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Characteristic Polynomial & Domination Energy of Some Special Class of Graphs

M.Kamal Kumar

(Department of Mathematics, CMR Institute of Technology, Bangalore, 560016, Karnataka, India)

E-mail: kamalmvz@gmail.com

Abstract: Representation of a set of vertices in a graph by means of a matrix was introduced by Sampath Kumar. Let $G(V, E)$ be a graph and $S \subseteq V$ be a set of vertices, we can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. In this paper we define set energy and find its properties and also study the special case of set S being a dominating set and corresponding domination energy of some special class of graphs.

Key Words: Adjacency matrix, Smarandachely k-dominating set, domination number, eigenvalues, energy of graph.

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§1. Introduction

A set $D \subseteq V$ of G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the *domination number* of G and denoted by $\gamma(G)$ in general.

The concept of graph energy arose in theoretical chemistry where certain numerical quantities, as the heat of formation of a hydrocarbon are related to total π electron energy that can be calculated as the energy of corresponding molecular graph. The molecular graph is representation of molecular structure of a hydrocarbon whose vertices are the position of carbon atoms and two vertices are adjacent, if there is a bond connecting them.

Eigenvalues and eigenvectors provide insight into the geometry of the associated linear transformation. The energy of a graph is the sum of the absolute values of the eigenvalues of its adjacency matrix. From the pioneering work of Coulson [2] there exists a continuous interest towards the general Mathematical properties of the total π electron energy ε as calculated within the framework of the Huckel Molecular Orbital (HMO) model. These efforts enabled one to get an insight into the dependence of ε on molecular structure. The properties of $\varepsilon(G)$ are discussed in detail in [7-10].

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The importance of eigenvalues is not only used in theoretical chemistry but also in analyze structures, car designers analyze eigenvalues in order to damp out the noise to reduce the vibration of the car due to music, eigenvalues can be used to test for cracks or deformities in a solid, oil companies frequently use eigenvalue analysis to explore land for oil, eigenvalues are also used to discover new and better designs for the future [23].

Representation of a set of vertices in a graph by means of a matrix was introduced by Sampath Kumar [5]. Let $G(V, E)$ be a graph and $S \subseteq V$ be a set of vertices we can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S, denoted by $A_S(G)$ and energy $E(G)$ obtained from the matrix $A_S(G)$ is called the set energy denoted by $E_S(G)$. In this paper we consider the special case of a set S being a dominating set and the corresponding matrix is domination matrix denoted by $A_\gamma(G)$ of G and energy $E(G)$ obtained from the domination matrix $A_\gamma(G)$ is defined as domination energy denoted by $E_\gamma(G)$. For any undefined terms or notation in this paper, we refer Harary [6]. In this paper we define set energy and find its properties and also study the special case of set S being a dominating set and corresponding domination energy of some special class of graphs.

Let the graph G be connected and let its vertices be labelled as $v_1, v_2, v_3, \ldots, v_n$. The domination matrix of G is defined to be the square matrix $A_\gamma(G)$ corresponding to the dominating set of G. The eigenvalues of the dominating matrix are denoted by $\kappa_1, \kappa_2, \kappa_3, \cdots, \kappa_n$ are said to be A_γ eigenvalues of G. Since the A_γ matrix is symmetric, its eigenvalues are real and can be ordered $\kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \cdots \geqslant \kappa_n$.

$$E_\gamma = E_\gamma(G) = \sum_{i=1}^{n} |\kappa_i|. \tag{1}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy [7-9]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \tag{2}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$ are the ordinary graph eigenvalues [7] that is, the eigenvalues of the adjacency matrix $A(G)$. Recall that in the few years, the graph energy $E(G)$ has been extensively studied in the mathematics [11-15] and mathematic-chemical literature [16-25].

§2. Main Results

Let $G(V, E)$ be a graph, $S \subseteq V$ and $A(G)$ be the adjacency matrix of G. Replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S.

