SMARANDACHE - R - MODULE AND MORITA CONTEXT

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ABSTRACT. In this paper we introduced Smarandache - 2 - algebraic structure of R-Module namely Smarandache - R - Module. A Smarandache - 2 - algebraic structure on a set N means a weak algebraic structure A_0 on N such that there exist a proper subset M of N, which is embedded with a stronger algebraic structure A_1, stronger algebraic structure means satisfying more axioms, by proper subset one understands a subset from the empty set, from the unit element if any , from the whole set. We define Smarandache - R - Module and obtain some of its characterization through S - algebra and Morita context. For basic concept we refer to Raul Padilla.

1. PRELIMINARIES

Definition 1.1. Let S be any field. An S-algebra A is an (R,R) - bimodule together with module morphisms : $\mu : A\ A \otimes_R A \to A$ and $\eta : R \to A$ called multiplication and unit linear maps respectively , such that

$A \otimes_R A \otimes_R A \xrightarrow{\mu\otimes 1_A, 1_A\otimes\mu} A \otimes_R A \xrightarrow{\mu} A$ with $\mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu)$ and

$R \xrightarrow{\mu\otimes 1_A, 1_A\otimes\mu} A \otimes_R A \xrightarrow{\mu} A$ with $\mu \circ (\eta \otimes 1_A) = \mu \circ (1_A \otimes \eta)$

Definition 1.2. Let A and B be S - algebras. Then $f : A \to B$ is an S - algebra homomorphism if $\mu_B(f \otimes f)f \circ \mu_A$ and $f \circ \eta_A = \eta_B$

Definition 1.3. Let S be a commutative field 1_R and A an S - algebra M is said to be a left A - module.

If for a natural map $\pi : A \otimes_R M \xrightarrow{\otimes} M$, we have $\pi \circ (1_A \otimes \pi) = \pi \circ (\mu \otimes 1_M)$

Definition 1.4. Let S be a commutative field . An S coalgebra is an (R,R) - bimodule C, with R - linear maps

$\Delta : C \to C \otimes_R C$ and $\varepsilon : C \to R$, called comultiplication and counit respectively such that

$C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{1_C\otimes\Delta, \Delta\otimes 1_C} C \otimes_R C \otimes_R C$

with $(1_C \otimes \Delta) \circ \Delta = (\Delta \otimes 1_C) \circ \Delta$ and

$C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{1_C\otimes\varepsilon, \varepsilon\otimes 1_C} R$ with $(1_C \otimes \varepsilon) \circ \Delta = 1_C = (\Delta \otimes 1_C) \circ \Delta$

Definition 1.5. Let C and D be S - coalgebras . A coalgebra morphism $f : C \to D$ is a module morphism satisfies $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\varepsilon_D \times f = \varepsilon_C$

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Definition 1.6. Let A be an S - algebra and C an S - coalgebra . Then the convolution product is defined by

$f * g = \mu \circ (f \otimes g) \circ \Delta$ with $1_{HomR(C,A)} = \eta \circ \varepsilon(1_R)$ for all $f, g \in Hom_R(C, A)$ with for all

Definition 1.7. For a commutative field S an S - bialgebra B is an R - module which is an algebra (B, μ, η) and a coalgebra (B, Δ, ε) such that Δ and ε are algebra morphisms, or equivalently , μ and η are coalgebra morphism

Definition 1.8. Let R, S be fields M an (R,S) - bimodule.Then $M^* = Hom_R(M, R)$ is an(S, R) bimodule and for every left R - module L, There is a canonical module morphism.

$\alpha_L^M : M^* \otimes_R L \to Hom_R(M, L)$

defined by $\alpha_L^M : (m^* \otimes l)(m) = m^*(m)l$ for all $m \in M, m^* \in M^*$ and $l \in L$. If α_L^M is an isomorphism for each left R-module L. Then ${}^R M_S$ is called a Cauchy module.

Definition 1.9. Let R,S be fields with multiplicative identities , M an (S,R)-bimodule and N an (R,S) - bimodule.

Then the six-tuple datum. $K = [R, S, M, N, \langle,\rangle_R, \langle,\rangle_S]$ is said to be a Morita context if the maps $\langle,\rangle_R : N \otimes_S M \to R$ and $\langle,\rangle_S : M \otimes_R N \to S$ are bimodule morphisms satisfying the following associativity conditions :

$m' \langle n, m\rangle_R = \langle m', n\rangle_S m$ and $\langle n, m\rangle_R n' = n\langle m, n'\rangle_S, \langle,\rangle_R$ and $\langle,\rangle_S$ are called the Morita maps.

2. CHARACTERISATION

Definition 2.1. The Smarandache R - module is defined to be an R - module R such that if there exist a proper subset A is an S - algebra with respect to the same induced operations on R.

