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The orthogonal planes split of quaternions and its relation to quaternion geometry of rotations[1]

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Abstract. Recently the general orthogonal planes split with respect to any two pure unit quaternions $f, g \in \mathbb{H}$, $f^2 = g^2 = -1$, including the case $f = g$, has proved extremely useful for the construction and geometric interpretation of general classes of double-kernel quaternion Fourier transformations (QFT) [7]. Applications include color image processing, where the orthogonal planes split with $f = g =$ the grayline, naturally splits a pure quaternionic three-dimensional color signal into luminance and chrominance components. Yet it is found independently in the quaternion geometry of rotations [3], that the pure quaternion units f, g and the analysis planes, which they define, play a key role in the spherical geometry of rotations, and the geometrical interpretation of integrals related to the spherical Radon transform of probability density functions of unit quaternions, as relevant for texture analysis in crystallography. In our contribution we further investigate these connections.

1. Introduction to quaternions

Gauss, Rodrigues and Hamilton's four-dimensional (4D) quaternion algebra $\mathbb{H}$ is defined over $\mathbb{R}$ with three imaginary units:

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},\ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j},\ \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \quad (1)$$

The explicit form of a quaternion $q \in \mathbb{H}$ is $q = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} \in \mathbb{H}$, $q_r, q_i, q_j, q_k \in \mathbb{R}$. The quaternion conjugate (equivalent to Clifford conjugation in $Cl(3,0)^+$ and $Cl(0,2)$) is defined as $\widetilde{q} = q_r - q_i\boldsymbol{i} - q_j\boldsymbol{j} - q_k\boldsymbol{k}$, $\widetilde{pq} = \widetilde{q}\,\widetilde{p}$, which leaves the scalar part q_r unchanged. This leads to the norm of $q \in \mathbb{H}$ $|q| = \sqrt{q\widetilde{q}} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}$, $|pq| = |p|\,|q|$. The part $\boldsymbol{q} = V(q) = q - q_r = \frac{1}{2}(q - \widetilde{q}) = q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k}$ is called a *pure* quaternion, it squares to the

[1] In memory of Hans Wondratschek, *07 Mar. 1925 in Bonn, †26 Oct. 2014 in Durlach.

negative number $-(q_i^2+q_j^2+q_k^2)$. Every unit quaternion $\in \mathbb{S}^3$ (i.e. $|q|=1$) can be written as: $q = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} = q_r + \sqrt{q_i^2+q_j^2+q_k^2}\,\widehat{\boldsymbol{q}} = \cos\alpha + \widehat{\boldsymbol{q}}\sin\alpha = \exp(\alpha\,\widehat{\boldsymbol{q}})$, where $\cos\alpha = q_r$, $\sin\alpha = \sqrt{q_i^2+q_j^2+q_k^2}$, $\widehat{\boldsymbol{q}} = \boldsymbol{q}/|q| = (q_i\boldsymbol{i}+q_j\boldsymbol{j}+q_k\boldsymbol{k})/\sqrt{q_i^2+q_j^2+q_k^2}$, and $\widehat{\boldsymbol{q}}^2=-1$, $\widehat{\boldsymbol{q}} \in \mathbb{S}^2$. The left and right *inverse* of a non-zero quaternion is $q^{-1} = \widetilde{q}/|q|^2 = \widetilde{q}/(q\widetilde{q})$. The *scalar part* of a quaternion is defined as $\mathrm{S}(q) = q_r = \frac{1}{2}(q+\widetilde{q})$, with *symmetries* $\forall p,q \in \mathbb{H}$: $\mathrm{S}(pq) = \mathrm{S}(qp) = p_rq_r - p_iq_i - p_jq_j - p_kq_k$, $\mathrm{S}(q) = \mathrm{S}(\widetilde{q})$, and *linearity* $\mathrm{S}(\alpha p + \beta q) = \alpha\,\mathrm{S}(p) + \beta\,\mathrm{S}(q) = \alpha p_r + \beta q_r$, $\forall p,q \in \mathbb{H}$, $\alpha,\beta \in \mathbb{R}$. The scalar part and the quaternion conjugate allow the definition of the $\mathbb{R}^4$ inner product of two quaternions p,q as $p\cdot q = \mathrm{S}(p\widetilde{q}) = p_rq_r + p_iq_i + p_jq_j + p_kq_k \in \mathbb{R}$.

Definition 1.1 (Orthogonality of quaternions)**.** *Two quaternions $p,q \in \mathbb{H}$ are* orthogonal $p \perp q$, *if and only if* $\mathrm{S}(p\widetilde{q}) = 0$.

