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Algorithmic and NP-Completeness Aspects of a Total Lict Domination Number of a Graph

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Abstract: A dominating set of a graph $\eta(G)$, is a total lict dominating set if the dominating set does not contain any isolates. The total lict dominating number $\gamma_t(\eta(G))$ of G is a minimum cardinality of total lict dominating set of G. The current paper studies total lict domination in graph from an algorithmic point of view. In particular we had obtained the algorithm for a total lict domination number of any graph. Also we had obtained the time complexity of a proposed algorithm. Further we discuss the NP-Completeness of a total lict domination number of the split graph, bipartite graph and chordal graph.

Key Words: Smarandachely k-dominating set, total lict dominating number, lict graph, vertex independence number, bipartite graph, split graph, chordal graph.

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§1. Introduction

All graphs considered here are finite, connected, undirected without loops or multiple edges and without isolated vertices. As usual 'p' and 'q' denotes the number of vertices and edges of a graph G.

The concept of domination in graph theory is a natural model for many location problems in operations research. In a graph G, a vertex is said to dominate itself and all of its neighbors.

A set $D \subseteq V$ of G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k=1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the *domination number* of G and denoted by $\gamma(G)$ in general.

A dominating set D of a graph G is a total dominating set if the dominating set D does not contain any isolates. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of total dominating set.

The lict graph $\eta(G)$ of a graph G is the graph whose vertex set is the union of the set

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of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident. A dominating set of a graph $\eta(G)$, is a total lict dominating set if the dominating set does not contain any isolates. The total lict dominating number $\gamma_t(\eta(G)$ of G is the minimum cardinality of total lict dominating set of G.

A vertex cover C of a graph $G = (V, E)$ is a subset $C \subseteq V$ such that for every edge $uv \in E$, we have $u \in C$ or $v \in C$. A cut-vertex of a connected graph G is a vertex v such that $G - \{v\}$ is disconnected.

A stable set in a graph G is a pair-wise non-adjacent vertices subset of $V(G)$, and a clique is a pairwise adjacent vertices subset of $V(G)$. A graph is split if its vertex set can be partitioned into a stable set and a clique. A graph is bipartite if its vertex set can be partitioned into two stable sets. A graph is chordal if every cycle of length at least 4 has at least one chord, which is an edge joining two non-consecutive vertices in the cycle.

In this paper, we obtain the algorithm for a total lict domination number of any graph. Also, we had obtained the time complexity of a proposed algorithm. Further we discuss the NP-Completeness of a total lict domination number of a graph with respect to split graph, bipartite graph and chordal graph.

§2. Algorithm

To find the algorithm for the minimum total lict domination set of a graph we use initially, the DFS algorithm to the find the cut vertices of a given graph [1], the VSA algorithm [2] to find the minimum vertex cover of a graph and shortest path algorithm [3] to find the shortest path in a graph. The edges in the shortest path gives a total lict domination set of graph G. Then we reduce this to a minimum set which gives the minimum total lict domination set of any graph G.

Algorithm to find the minimum total lict domination set of a given graph:

Input: A graph $G = (V, E)$.

Output: A minimum total lict domination set D of a graph $G = (V, E)$.

Step 1: Initialize $D = \phi$.

Step 2: Label the vertices of a graph G as $\{v_i/i = 1, 2, 3, 4, 5, \cdots, n\}$ and label the edges of a graph G as $\{e_j/j = 1, 2, 3, 4, 5, \cdots, m\}$.

Step 3: Let A=$\{v_i/v_i$ is a cut vertex of a graph $G(V, E)\}$.

Step 4: Compute the set C of all minimal vertex covers in G, such that C does not contain vertex of degree one.

Step 5: FOR the minimal vertex cover set $c \in C$, DO

Step 6: IF $|V(c)| = 1$.
GOTO Step 7.
ELSE

IF $|V(c)| = 2$ and they are adjacent
GOTO Step 8.
ELSE
GOTO Step 9.
END IF.

Step 7: $D = D \cup \{$ any two adjacent edges of $E(G)\}$.
GOTO Step 13.

Step 8: $D = D \cup \{(e_i, e_j), e_i$ is a common edge incident with $V(c)$ and $e_j \in N(e_i)\}$
GOTO Step 13.

Step 9: Let $E_1 = \{e_q/e_q \in E(G)$, where e_q is the set of edges in the shortest path connecting all the vertices of $V(c)$ and $\langle E_1 \rangle \neq K_{1,n}$ if there is any other shortest path $\}$.
$K = \{e_l/e_l$ is an end edge $\in E_1\}$.
$R = \{e_j/e_j \in E(G) - E_1/e_j$ is adjacent to $K\}$
FOR $|E_1| \neq$1 or 0 DO,
Let two edges $E_2 = (e_i, e_j) \in E_1$ such that $e_j \in N(e_i)$.
IF $e_i \in N(e_j)$ and $e_i \in N(e_k)$, where e_k or e_j is and end edge.
Then $E_2 = (e_i$,an end edge)
ELSE IF $e_i \in N(e_j, e_k)$ and $e_j \in N(e_l, e_m), (e_l, e_m) \neq e_i$
Then E_2=(e_i, e_k)
END IF
END IF
$D = D \cup E_2$.
$B = \{e_p/e_p \in N(e_i, e_j)$ in $E_1\}$.
$C_1 = \{e_r/e_r \in N(B) \cap E_1 - (D \cup B), e_r$ is not incident with A, $e_r \neq (v_i, v_j), v_i, v_j \in C\}$.
$E_1 = E_1 - (B \cup C_1)$.
END FOR.