Theorem 2.1. *Let A be S-coalgebra and Cauchy R-module iff A^* is an S-algebra*

Proof. Let us assume A^* is an S-algebra. Now to prove that A is an S-coalgebra .

we first check the counit conditions as follows. $\varepsilon : A \approxeq A \otimes_R S \xrightarrow{1_A \otimes \mu} A \otimes_R A^* \xrightarrow{\psi_A} R$

Next , We check comultiplication conditions as follows

$\Delta : A \approxeq A \otimes_R S \otimes_R S$

$\xrightarrow{1_A \otimes \eta End_S(A)} A \otimes_R (A^* \otimes_R A) \otimes_R A^*$

$\xrightarrow{1_A \otimes A \otimes A^*} (A \otimes_R A) \otimes_R (A \otimes_R A)^*$

$\xrightarrow{1_A \otimes_R A} (A \otimes_R A) \otimes_R (A \otimes_R A)^*$

$\xrightarrow{1_A \otimes A} (A \otimes_R A) \otimes_R A^*$

$\xrightarrow{\approxeq} (A \otimes_R A) \otimes_R A$

$\xrightarrow{\approxeq} A \otimes_R A$

$\xrightarrow{\approxeq} A$

$\Rightarrow A$ is an S - coalgebra.

Conversely , Let us assume A is an S - coalgebra. Now to prove that A^* is an S - algebra . Now we first check the unit conditions as follows

$\eta : R \xrightarrow{1_A \otimes \eta End_S(A)} A \otimes_R A^* \to 1_A \otimes A \xrightarrow{\approxeq} A$

we check the multiplication conditions as follows A is a Cauchy module, We have $A \otimes_R A \to R$

$A \approxeq A \otimes_R A \otimes_R R \xrightarrow{1_A \otimes \eta Ends_S(A)} A \otimes_R \otimes_R A^* \to R \otimes_R A^* \xrightarrow{\approxeq} A^*$

$\mu : A \otimes_R A \xrightarrow{\varepsilon \otimes 1_A} A^* \otimes_R A \xrightarrow{\approxeq} R \otimes_R A^* \xrightarrow{\approxeq} A^*$

$\Rightarrow A^*$ is an S - algebra. □

Theorem 2.2. *Let R be a R - module , if there exist a proper subset $End_S(M)^*$ of R Where S is a commutative field and M a Cauchy R - module. Then R is a smarandache R - module.*

Proof. Let us assume that R be a R - module.

Now to prove that $End_S(M)^*$ is an S - coalgebra which satisfies multiplication and unit conditions as follows.

$\mu : End_S(M) \otimes_R End_S(M) \to End_S(M)$ and

$\eta : R \to End_S(M)$, We first check the comultiplication as follows.

$\Delta : End_S(M) \approxeq End_S(M) \otimes_R \xrightarrow{1End_S(M) \otimes \eta} End_S(M) \otimes_R End_S(M)$

Next , we check the counit conditions as follows.

$\varepsilon : End_S(M) \approxeq End_S(M) \otimes_R R$

$\xrightarrow{1_{End(M)} \otimes \eta} End_S(M) \times_R End_S(M)$

$\xrightarrow{\approxeq \otimes \approxeq} Hom_R(M, M) \otimes_R Hom_R(M, M)$

$\xrightarrow{\approxeq \otimes \approxeq} (M^* \otimes_R M) \otimes_R (M^* \otimes_R M)$

$\xrightarrow{\psi M \otimes \psi M} R \otimes_R R$

$\xrightarrow{\approxeq} R$

$\Rightarrow End_S(M)$ is an S - coalgebra. By theorem 2.1, $End_S(M)^*$ is an S - algebra .

$\therefore$ By definition R is an smarandacha - R - module.

□

Theorem 2.3. *Let R be a R - module , if there exist a proper subset $M \otimes_R M^*$ of R Where M is a Cauchy R - module. Then R is a smarandache R-module.*

Proof. Now to prove that $M \otimes_R M^*$ is an S - algebra . we check the multiplication and unit conditions as follows.

$\mu : (M \otimes_R M^*) \otimes_R (M \otimes_R M^*) \xrightarrow{\approxeq} M \otimes_R (M^* \otimes_R M) \otimes_R M^*$

$\xrightarrow{1_M \psi_M \otimes 1_M} M \otimes_R R \otimes_R M^*$

$\xrightarrow{1_M \otimes \psi_M}_R M^*$

As M is a Cauchy module , we have

$\eta : R \to End_S(M) \xrightarrow{\approxeq} M \otimes_R M^*$

$\Rightarrow M \otimes_R M^*$ is an S - algebra.
$\therefore$ By definition , R is an smarandache R - module. □

Theorem 2.4. *Let R be a R - module, if there exist a proper subset the datum* $[R, M, N, \langle . \rangle_R]$ *a morita context* $(M \otimes_R N)$ *of R is a S - algebra let S be a commutative field and M, N are Cauchy R - modules. Then R is a smarandache - R- module.*

Proof. Let us assume that R be a R-module. Now to prove that $M \otimes_R N$ is an S-algebra.