2. Motivation for quaternion split

2.1. Splitting quaternions and knowing what it means

We deal with a split of quaternions, motivated by the consistent appearance of two terms in the *quaternion Fourier transform* [4]. This observation (note that in the following always $\boldsymbol{i}$ is on the left, and $\boldsymbol{j}$ is on the right) and that every quaternion can be rewritten as $q = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{i}\boldsymbol{j}$, motivated the quaternion *split*[2] with respect to the pair of orthonormal pure unit quaternions $\boldsymbol{i},\boldsymbol{j}$

$$q = q_+ + q_-, \quad q_\pm = \frac{1}{2}(q \pm \boldsymbol{i}q\boldsymbol{j}). \tag{2}$$

Using (1), the detailed results of this split can be expanded in terms of real components $q_r, q_i, q_j, q_k \in \mathbb{R}$, as

$$q_\pm = \{q_r \pm q_k + \boldsymbol{i}(q_i \mp q_j)\}\frac{1\pm\boldsymbol{k}}{2} = \frac{1\pm\boldsymbol{k}}{2}\{q_r \pm q_k + \boldsymbol{j}(q_j \mp q_i)\}. \tag{3}$$

The analysis of these two components leads to the following Pythagorean *modulus identity* [5].

Lemma 2.1 (Modulus identity)**.** *For $q \in \mathbb{H}$, $|q|^2 = |q_-|^2 + |q_+|^2$.*

Lemma 2.2 (Orthogonality of OPS split parts [5])**.** *Given any two quaternions $p,q \in \mathbb{H}$ and applying the OPS split of (2) the resulting two parts are* orthogonal, *i.e., $p_+ \perp q_-$ and $p_- \perp q_+$,*

$$\mathrm{S}(p_+\widetilde{q_-}) = 0, \qquad \mathrm{S}(p_-\widetilde{q_+}) = 0. \tag{4}$$

Next, we discuss the map $\boldsymbol{i}(\,)\boldsymbol{j}$, which will lead to an adapted orthogonal basis of $\mathbb{H}$. We observe, that $\boldsymbol{i}q\boldsymbol{j} = q_+ - q_-$, i.e. under the map $\boldsymbol{i}(\,)\boldsymbol{j}$ the q_+ part is *invariant*, but the q_- part *changes sign*. Both parts are *two-dimensional* (3),

[2] Also called *orthogonal planes split* (OPS) as explained below.

and by Lemma 2.2 they span *two completely orthogonal planes*, therefore also the name *orthogonal planes split (OPS)*. The q_+ plane has the orthogonal quaternion basis $\{\boldsymbol{i}-\boldsymbol{j}=\boldsymbol{i}(1+\boldsymbol{ij}), 1+\boldsymbol{ij}=1+\boldsymbol{k}\}$, and the q_- plane has orthogonal basis $\{\boldsymbol{i}+\boldsymbol{j}=\boldsymbol{i}(1-\boldsymbol{ij}), 1-\boldsymbol{ij}=1-\boldsymbol{k}\}$. All four basis quaternions (if normed: $\{q_1, q_2, q_3, q_4\}$)

$$\{\boldsymbol{i}-\boldsymbol{j}, 1+\boldsymbol{ij},\ \boldsymbol{i}+\boldsymbol{j}, 1-\boldsymbol{ij}\}, \tag{5}$$

form an orthogonal basis of $\mathbb{H}$ interpreted as $\mathbb{R}^4$. Moreover, we obtain the following geometric picture on the left side of Fig. 1. The map $\boldsymbol{i}()\boldsymbol{j}$ *rotates* the q_- plane by $180°$ around the two-dimensional q_+ axis plane. This interpretation of the map $\boldsymbol{i}()\boldsymbol{j}$ is in perfect agreement with Coxeter's notion of *half-turn* [2]. In agreement with its geometric interpretation, the map $\boldsymbol{i}(\,)\boldsymbol{j}$ is an *involution*, because applying it twice leads to identity

$$\boldsymbol{i}(\boldsymbol{i}q\boldsymbol{j})\boldsymbol{j} = \boldsymbol{i}^2 q \boldsymbol{j}^2 = (-1)^2 q = q. \tag{6}$$

We have the important exponential factor identity

$$e^{\alpha \boldsymbol{i}} q_{\pm} e^{\beta \boldsymbol{j}} = q_{\pm} e^{(\beta \mp \alpha)\boldsymbol{j}} = e^{(\alpha \mp \beta)\boldsymbol{i}} q_{\pm}. \tag{7}$$

This equation should be compared with the kernel construction of the quaternion Fourier transform (QFT). The equation is also often used in our present context for values $\alpha = \pi/2$ or $\beta = \pi/2$.