Step 10: IF $|E(E_1)| = 0$ then
GOTO Step 11.
ELSE
$D = D \cup \{E_1 \cup e_i, e_i \in E_1$ and $e_i \in N(D)\}$.
GOTO Step 11.
END IF.

Step 11: FOR $R \neq \phi$ DO,
Let any edge in R
$D = D \cup \{e_k, e_k \in E_1$ and $e_k \in N(e_i)\}$.
$R = R - \{e_i\} \cup \{e_s/e_s \in N(D)\}$.
END FOR

Step 12: END FOR (from Step 4)

Step 13: RETURN D, a minimum total lict domination set of a graph G.

Step 14: STOP.

§3. Time Complexity

The worst case time complexity of finding the solution of the minimum total lict domination problem of a graph using the proposed algorithm can be obtained as follows:

Assume that there are n vertices and m edges in the proposed algorithm.

(i) DFS algorithm [1] to find the cut vertices of a given graph which requires a running time of O(mn).

(ii) VSA algorithm [2] to find the minimum vertex cover of a given graph which requires the running time of $0(mn^2)$.

(iii) Shortest path algorithm [3] to find the shortest path connecting the vertices of $V(c)$ which requires the worst case of running time of O$(m+n)$.

(iv) For a FOR loop in step 9 requires the worst case running time of $0\left(\frac{m-1}{3}\right)$.

(v) For a FOR loop in step 11 requires the worst case running time of $0\left(\frac{2n}{3}-2\right)$.

(vi) So the overall time is

$$O(mn)+0(mn^2)+O(m+n)+0\left(\frac{m-1}{3}\right)+0\left(\frac{2n}{3}-2\right)=0(mn^2).$$

§4. NP-Completeness of total lict domination number of a graph

This section establishes NP-Complete results for the total lict domination problem in bipartite graph, split graph and in chrodal graph. The transformation is from the vertex cover problem, which is known to be NP-Complete.

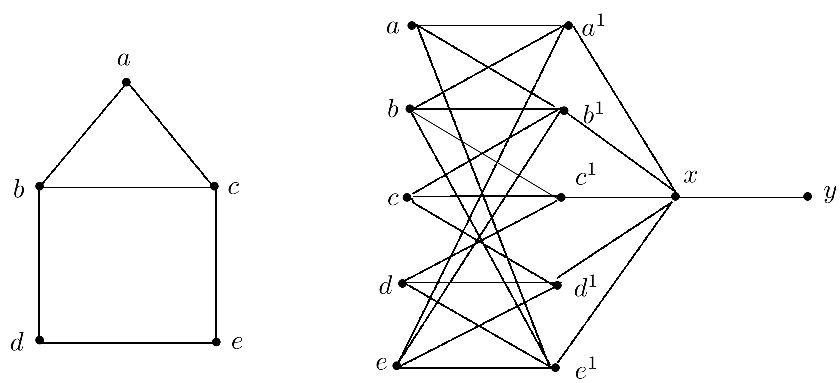

Fig.1 A constructed bipartite graph G' from the graph G

Theorem 4.1 *The total lict domination number problem is NP-Complete for bipartite graph.*

Proof The total lict domination number problem for bipartite graph is NP-Complete as we can transform the vertex cover problem to it as follows. Given a non-trivial graph $G=(V,E)$, construct the graph $G^{'} = (V^{'}, E^{'})$ with the vertex set $V^{'}$ consists of two copies of V denoted by V and $V^{'}$, together with two special vertices x and y and whose edges $E^{'}$ consists of

(i) edges $uv^{'}$ and $u^{'}v$ for each edge $uv \in E(G)$.

(ii) edges of the form $uu^{'}$ for each vertex $u \in V$.

(iii) edges of the form $u^{'}x$ for every vertex $u \in V$.

(iv) the one additional edge xy.

We claim that $G = (V, E)$ has a vertex cover of size k if and only if $G^{'} = (V^{'}, E^{'})$ has a minimal total lict domination set of size $k + (p - k)$. Let C be the vertex cover of G of size k. Let $B = \{u^{'}x/u \in V\}$ such that $|B| = k$. Let $D = B \cup R$, where $R = \{u^{'}x/u \in V - C\}$ with $|R| = p - k$. Then it is clear that, D is a total lict dominating number of a bipartite graph with cardinality $k + (p - k)$.

