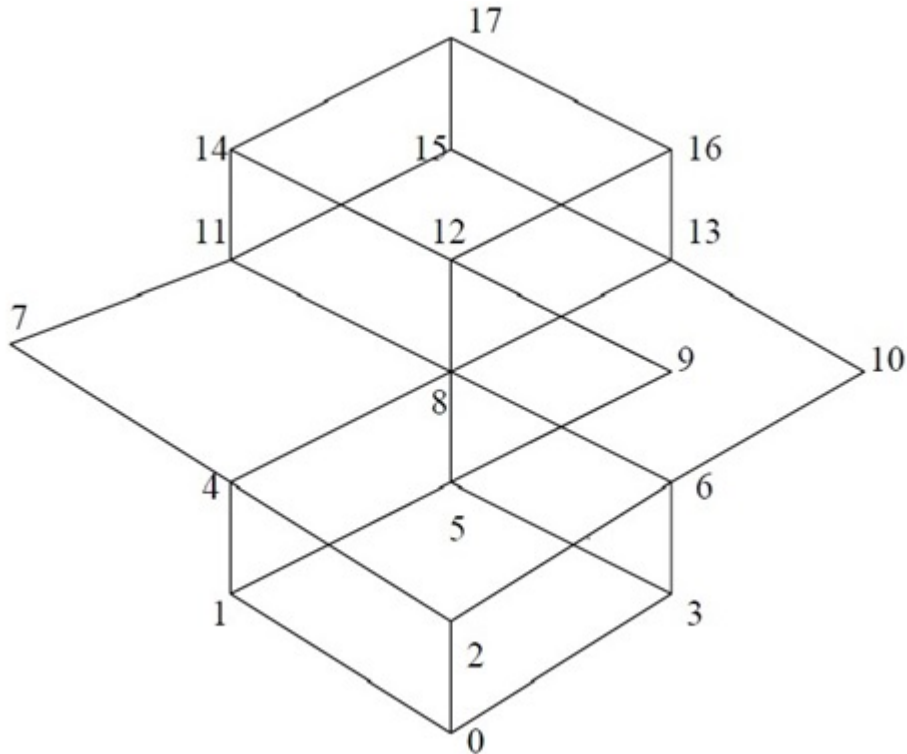
So N_P is a bounded complemented lattice.

(v) Since $x \leq_N (x \vee_N (y \wedge_N z))$, $(x \wedge_N z) \leq_N x \vee_N (y \wedge_N z)$
 Also $(y \wedge_N z) \leq_N x \vee_N (y \wedge_N z)$
 Obviously, if $a \leq_N b$, then $a \wedge_N b^* = 0_N$, Since $b \wedge b^* = 0_N$,
 Hence, $(x \wedge_N z) \wedge_N (x \vee_N (y \wedge_N z))^* = 0_N$ and
 $(y \wedge_N z) \wedge_N (x \vee_N (y \wedge_N z))^* = 0_N$,
 $x \wedge_N (z \wedge_N (x \vee_N (y \wedge_N z))^*) = 0_N$,
 $y \wedge_N (z \wedge_N (x \vee_N (y \wedge_N z))^*) = 0_N$.

By definition of Pseudo complement: $z \wedge_N (x \vee_N (y \wedge_N z))^* \leq_N x^*$,
 $z \wedge_N (x \vee_N (y \wedge_N z))^* \leq_N y^*$,
 Hence $z \wedge_N (x \vee_N (y \wedge_N z))^* \leq_N x^* \wedge_N y^*$
 Once again, If $a \leq_N b$, then $a \wedge_N b^* = 0_N$,
 Hence, $z \wedge_N (x \vee_N (y \wedge_N z))^* \wedge_N (x^* \wedge_N y^*)^* = 0_N$
 $z \wedge_N (x^* \wedge_N y^*)^* \leq_N (x \vee_N (y \wedge_N z))^{**}$
 Now, by definition of \wedge_N : $z \wedge_N (x^* \vee_N y^*)^* = z \wedge_N (x \vee_N y)$
 And by $N_P = \{ x \in P, : x = x^{**} \}$: $(x \vee_N (y \wedge_N z))^{**} = x \vee_N (y \wedge_N z)$,
 Hence: $z \wedge_N (x \vee_N y) \leq_N x \vee_N (y \wedge_N z)$.