Finally, we note the interpretation [7] of the QFT integrand $e^{-\boldsymbol{i}x_1\omega_1} h(\boldsymbol{x})\, e^{-\boldsymbol{j}x_2\omega_2}$ as a *local rotation* by phase angle $-(x_1\omega_1 + x_2\omega_2)$ of $h_-(\boldsymbol{x})$ in the two-dimensional q_- plane, spanned by $\{\boldsymbol{i}+\boldsymbol{j}, 1-\boldsymbol{ij}\}$, and a *local rotation* by phase angle $-(x_1\omega_1 - x_2\omega_2)$ of $h_+(\boldsymbol{x})$ in the two-dimensional q_+ plane, spanned by $\{\boldsymbol{i}-\boldsymbol{j}, 1+\boldsymbol{ij}\}$. This concludes the geometric picture of the OPS of $\mathbb{H}$ (interpreted as $\mathbb{R}^4$) with respect to two orthonormal pure quaternion units.

2.2. *Even one pure unit quaternion can do a nice split*

Let us now analyze the involution $\boldsymbol{i}(\,)\boldsymbol{i}$. The map $\boldsymbol{i}(\,)\boldsymbol{i}$ gives

$$\boldsymbol{i}q\boldsymbol{i} = \boldsymbol{i}(q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k})\boldsymbol{i} = -q_r - q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k}. \tag{8}$$

The following orthogonal planes split (OPS) with respect to the *single quaternion unit* $\boldsymbol{i}$ gives

$$q_{\pm} = \frac{1}{2}(q \pm \boldsymbol{i}q\boldsymbol{i}), \quad q_+ = q_j\boldsymbol{j} + q_k\boldsymbol{k} = (q_j + q_k\boldsymbol{i})\boldsymbol{j}, \quad q_- = q_r + q_i\boldsymbol{i}, \tag{9}$$

where the q_+ plane is two-dimensional and manifestly *orthogonal* to the two-dimensional q_- plane. The basis of the two planes are (if normed: $\{q_1, q_2\}$, $\{q_3, q_4\}$)

$$q_+\text{-basis: } \{\boldsymbol{j}, \boldsymbol{k}\}, \qquad q_-\text{-basis: } \{1, \boldsymbol{i}\}. \tag{10}$$

The geometric interpretation of $\boldsymbol{i}(\,)\boldsymbol{i}$ as Coxeter *half-turn* is perfectly analogous to the case $\boldsymbol{i}(\,)\boldsymbol{j}$. This form (9) of the OPS is identical to the quaternionic *simplex/perplex* split applied in quaternionic signal processing, which leads in color image processing to the *luminosity/chrominance* split [6].

3. General orthogonal two-dimensional planes split (OPS)

Assume in the following an arbitrary pair of pure unit quaternions f, g, $f^2 = g^2 = -1$. The *orthogonal 2D planes split (OPS)* is then defined with respect to *any two pure unit quaternions* f, g as

$$q_{\pm} = \frac{1}{2}(q \pm fqg) \qquad \Longrightarrow \qquad fqg = q_+ - q_-, \tag{11}$$

i.e. under the map $f()g$ the q_+ part is invariant, but the q_- part changes sign.

Both parts are two-dimensional, and span two completely orthogonal planes. For $f \neq \pm g$ the q_+ plane is spanned by two orthogonal quaternions $\{f - g, 1 + fg = -f(f - g)\}$, the q_- plane is e.g. spanned by $\{f + g, 1 - fg = -f(f+g)\}$. For $g = f$ a fully *orthonormal* four-dimensional basis of $\mathbb{H}$ is (R acts as rotation operator (rotor))

$$\{1, f, \boldsymbol{j}', \boldsymbol{k}'\} = R^{-1}\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}R, \qquad R = \boldsymbol{i}(\boldsymbol{i} + f), \tag{12}$$

and the two orthogonal two-dimensional planes basis:

$$q_+\text{-basis: } \{\boldsymbol{j}', \boldsymbol{k}'\}, \qquad q_-\text{-basis: } \{1, f\}. \tag{13}$$

Note the notation for normed vectors in [3] $\{q_1, q_2, q_3, q_4\}$ for the resulting total *orthonormal basis of* $\mathbb{H}$.