On the other hand suppose D is a minimal total lict domination set of the graph $G^{'}$ with cardinality $k + (p - k)$. Let $A = \{v_i/v_i \in V^{'}, v_i$ is incident with $e_i \in D\}$ with $|A| = |D|$. The vertex set A in $G^{'}$ is $V(G)$, such that A consists of copies of V and $V - C$ and whose vertices are adjacent to atleast one vertex of C. So, the graph G has a vertex cover of size k. □

Theorem 4.2 *The total lict domination number problem is NP-Complete for split graph.*

Proof The total lict domination number problem for split graph is NP-Complete as we can transform the vertex cover problem to it as follows.

Given a non-trivial graph $G = (V, E)$ construct the graph $G^{'} = (V^{'}, E^{'})$ with the vertex set $V^{'} = V \cup E$and $E^{'} = \{uv : u \neq v, u, v \in V\} \cup \{ve : v \in V, e \in E, v \in e\}$.

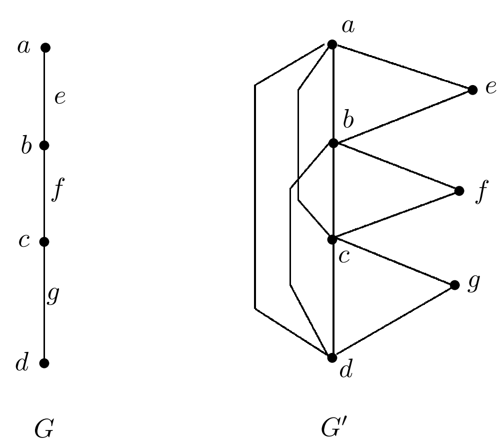

Fig.2 A constructed split graph G^1 from a graph G

We claim that $G = (V, E)$ has a vertex cover of size k if and only if $G^{'} = (V^{'}, E^{'})$ has a total lict domination set of size $k + (p - k) - 1$. Let C be the vertex cover of G of size k. Let $B = \{e_i/e_i \in E(G^{'}) \cap E(G), e_i$ is incident with $V^{'} \in C$ and $V^{'} \in V - C$ in $G\}$. Then it is clear that B is a total lict dominating set of a split graph with cardinality $k + (p - k) - 1$.

On the other hand, suppose D is the total lict domination number of the graph $G^{'}$ with cardinality $k+(p-k)-1$. Let $A=\{v_i/v_i \in V', v_i$ is incident with $e_i \in D \cap E(G)\}$ with cardinality equal to $|D|+1=k+(p-k)$. The vertex set A in G' is $V(G)$ such that A consists copies of V and $V-C$ whose vertices are adjacent to at least to one vertex of C. So, the graph Ghas a vertex cover of size k. □

Theorem 4.3 *The total lict domination number problem is NP-Complete for chordal graph.*

Proof we shall transform the vertex cover problem in general graph to the total lict domination in chordal graph. Therefore, the NP-Completeness of the total lict domination problem in chordal graph follows from that of the vertex cover problem in general graph. For any graph G consider the chordal graph $G'=(V', E')$ with vertex set $V'=\{v_1, v_2, v_3, v_4/v \in V\}$ and the edge set $E'=\{v_1v_2, v_2v_3, v_3v_4/v \in V\} \cup \{u_3v_4/uv \in E\} \cup \{u_4v_4/uv \in V, u \neq v\}$.

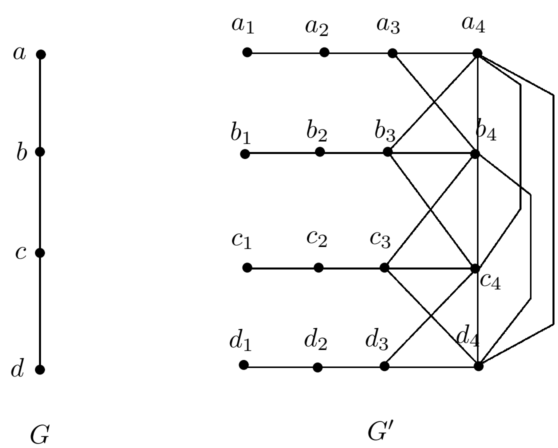

Fig.3 A constructed chordal graph G^1 from a graph G

We claim that $G=(V,E)$ has vertex cover of size k if and only if $G'=(V', E')$ has a minimal total lict domination set of size $2(k+(p-k))$. Let C be the vertex cover of G of size k. Let $B=\{v_2v_3, v_3v_4/v \in V\}$. Then it is clear that B is a minimal total lict dominating set of a chordal graph with cardinality $2(k+(p-k))$.

On the other hand suppose D is the minimal total lict domination number of the graph G' with cardinality $2(k+(p-k))$. Let $A=\{v_3/v_3 \in V', v_3$ is incident with $v_2v_3, v_3v_4 \in D\}$ with $|A|=\frac{D}{2}$=$k+(p-k)$. The vertex set A in G' is $V(G)$ such that A consists copies of V and $V-C$ whose vertices are adjacent to at least to one vertex of C. So, the graph G has a vertex cover of size k. □

§4. Conclusion

The main purpose of this paper is to establish an algorithm for the total lict domination problem in general graph. NP-Complete results for the problem are also shown for split graph, chordal graph and for bipartite graphs.

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