Hence, indeed N_P is a Boolean Algebra. Therefore by definition, S is a Smarandache lattice

Example: Distributive lattice D_3 Figure 1.1



D_3 is pseudocomplemented:

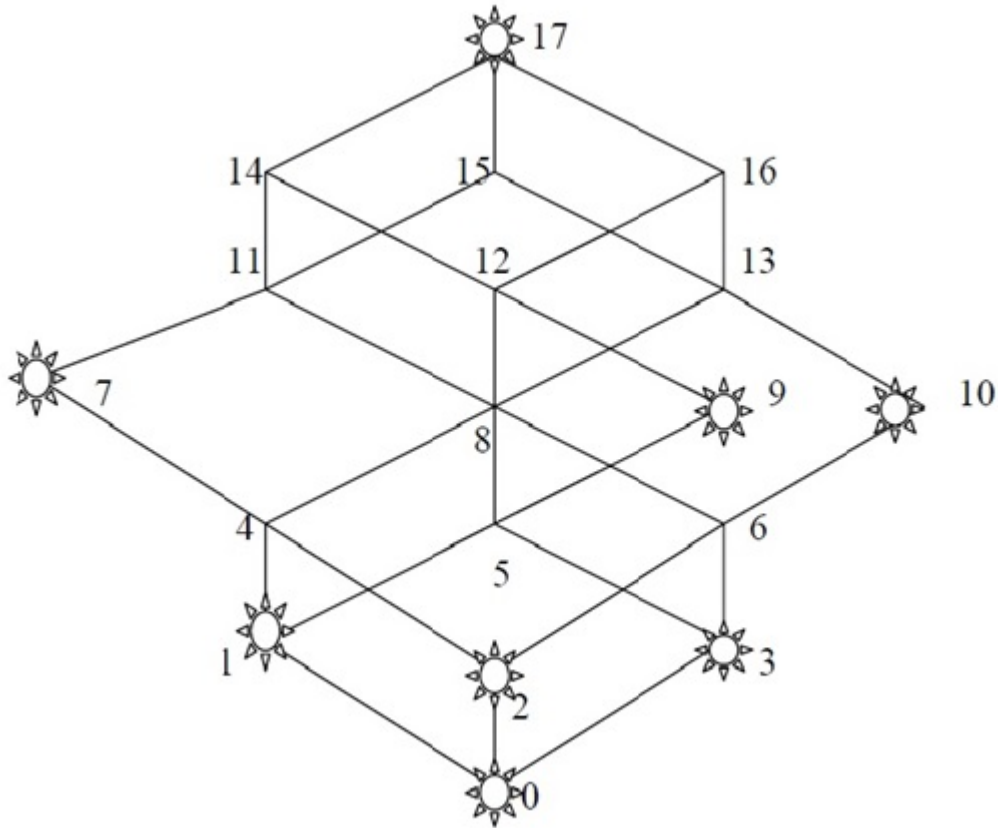
$$0^* = 17 \quad 8^* = 11^* = 12^* = 13^* = 14^* = 15^* = 16^* = 17^* = 0$$

$$1^* = 10 \quad 6^* = 10^* = 1$$

$$2^* = 9 \quad 5^* = 9^* = 2$$

$$3^* = 7 \quad 4^* = 7^* = 3$$

Figure: 1.2 Smarandache lattice



□

Theorem 3.2. *Let S be a distributive lattice with 0 . If there exist a proper subset NI_P of S , defined Definition 2.4. Then S is a smarandache lattice.*

Proof. By hypothesis, let S be a distributive lattice with 0 and whose proper subset $NI_P = \{ I^* \in I_P, I \in I_P \}$ is the set of normal ideals in P .