Lemma 3.1 (Orthogonality of two OPS planes)**.** *Given any two quaternions q, p and applying the OPS with respect to any two pure unit quaternions f, g we get zero for the scalar part of the mixed products*

$$Sc(p_+\widetilde{q}_-) = 0, \qquad Sc(p_-\widetilde{q}_+) = 0. \tag{14}$$

Note, that the two parts $x_{\pm}$ can be *represented* as

$$x_{\pm} = x_{+f}\,\frac{1 \pm fg}{2} + x_{-f}\,\frac{1 \mp fg}{2} = \frac{1 \pm fg}{2}\,x_{+g} + \frac{1 \mp fg}{2}\,x_{-g}, \tag{15}$$

with commuting and anticommuting parts $x_{\pm f}f = \pm f x_{\pm f}$, etc.

Next we mention the possibility to perform a split along any given set of two (two-dimensional) analysis planes. It has been found, that any two-dimensional plane in $\mathbb{R}^4$ determines in an elementary way an OPS split and vice versa, compare Theorem 3.5 of [7].

Let us turn to the geometric interpretation of the map $f()g$. It *rotates* the q_- plane by 180° around the q_+ axis plane. This is in perfect agreement with Coxeter's notion of *half-turn* [2], see the right side of Fig. 1. The following *identities* hold

$$e^{\alpha f}q_{\pm}e^{\beta g} = q_{\pm}e^{(\beta \mp \alpha)g} = e^{(\alpha \mp \beta)f}q_{\pm}. \tag{16}$$

This leads to a straightforward geometric interpretation of the integrands of the quaternion Fourier transform (OPS-QFT) with two pure quaternions f, g, and of the orthogonal 2D planes phase rotation Fourier transform [7].

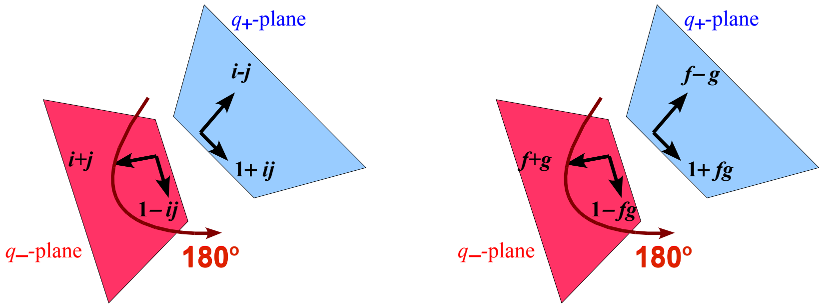

Figure 1. Geometric pictures of the involutions $\boldsymbol{i}(\,)\boldsymbol{j}$ and $f()g$ as half turns.

We can further incorporate quaternion conjugation, which consequently provides a geometric interpretation of the QFT involving quaternion conjugation of the signal function. For $d = e^{\alpha g}, t = e^{\beta f}$ the map $d\widetilde{(\,)}t$ represents a *rotary-reflection* in four dimensions with pointwise *invariant line* $d + t$, a rotary-reflection *axis* $d - t$: $d\widetilde{(d-t)}t = -(d-t)$, and rotation *angle* $\Gamma = \pi - \arccos \mathrm{S}\left(\widetilde{d}t\right)$ in the plane $\perp \{d+t, d-t\}$. (The derivation of Γ will be shown later.) We obtain the following Lemma.

Lemma 3.2. *For OPS* $q_\pm = \frac{1}{2}(q \pm fqg)$, *and left and right exponential factors we have the identity*

$$e^{\alpha g}\,\widetilde{q_\pm}e^{\beta f} = \widetilde{q_\pm}e^{(\beta\mp\alpha)f} = e^{(\alpha\mp\beta)g}\,\widetilde{q_\pm}. \tag{17}$$

4. Coxeter on Quaternions and Reflections [2]

The four-dimensional *angle* Θ between two unit quaternions $p, q \in \mathbb{H}$, $|p| = |q| = 1$, is defined by

$$\cos\Theta = Sc(p\widetilde{q}). \tag{18}$$

The right and left *Clifford translations* are defined by Coxeter [2] as

$$q \to q' = qa, \quad q \to q'' = aq, \quad a = e^{\widehat{a}\Theta}, \quad \widehat{a}^2 = -1. \tag{19}$$

Both Clifford translations represent *turns by constant angles* $\Theta_{q,q'} = \Theta_{q,q''} = \Theta$. We analyze the following special cases, assuming the split $q_\pm$ w.r.t. $f = g = \widehat{a}$:

- For $\widehat{a} = \boldsymbol{i}$, $aq_- = q_-a = (q_-a)_-$, is a mathematically *positive* (anti clockwise) *rotation* in the q_- plane $\{1, \boldsymbol{i}\}$.
- Similarly, $aq_+ = (aq_+)_+$, is a mathematically *positive rotation* in the q_+ plane $\{\boldsymbol{j}, \boldsymbol{k}\}$.
- Finally, $q_+a = \widetilde{a}q_+ = (q_+a)_+$, is a mathematically *negative rotation* (clockwise) by Θ in the q_+ plane $\{\boldsymbol{j}, \boldsymbol{k}\}$.