We have $\mu : (M \otimes_R N) \otimes_R (M \otimes_R N) \to M \otimes_R (N \otimes_R M) \otimes_R N$
$\xrightarrow{1_M \otimes \langle , \rangle \otimes 1_N} M \otimes_R R \otimes_R N$
$\xrightarrow{\approxeq} M \otimes_R N$

Which shows that the multiplication condition is satisfied. Also, since M and N are Cauchy R - modules, there exist maps

$\eta : R \approxeq R \otimes_R \xrightarrow{\eta End_S(M) \otimes \eta End_S(N)} (M^* \otimes_R M) \otimes_R (N^* \otimes_R N)$
$\xrightarrow{\approxeq \otimes 1_{M \otimes N}} (M^* \otimes_R N^*) \otimes_R (M \otimes_R N)$
$\xrightarrow{\approxeq \otimes 1_{M \otimes N}} (M \otimes_R N)^* \otimes_R (M \otimes_R N)$
$\xrightarrow{\approxeq \otimes 1_{M \otimes N}} R^* \otimes_R (M \otimes_R N)$
$\xrightarrow{\approxeq} R \otimes_R (M \otimes_R N)$
$\xrightarrow{\approxeq} (M \otimes_R N)$
$\Rightarrow M \otimes_R N$ is an S - algebra.
$\therefore$ By definition , R is an smarandache R - module.

□

Theorem 2.5. *Let R be a R - module , if there exist a proper subset datum* $[R, M, N, \langle . \rangle_R]$ *a morita context* $M \otimes_R N$ *of R is a S - coalgebra let S be a commutative field, M, N are Cauchy R - modules. Then R is a smarandache R - module.*

Proof. Let us assume that R be a R - module .

Now to prove that $(M \otimes_R N)$ is an S - coalgebra.

We have

$\Delta : M \otimes_R N = (M \otimes_R N) \otimes_R (R \otimes_R R)$
$\xrightarrow{1_{M \otimes N} \otimes \eta End_S(M) \otimes \eta End_S(N)} (M^* \otimes_R M) \otimes_R (N^* \otimes_R N)$
$\xrightarrow{1_{M \otimes N} \otimes \approxeq} (M \otimes_R N) \otimes_R (M \otimes_R N) \otimes_R (M^* \otimes_R N^*)$
$\xrightarrow{1_{M \otimes N} \otimes \approxeq} (M \otimes_R N) \otimes_R (M \otimes_R N) \otimes_R (M \otimes_R N)^*$
$\xrightarrow{1_{M \otimes N} \otimes \langle , \rangle R^*} (M \otimes_R N) \otimes_R (M \otimes_R N) \otimes_R R$
$\xrightarrow{\approxeq} (M \otimes_R N) \otimes_R (M \otimes_R N)$

Also we have the counit condition as follows.

$\varepsilon : M \otimes_R N \approxeq (M \otimes_R N) \otimes_R R \xrightarrow{1_{M \otimes N} \otimes \eta End_S(M)} (M \otimes_R N) \otimes_R (M^* \otimes_R M)$
$\xrightarrow{\langle , \rangle_R \otimes 1_{M^* \otimes M}} R \otimes_R M^* \otimes_R M$

$\xrightarrow{\approxeq} M^* \otimes_R M$
$\xrightarrow{\psi_M} R$
$\Rightarrow M \otimes_R N$ is an S - coalgebra.
By theorem $M \otimes_R N$ is an S - algebra
$\therefore$ By definition , R is a smarandache R - module.
Hence the proof.

□

Theorem 2.6. *Let R be a R - module, Let S be a commutative field and M, N Cauchy R - Modules then the datum $[R, M, N, \langle . \rangle_R]$ a morita context iff R is a smarandache R - module where $M \otimes_R N$ is a S - bialgebra, a proper subset of R.*

Proof. Part I : We assume that $M \otimes_R N$ is a S - bialgebra, a proper subset of R.
To prove that R is a smarandache R-module.
By theorem, $M \otimes_R N$ is a S - algebra and $M \otimes_R N$ is a S- coalgebra.
Hence $M \otimes_R N$ is a S - bialgebra. By definition, R is a smarandache R - module.
Part II : We assume that R is a smarandache R - module. To prove that $M \otimes_R N$ is a S - bialgebra. By known theorem assume that $M \otimes_R N$ is an S - bialgebra. Then we have the map.
$\varepsilon = \langle . \rangle_R : M \otimes_R N \to R$
Associativity of the map $\varepsilon = \langle . \rangle_R$ holds because the diagram
$M \otimes_R N) \otimes_R M \xrightarrow{\approx} M \otimes_R (N \otimes_R M)$
$\varepsilon \otimes 1_M \searrow M \swarrow 1_M \otimes \varepsilon$
is commutative. Hence the datum $[R, M, N, \langle . \rangle_R]$ is a Morita context.
Hence the proof.

□

3. References

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