Definition 2.1(Minimal dominating energy) *A dominating set D in G is minimal dominating set, if no proper subset of D is a dominating set. The domination energy $E_\gamma(G)$ obtained for* *minimal dominating set is called Minimal Dominating Energy denoted by $E_{\gamma-\min}(G)$.*

Definition 2.2(Minimal dominating energy) *A dominating set D in G is maximal dominating set, if D contains all the vertices of G. The domination energy $E_\gamma(G)$ obtained for maximal dominating set is called Maximal Dominating Energy denoted by $E_{\gamma-\max}(G)$.*

Observation 2.3 If $A(G)$ is the adjacency matrix corresponds to the graph $G(V,E)$, $A_{\gamma-\min}(G)$ is the adjacency matrix corresponding to the minimal dominating set $S_{\min}$ and $A_{\gamma-\max}(G)$ is the adjacency matrix corresponding to the maximum dominating set $S_{\max}$. Cardinality $|S_{\min}| \leqslant |S| \leqslant |S_{\max}|$ where set S is the dominating set whose cardinality is in between minimal and maximal dominating set. A graph $G(V,E) \neq K_n$, $n \geqslant 3$ then $E_{\gamma-\min}(G) \pm \varepsilon \leqslant E_\gamma(G) \leqslant E_{\gamma-\max}(G) \pm \varepsilon$, where ε is the error factor such that $|\varepsilon| \leqslant 1$.

Corollary 2.4 *A graph $G(V,E) \neq K_n$, $n \geqslant 3$ then $E(G) \leqslant E_{\gamma-\min}(G)$.*

Observation 2.5 A graph $G(V,E) = K_n$, $n \geqslant 3$ then $E_{\gamma-\min}(G) \pm \varepsilon \geqslant E_\gamma(G) \geqslant E_{\gamma-\max}(G) \pm \varepsilon$, where ε is the error factor such that $|\varepsilon| \leqslant 1$.

Corollary 2.6 *If graph $G(V,E) = K_n$, $n \geqslant 3$ then $E(G) \geqslant E_{\gamma-\min}(G)$. $E(K_n) = 2(n-1) \geqslant E_{\gamma-\min}(K_n) = (n-2) + \sqrt{n^2-2n+5}$ (Theorem 4.3).*

Observation 2.7[Set Energy] Domination energy is the energy calculated w.r.t. the dominating set, but in order to understand the spectra of dominating set we generalize the concept as set energy. That is w.r.t. the set of different cardinality the energy were found. Energy for the $|S| = 0$ is the energy of the Graph $E(G)$. Similarly we find the energy for $|S| = 1$ to n. The particular case of set energy is the domination energy.

1. P_2 and C_6 are the only graphs with set energy of $|\varphi| = |2|$ and $|\varphi| = |6|$, $|1| = |5|$, $|2| = |4|$ respectively. Spectra are different but energy is same.

2. In energy of graph $\sum_{i=1}^n \lambda_i^2 = 2m$, m is the number of edges where as for Set energy $\sum_{i=1}^n \kappa_i^2 = 2m + |S|$, $|S|$ is the cardinality of set for which energy is calculated.

3. Set energies are symmetry in nature i.e., w.r.t. the shape of the graph (molecule). This can be proved by showing the matrix for the respected set will be same with the corresponding operation$R_i \leftrightarrow R_j, C_i \leftrightarrow C_j$. Example in a cycle of order n, label the vertices as $v_1, v_2, v_3, \ldots, v_n$ clockwise then $E_S(v_i, v_{i+1}) = E_S(v_{i+1}, v_{i+2})$ $E_S(v_i, v_{i+2}) = E_S(v_{i+1}, v_{i+3})$ etc. where $i = 1$ to $n-1$ and $j = 1$ to n.

4. In energy of graph $\sum_{i<j} \lambda_i\lambda_j = -m$, m is the number of edges where as for Set energy, $\sum_{i<j} \kappa_i\kappa_j \geqslant -m$, for $|S| \neq 1$, $\sum_{i<j} \kappa_i\kappa_j > -m$.

5. In energy of graph $\sum_{i=1}^n \lambda_i = 0$ where as for the set energy $\sum_{i=1}^n \lambda_i = |S|$, $i = 1$ to n.

6. It was found that there are same spectra for different sets of same cardinality (symmetry w.r.t. shape). Different spectra for different sets of same cardinality. Different spectra with set energy being same for the set with different cardinality.

7. If λ_1 is the highest eigenvalue w.r.t. energy of graph then $\sqrt{\Delta} \leqslant \lambda_1 \leqslant \Delta$. If κ_1 is the highest eigenvalue w.r.t. set energy of graph then $\sqrt{\Delta+1} \leqslant \kappa_1 \leqslant \Delta + 1$.