We claim that NI_P is Boolean algebra.

Since $NI_P = \{ I^* \in I_P : I \in I_P \}$ is the set of normal ideals in P .

Alternatively $NI_P = \{ I \in I_P : I = I^{**} \}$.

Let $I \in I_P$ Take $I^* = \{ y \in P : \text{for every } i \in I, y \wedge i = 0 \}$, $I^* \in I_P$

Namely if $a \in I^*$ then for every $i \in I: a \wedge i = 0$,

Let $b \leq a$, Then, obviously, for every $i \in I: b \wedge i = 0$ hence $b \in I^*$.

If $a, b \in I^*$ then for every $i \in I: a \wedge i = 0$, and for every $i \in I: b \wedge i = 0$,

Hence for every $i \in I: (a \wedge i) \vee (b \wedge i) = 0$.

With distributive, for every $i \in I: i \wedge (a \vee b) = 0$, hence $a \vee b \in I^*$.

Hence $I^* \in I_P, I \cap I^* = I \{ y \in P: \text{for every } i \in I: y \wedge i = 0 \} = \{ 0 \}$.

Let $I \cap J = \{ 0 \}$, let $j \in J$

Suppose that for some $i \in I: i \wedge j \neq 0$. Then $i \wedge j \in I \cap J$,

Since I and J are ideals, hence $I \cap J = \{ 0 \}$.

Hence for every $i \in I: j \wedge i = 0$, and hence $j \subseteq I$.

Consequently, I^* is a pseudo complement of I and I_P is a pseudo complemented.

Therefore I_P is a Boolean algebra.

Thus NI_P is the set of all Pseudo complements lattice in I_P .

In Theorem 3.1 we have proved that pseudo complemented form a Boolean algebra.

Therefore NI_P is a Boolean algebra.

Hence by definition, S is a smarandache lattice.

□

Theorem 3.3. *Let S be a lattice. If there exist a Pseudo complemented distributive lattice P , $X^*(P)$ is a sub lattice of the lattice $I^\delta(P)$ of all δ -ideals of P , which is the proper subset of S . Then S is a Smarandache lattice.*

Proof. By hypothesis, let S be a lattice and there exist a Pseudo complemented distributive lattice P , $X^*(P)$ is a sub lattice of the lattice $I^\delta(P)$ of all δ -ideals of P , which is the proper subset of S .

Let $(a^*), (b^*) \in X^*(P)$, for some $a, b \in P$.

Then clearly $(a^*) \cap (b^*) \in X^*(P)$.

Again, $(a^*) \cup (b^*) = \delta([a]) \cup \delta([b]) = \delta([a] \cup [b]) = \delta([a \cap b]) = ((a \cap b)^*) \in X^*(P)$.

Hence $X^*(P)$ is a sub lattice of $I^\delta(P)$ and hence a distributive lattice.

Clearly (0^{**}) and (0^*) are the least and greatest elements of $X^*(P)$.

Now for any $a \in P$, $(a^*) \cap (a^{**}) = [0]$ and

$(a^*) \cup (a^{**}) = \delta([a]) \cup \delta([a^*]) = \delta([a]) \cup ([a^*]) = \delta([a \cap a^*]) = \delta([0]) = \delta(P) = P$.

Hence (a^{**}) is the complement of (a^*) in $X^*(P)$.

Therefore $\{ X^*(P), \cup, \cap \}$ is a bounded distributive lattice in which every element is complemented.

Thus $X^*(P)$ is a Boolean algebra.

By definition, S is a Smarandache lattice.