Next, we compose Clifford translations, assuming the *split* $q_\pm$ w.r.t. $f = g = \widehat{a}$. For the unit quaternion $a = e^{\widehat{a}\Theta}$, $\widehat{a}^2 = -1$ we find that

$$q \to aqa = a^2 q_- + q_+ \tag{20}$$

is a *rotation* only in the q_- plane by the *angle* 2Θ, and

$$q \to aq\tilde{a} = q_- + a^2 q_+ \tag{21}$$

is a *rotation* only in the q_+ plane by the *angle* 2Θ.

Let us now revisit Coxeter's **Lemma 2.2** in [2]: For any two quaternions $a, b \in \mathbb{H}$, $|a| = |b|$, $a_r = b_r$, we can find a $y \in \mathbb{H}$ such that

$$ay = yb. \tag{22}$$

We now further ask for the set of *all* $y \in \mathbb{H}$ such that $ay = yb$? Based on the OPS, the answer is straightforward. For $a = |a|e^{\Theta\hat{a}}$, $b = |a|e^{\Phi\hat{b}}$, $\Phi = \pm\Theta$, $\hat{a}^2 = \hat{b}^2 = -1$ we use the split $q_\pm = \frac{1}{2}(q \pm \hat{a}q\hat{b})$ to obtain:

- For $\Theta = \Phi$: The set of all y spans the q_- plane. Moreover,

$$aq_+b = q_+, q_+b = \tilde{a}q_+, aq_+ = q_+\tilde{b}, aq_-b = a^2q_- = q_-b^2, aq_- = q_-b. \tag{23}$$

- For $\Theta = -\Phi$: The set of all y spans the q_+ plane. Moreover,

$$aq_-b = q_-, q_-b = \tilde{a}q_-, aq_- = q_-\tilde{b}, aq_+b = a^2q_+ = q_+b^2, aq_+ = q_+b. \tag{24}$$

Let us turn to a reflection in a hyperplane. **Theorem 5.1** in [2] says: The reflection in the hyperplane $\perp a \in \mathbb{H}$: $Sc(a\tilde{q}) = 0$, $|a|^2 = 1$, $a = |a|e^{\Theta\hat{a}}$, $\hat{a}^2 = -1$, is represented by

$$q \to -a\tilde{q}a. \tag{25}$$

We analyze the situation using the OPS. We define the split $q_\pm = \frac{1}{2}(q \pm \widehat{a}q\widehat{a})$ to obtain

$$q_+ \to -a\widetilde{q_+}a = q_+, \qquad a \to -a\tilde{a}a = -a. \tag{26}$$

and for $a' = ae^{-\frac{\pi}{2}\hat{a}}$

$$a' \to -a\widetilde{a'}a = a'. \tag{27}$$

We further consider a general rotation. **Theorem 5.2** in [2] states: The general rotation through 2Φ (about a plane) is $q \to aqb$, $a = |a|e^{\Phi\hat{a}}$, $b = |a|e^{\Theta\hat{b}}$, $\Phi = \pm\Theta$, $\hat{a}^2 = \hat{b}^2 = -1$.

We again apply the OPS. We define the split $q_\pm = \frac{1}{2}(q \pm \hat{a}q\hat{b})$ to obtain:

- For $\Theta = \Phi$: Rotation of q_- plane by 2Φ around q_+-plane.
- For $\Theta = -\Phi$: Rotation of q_+ plane by 2Φ around q_--plane.

Let us illustrate this with an *example*: $\hat{a} = \hat{b} = \boldsymbol{i}$, $\Phi = -\Theta$,

$$aq_-b = q_-. \tag{28}$$

For $q_+ = \boldsymbol{j}$:

$$aq_+b = a\boldsymbol{j}b = \boldsymbol{j}b^2 = \boldsymbol{j}e^{-2\theta\boldsymbol{i}} = \boldsymbol{j}\cos 2\Phi - \boldsymbol{k}\sin 2\Phi, \tag{29}$$

a rotation in the q_+-plane around the q_- plane. Note, that the detailed analysis of general $q \to aqb$, $q_\pm = \frac{1}{2}(q \pm \hat{a}q\hat{b})$, $|a| = |b| = 1$, can be found in [7].