8. In energy of a graph, characteristic polynomial is given by $\varphi(G:\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$, $a_1 = 0$, $-a_2$ is number of edges, $-a_3$ is twice the number of triangles in G. For set energy the characteristic polynomial is given by $\varphi(G:\kappa) = \kappa^n + a_1\kappa^{n-1} + a_2\kappa^{n-2} + \ldots + a_n$, a_1 and a_2 are same for all characteristic polynomial with same cardinality of the set S, where $a_1 = -|S|$, but a_2 varies w.r.t. the cardinality of the set, i.e., when $|S| = 1$, $a_2 = -e = u_1$, e is the number of edges for a given graph. When $|S| = 2$, $a_2 = u_1 + 1 = u_2$, when $|S| = 3$, $a_2 = u_2 + 2 = u_3, \ldots$. When $|S| = n$, $a_2 = u_{n-1} + (n-1)$. Finding a_i for $i > 2$ is difficult for different cardinality of the set for the same graph.

§3. Preliminary Results

The following results comes from [22].

1. A graph $G(V,E)$ with $n \geqslant 3$ and $G \neq K_n$ then

$$\sqrt{2m + n(n-1)\left(\det A\right)^{2/n}} \leqslant E_{\gamma-\min}(G) \leqslant \sqrt{2mn},$$

where m is the number of edges and n is the number of vertices in G.

2. A graph G with n vertices without isolated vertices, with $n \geqslant 3$ and $G \neq K_n$ then $E_{\gamma-\min}(G) \geqslant 2\sqrt{n} + \varepsilon$.

3. $K_{n,n}$ is a Complete regular bipartite graph with $n \geqslant 3$, then

$$E_{\gamma-\min}\left(K_{n,n}\right) \leqslant 2\left|V\right| - 2,$$

where $|V|$ is the cardinality of vertices in G.

4. A graph $G(V,E)$ with $n \geqslant 3$ then $E_{\gamma-\min}(G) \leqslant \frac{n}{2}\left(\sqrt{n+1}\right) + \varepsilon$ where n is the number of vertices in G.

5. A graph $G(V,E)$ is a complete graph with $n \geqslant 3$ then $E_{\gamma}\left(K_n\right) \leqslant \sqrt{mn}$ where m is the number of edges and n is the number of vertices in G.

6. A graph $G(V,E)$ with $n \geqslant 3$ then $E_{\gamma-\min}\left(K_{1,n-1}\right) \leqslant E_{\gamma-\min}\left(T_n\right) \leqslant E_{\gamma-\min}\left(P_n\right)$ where $K_{1,n-1}$ is star graph with n vertices, T_n is tree with n vertices and P_n is path with n vertices.

§4. Characterizing Graphs w.r.t. to the Unique Dominating Set

Case 1 $\gamma(G) = 1$.

The characteristic polynomial is found using the method of Souriau (Faddeev & Frame) [21] which is also a modified method of Leverrier's method.

Theorem 4.1 *For any given star $K_{1,n-1}$ with $n \geqslant 3$, the characteristic polynomial is given by*

$\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} = 0$ *with*

$$E_{\gamma-\min}(K_{1,n-1}) = \sqrt{4n-3}$$
$$E(K_{1,n-1}) = 2\sqrt{n-1} \leqslant E_{\gamma-\min}(K_{1,n-1}) = \sqrt{4n-3}.$$

Proof Consider a star, $K_{1,n-1}$. Label the vertices $v_1, v_2, v_3, \ldots, v_n$ such that v_1 has the maximum degree, hence in the domination matrix $a_{11} = 1$ and all other $a_{ii} = 0$, the characteristic polynomial is found using the method of Souriau (Faddeev & Frame) [21] which is also an modified method of Leverrier's method. That is, the characteristic polynomial is given by $\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \cdots + q_{n-1}\kappa + q_n = 0$ where,

$$A_1 = A, \quad q_1 = -\text{Trace}A_1, \quad B_1 = A_1 + q_1 I_n,$$

where I_n is the unit matrix of order n.