□

Theorem 3.4. *Let S be a lattice and P be a pseudo complemented distributive lattice. If S is a Smarandache lattice. Then the following conditions are equivalent:*

- (a) P is a Boolean algebra.
- (b) every element of P is closed,
- (c) every principal ideal is a δ -ideal,
- (d) for any ideal I , $a \in I$ implies $a^{**} \in I$,
- (e) for any proper ideal I , $I \cap D(P) = \phi$,
- (f) for any prime ideal A , $A \cap D(P) = \phi$,
- (g) every prime ideal is a minimal prime ideal,
- (h) every prime ideal is a δ -ideal,
- (i) for any $a, b \in P$, $a^* = b^*$ implies $a = b$,
- (j) $D(P)$ is a singleton set.

Proof. Since S is a Smarandache lattice.

Then by definition and above theorem, we observe that, there exists a proper subset P of S such that which is a Boolean algebra.

Therefore P is a Boolean algebra.

Now to prove (a) \implies (b):

Then clearly P has a unique dense element, precisely the greatest element.

Let $a \in P$ Then $a^* \wedge a = 0 = a^* \wedge a^{**}$.

Also $a^* \vee a, a^* \vee a^{**} \in D(P)$.

Hence $a^* \vee a = a^* \vee a^{**}$.

By the cancellation property of P , we get $a = a^{**}$.

Therefore every element of P is closed.

(b) \implies (c): Let I be a principal ideal of P .

Then $I = (a]$ for some $a \in P$.

Then by condition (b), $a = a^{**}$.

Now, $(a] = (a^{**}] = \delta([a^*])$.

Therefore $(a]$ is a δ -ideal.

(c) \implies (d): I be a proper ideal of P .

Let $a \in I$. Then $(a] = \delta(F)$ for some filter F of P .

Hence We get $a^{***} = a^* \delta F$.

Therefore $a^{**} \in \delta(F) = (a] \subseteq I$.

(d) \implies (e): Let I be a proper ideal of P .

Suppose $a \in I \cap D(P)$.

Then $a^{**} \in P$ and $a^* = 0$.

Therefore $1 = 0^* = a^{**} \in P$,

which is a contradiction.

(e) \implies (f): Let I be a proper ideal of P , $I \cap D(P) = \phi$,
then P be a prime ideal of P , $A \cap D(P) = \phi$.

(f) \implies (g): Let A be a prime ideal of P such that $A \cap D(P) = \phi$.
Let $a \in A$. Then clearly $a \wedge a^* = 0$ and $a \vee a^* \in D(P)$.
Hence $a \vee a^* \notin A$ Thus $a^* \notin A$
Therefore A is a minimal prime ideal of P .

(g) \implies (h): Let A be a minimal prime ideal of P .
Then clearly $P-A$ is a filter of P . Let $a \in A$.
Since A is minimal, there exists $b \notin A$ such that $a \wedge b = 0$.
Hence $a^* \wedge b = b$. $a^* \notin A$.
Thus $a^* \in (P-A)$ which yields $a \in \delta(P-A)$.
Conversely, let $a \in \delta(P-A)$.
Then we get $a^* \notin A$.
Hence we have $a \in A$.
Thus $P = \delta(P-A)$ and therefore A is δ -ideal of P .

(h) \implies (i): Assume that every prime ideal of P is a δ -ideal.
Let $a, b \in P$ be such that $a^* = b^*$. Suppose $a \neq b$.
Then there exists a prime ideal A of P such that $a \in A$ and $b \notin A$.
By Hypothesis, A is a δ -ideal of P .
Hence $A = \delta(F)$ for some filter F of P .
Since $a \in A = \delta(F)$, We get $b^* = a^* \in F$.
Hence $b \in F = A$, which is a contradiction.
Therefore $a = b$.

(i) \implies (j): Suppose x, y be two elements of $D(P)$.
Then $x^* = 0 = y^*$.
Hence $x = y$. Therefore $D(P)$ is a singleton set.

(j) \implies (a): Assume that $D(P) = \{d\}$ is singleton set. Let $a \in P$.
We have always $a \vee a^* \in D(P)$.
Therefore $a \wedge a^* = 0$ and $a \wedge a^* = d$.
This true for all $a \in P$. Also $0 \leq a \leq a \vee a^* = d$.
Hence the above conditions are equivalent.

□

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