As for the *rotary inversion*, we follow the discussion in [7], sec. 5.1, but add a simple formula for determining the rotation angle. The rotary inversion is given by, $d, t \in \mathbb{H}$, $|d| = |t| = 1$, $q \to d\widetilde{q}t$. For $d \neq \pm t$, $[d,t] = dt - td$, we obtain two vectors in the rotation plane $v_{1,2} = [d,t](1 \pm \tilde{d}t)$, with $d\widetilde{v_{1,2}}t = -v_{1,2}\tilde{d}t$. The angle Γ of rotation can therefore be simply found from

$$\cos\Gamma = Sc(\frac{1}{|v_1|^2}\widetilde{v_1}d\widetilde{v_1}t) = Sc(-\tilde{d}t) = \cos(\pi - \gamma), \tag{30}$$

with γ the angle between d and $t : \cos\gamma = \tilde{d}t$.

5. Quaternion geometry of rotations [3] analyzed by 2D OPS

According to [3] the circle $C(q_1, q_2)$ of all unit quaternions, which rotate $g \to f$, $f \neq \pm g$ is given by

$$q(t) = \frac{1 - fg}{|1 - fg|}e^{\frac{t}{2}g} = q_1 e^{\frac{t}{2}g}, \quad t \in [0, 2\pi), \quad q(t)g\widetilde{q(t)} = f, \quad q_2 = \frac{f+g}{|f+g|}.$$

The two-dimensional OPS $q_\pm^{f,g} = \frac{1}{2}(q \pm fqg)$ tells us, that all $q(t), t \in [\,0, 2\pi\,)$ are elements of the q_- plane. And in deed, $fq_-g = -q_-$ for all $q_- \in \mathbb{H}$ leads to

$$f = q_- g q_-^{-1}, \tag{31}$$

for all q_- in the q_--plane. Note, that this is valid for all $f, g \in \mathbb{H}$, $f^2 = g^2 = -1$, even for $f = \pm g$! We therefore get a *one line proof*, which at the same time generalizes from the unit circle to the whole plane.

Meister and Schaeben [3] state that for $q \in C(q_1, q_2)$: $fq, qg, fqg \in C(q_1, q_2)$. This can easily be generalized to the whole q_--plane, because

$$(fq_-)_- = fq_-, \quad (q_-g)_- = q_-g, \quad (fq_-g)_- = fq_-g. \tag{32}$$

We can use the exponential form, and show that the circle $C(q_1, q_2)$ parametrization of (34), (35) in [3] is a *specialization of the general relation*

$$e^{\frac{t}{2}f}q_- = q_- e^{\frac{t}{2}g} \tag{33}$$

which means that the two parametrizations are element wise identical.

Now we look at the quaternion circles for the rotations $g \to \pm f$. **Prop. 5** of [3] states: Two circles $C(q_1, q_2) = G(g, f)$ and $C(q_3, q_4) = G(g, -f) = G(-g, f)$, representing all rotations $g \to f$ and $g \to -f$, respectively, are orthonormal to each other. Here four orthogonal unit quaternions are defined as:

$$q_1 = \frac{1 - fg}{|1 - fg|}, \;\; q_2 = \frac{f + g}{|f + g|}, \;\; q_3 = \frac{1 + fg}{|1 + fg|}, \;\; q_4 = \frac{f - g}{|f - g|}. \tag{34}$$

We provide a simple proof: We already know that all q_1, q_2, span the q_- plane of the split $q_\pm^{f,g} = \frac{1}{2}(q \pm fqg)$, and q_3, q_4 span the q_+-plane. And that

$$fq_\pm g = \pm q_\pm \qquad \Leftrightarrow \qquad f = q_\pm(\mp g)q_\pm^{-1}. \tag{35}$$

QED. Note, the proof is again much faster than in [3]. We see that $G(g,f) = \{q_-^{f,g}/|q_-^{f,g}|, \forall q \in \mathbb{H}\}$, and $G(g,-f) = \{q_+^{f,g}/|q_+^{f,g}|, \forall q \in \mathbb{H}\}$.

For later use, we translate the notation of [3] (38),(39):

$$n_3 = -n_1 = \frac{[f,g]}{|[f,g]|}, \;\; n_4 = q_4, \;\; n_2 = n_4 n_1, \;\; n_4 = n_1 n_2, \;\; n_1 = n_2 n_4, \tag{36}$$

which shows that $\{n_1, n_2, n_4\}$ is a right handed set of three orthonormal pure quaternions, obtained by rotating $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

The two circles $G(g,f), G(g',f)$ do not intersect for $g \neq g'$, see Cor. 1(i) of [3]. We provide a simple proof: Assume $\exists_1 q \in H : fqg = -q, fqg' = -q$ for $g \neq g'$. Then

$$fqg = fqg' \Leftrightarrow g = g' \Rightarrow G(g,f) \bigcap G(g',f) = \emptyset. \tag{37}$$

QED.

Cor. 1 (iii) of [3] further states that for every 3D rotation R and given g_0, $g_0^2 = -1$ we can always find f, $f^2 = -1$, such that R is represented by a (unit) quaternion q in $G(g_0, f)$. We can equivalently ask for f, such that q representing the rotation R is $\in q_-^{f,g_0}$-plane. We find

$$fqg_0 = -q \Leftrightarrow f = qg_0q^{-1}. \tag{38}$$

The left side of Fig. 2 shows two small circles $C(g,\rho), C(f,\rho) \subset \mathbb{S}^2$ [3]. We now analyze the mapping between pairs of small circles. A small circle with center g and radius ρ is defined as $C(g,\rho) = \{g' \in \mathbb{S}^2 : g \cdot g' = \cos\rho\}$, and all $q \in q_-^{f,g}$-plane map $C(g,\rho)$ to the small circle $C(f,\rho)$ of the same radius (a slight generalization of [3], Prop. 6), with the correspondence

$$\begin{aligned} q(t)g'(u)q(t)^{-1} &= f'(u+2t), \\ q(t) = q_1 e^{tg}, \quad g'(u) = e^{\frac{u}{2}g} g'_0 e^{-\frac{u}{2}g}, &\quad f'(u) = e^{\frac{u}{2}f} f'_0 e^{-\frac{u}{2}f} \end{aligned} \tag{39}$$

starting with the corresponding circle points $q_1 g'_0 = f'_0 q_1$.

We provide the following direct proof: We repeatedly apply (16) to obtain

$$\begin{aligned} q_1 g'_0 = f'_0 q_1 &\Leftrightarrow e^{\frac{u}{2}f} q_1 g'_0 e^{-\frac{u}{2}g} = e^{\frac{u}{2}f} f'_0 q_1 e^{-\frac{u}{2}g} \\ &\Leftrightarrow q_1 e^{\frac{u}{2}g} g'_0 e^{-\frac{u}{2}g} = e^{\frac{u}{2}f} f'_0 e^{-\frac{u}{2}f} q_1 \\ &\Leftrightarrow e^{tf} q_1 e^{\frac{u}{2}g} g'_0 e^{-\frac{u}{2}g} = e^{tf} e^{\frac{u}{2}f} f'_0 e^{-\frac{u}{2}f} e^{-tf} e^{tf} q_1 \\ &\Leftrightarrow q_1 e^{tg}\, e^{\frac{u}{2}g} g'_0 e^{-\frac{u}{2}g} = e^{tf} e^{\frac{u}{2}f} f'_0 e^{-\frac{u}{2}f} e^{-tf}\, q_1 e^{tg}. \end{aligned} \tag{40}$$

QED. Note, that this proof is much shorter than in [3], and we do not need to use addition theorems.

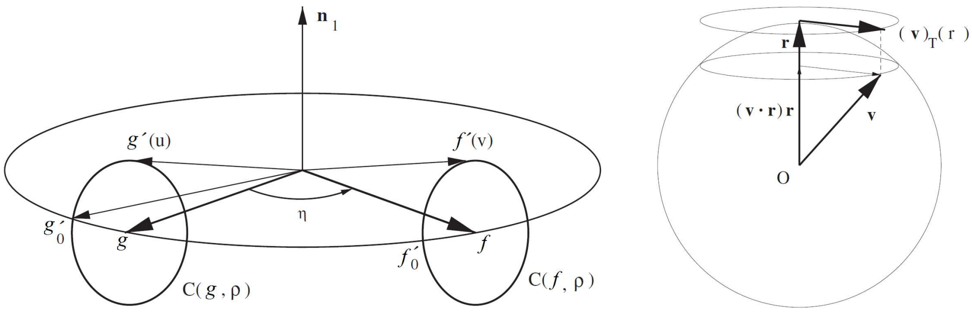

Figure 2. Small circles and tangential plane projection. Adapted from Figs. 2 and 3 of [3].

We consider the *projection onto the tangential plane* of a $\mathbb{S}^2$ vector, see the right side of Fig. 2. Assume $v, r \in \mathbb{S}^2$. Note, that $(v)_T(r) = v - (v \cdot r)r$ of [3] can be simplified to $(v)_T(r) = V(vr)r^{-1}$, valid for all pure (non-unit) quaternions r.

Finally we consider the *torus theorem* for all maps $g \to$ small circle $C(f, 2\Theta)$. We slightly reformulate the theorem **Prop. 13** of [3]. We will use the two-dimensional OPS with respect to $f, g \in \mathbb{S}^2$, and the corresponding orthonormal basis $\{q_1, q_2, q_3, q_4\}$ of (34). The theorem says, that the circle $C(q_1, q_2) \in q_-$-plane: $q_-(s) = q_1 \exp(sg/2)$, $s \in [0, 2\pi)$, represents all rotations $g \to f$, while the orthogonal circle $C(q_3, q_4) \in q_+$-plane: $q_+(t) = q_3 \exp(-tg/2)$, $t \in [0, 2\pi)$, represents all rotations $g \to -f$. Then the *spherical torus* $T(q_-(s), q_+(t); \Theta)$ is defined as the set of quaternions

$$q(s,t;\Theta) = q_-(2s)\cos\Theta + q_+(2t)\sin\Theta, \quad s,t \in [0,2\pi), \quad \Theta[0,\pi/2], \tag{41}$$

and represents all rotations $g \to C(f, 2\Theta) \subset \mathbb{S}^2$.

In particular, the set $q(s, -s; \Theta)$ maps g for all $s \in [0, 2\pi)$ onto f_0' in the f, g plane with $g \cdot f_0' = \cos(\eta - 2\Theta)$, $g \cdot f = \cos\eta$,

$$q(s,-s;\Theta) g q(s,-s;\Theta)^{-1} = f_0' \quad \forall s \in [0,2\pi). \tag{42}$$

Moreover, for arbitrary $s_0 \in [0, 2\pi)$, the set $q(s_0, t - s_0; \Theta)$ (or equivalently $q(s_0 + t, s_0; \Theta)$) maps $g \to f' \in C(f, 2\Theta)$, which results from positive rotation (counter-clockwise) of f_0' about f by the angle $t \in [0, 2\pi)$,

$$q(s_0, t-s_0;\Theta) g q(s_0, t-s_0;\Theta)^{-1} = e^{\frac{t}{2}f} f_0' e^{-\frac{t}{2}f} \quad \forall s_0 \in [0,2\pi). \tag{43}$$

We state the following *direct proof* of the torus theorem.

$$\begin{aligned}
q(s,t;\Theta) &= q_-(2s)\cos\Theta + q_+(2t)\sin\Theta = q_1 e^{sg}\cos\Theta + q_3 e^{-tg}\sin\Theta \\
&= (q_1\cos\Theta + q_3 e^{-(t+s)g}\sin\Theta)e^{sg} \\
&= (\cos\Theta + (q_3/q_1)e^{-(t+s)f}\sin\Theta)q_1 e^{sg} \\
&= (\cos\Theta + (-n_1)e^{-(t+s)f}\sin\Theta)q_1 e^{sg} \\
&= (\cos\Theta + e^{(t+s)f}(-n_1)\sin\Theta)e^{sf} q_1 \\
&= e^{-n_1'\Theta}e^{sf}q_1, \qquad n_1' = e^{(t+s)f}(-n_1), (n_1')^2 = -1.
\end{aligned} \tag{44}$$

We observe, that n_1' is n_1 rotated around f by angle $s+t$. Application to g gives

$$q(s,t;\Theta)g\; q(s,t;\Theta)^{-1} = e^{-n_1'\Theta} f e^{n_1'\Theta}, \tag{45}$$

so geometrically g is rotated into $f = e^{sf} q_1 g\; q_1^{-1} e^{-sf}$, which in turn is rotated around n_1' on the circle $C(f, 2\Theta)$. For $t = -s$ obviously

$$q(s,-s;\Theta)gq(s,-s;\Theta)^{-1} = e^{-n_1\Theta} f e^{n_1\Theta} = f_0', \tag{46}$$

is a rotation in the f, g plane of g into f_0', with $g \cdot f_0' = \cos\eta - 2\Theta$. We further note, that for $s = s_0$, $t \to t - s_0$: $n_1' = e^{tf}(-n_1)$, such that

$$q(s_0, t-s_0;\Theta)gq(s_0, t-s_0;\Theta)^{-1} = e^{-n_1'\Theta} f e^{n_1'\Theta} = e^{tf} f_0' e^{-tf}, \tag{47}$$

describes the small circle $C(f, 2\Theta)$. QED.

Our proof is very *compact*, obtained by *direct* computation of *monomial* results, which in turn permit direct *geometric interpretation*.

6. Conclusions

We have exposed the geometric understanding of the *general OPS split of quaternions* [7] into two *orthogonal planes* ($\mathbb{R}^4$ interpretation). Moreover, we have consolidated the OPS with the geometric understanding by Altmann [1], Coxeter [2], and Meister and Schaeben [3].

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