$$A_2 = AB_1, \quad q_2 = -\frac{1}{2}\text{Trace}A_2,\ B_2 = A_2 + q_2 I_n,$$
$$A_3 = AB_2, \quad q_3 = -\frac{1}{3}\text{Trace}A_3,\ B_3 = A_3 + q_3 I_n$$
$$\cdots\cdots$$
$$A_n = AB_{n-1}, \quad q_n = -\frac{1}{n}\text{Trace}A_n, \quad B_n = A_n + q_n I_n.$$

Now consider an domination matrix of $K_{1,n-1}$, whose $\gamma(K_{1,n-1}) = 1$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & - & - & 1 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 1 & 0 & 0 & 0 & - & - & 0 \end{bmatrix}, \quad A_1 = A, \quad q_1 = -\text{Trace}A_1 = -1.$$

$$B_1 = A_1 + q_1 I_n = A_1 - I_n$$
$$= \begin{bmatrix} 0 & 1 & 1 & 1 & - & - & 1 \\ 1 & -1 & 0 & 0 & - & - & 0 \\ 1 & 0 & -1 & 0 & - & - & 0 \\ 1 & 0 & 0 & -1 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 1 & 0 & 0 & 0 & - & - & -1 \end{bmatrix}$$

$$A_2 = AB_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & - & - & 1 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 1 & 0 & 0 & 0 & - & - & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 & - & - & 1 \\ 1 & -1 & 0 & 0 & - & - & 0 \\ 1 & 0 & -1 & 0 & - & - & 0 \\ 1 & 0 & 0 & -1 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 1 & 0 & 0 & 0 & - & - & -1 \end{bmatrix}$$

$$= \begin{bmatrix} n-1 & 0 & 0 & 0 & - & - & 0 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & - & - & - & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \end{bmatrix}$$

$$q_2 = -\frac{1}{2}\text{Trace}A_2 = -\frac{1}{2}\left(n-1+n-1\right) = -\frac{1}{2}\left(2n-2\right) = -\left(n-1\right).$$

$$B_2 = A_2 + q_2 I_n = A_2 - (n-1)I_n.$$

$$= \begin{bmatrix} n-1 & 0 & 0 & 0 & - & - & 0 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & - & - & - & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \end{bmatrix} - \begin{bmatrix} n-1 & 0 & 0 & 0 & - & - & 0 \\ 0 & n-1 & 0 & 0 & 0 & -0 & 0 \\ 0 & 0 & n-1 & 0 & - & - & 0 \\ 0 & 0 & 0 & n-1 & 0 & - & 0 \\ - & 0 & - & - & - & - & 0 \\ - & - & - & - & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & n-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & -n+2 & 1 & 1 & - & - & 1 \\ 0 & 1 & -n+2 & 1 & - & - & 1 \\ 0 & 1 & 1 & -n+2 & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & - & - & - & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & -n+2 \end{bmatrix}$$

$$A_3 = AB_2$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & - & - & 1 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 1 & 0 & 0 & 0 & - & - & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & -n+2 & 1 & 1 & - & - & 1 \\ 0 & 1 & -n+2 & 1 & - & - & 1 \\ 0 & 1 & 1 & -n+2 & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & 1 \\ 0 & - & - & - & - & - & 1 \\ 0 & 1 & 1 & 1 & - & - & -n+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & 0 \end{bmatrix},$$

$$q_3 = -\frac{1}{3}\text{Trace}A_3 = 0$$

$$B_3 = A_3 + q_3 I_n = A_3 + 0I_n = A_3.$$

$$A_4 = AB_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & 0 \end{bmatrix},$$

$$q_4 = -\frac{1}{4}\text{Trace}A_4 = 0$$

Similarly all the further A_i will have $a_{ii} = 0$ hence $q_i = 0$, for $i = 3$ to n.

Hence the resultant characteristic polynomial is $\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} = 0$ implies $\kappa^n - \kappa^{n-1} - (n-1)\,\kappa^{n-2} = 0$. Solving this equation we get roots (eigenvalues). $\kappa^{n-2}\left(\kappa^2 - \kappa - (n-1)\right) = 0. Whence, \kappa^{n-2} = 0$ or $\left(\kappa^2 - \kappa - (n-1)\right) = 0$.

Notice that $\kappa^{n-2} = 0$ implies $n-2$ roots are zero and solving $\kappa^2 - \kappa - (n-1) = 0$ enables one knowing that

$$\kappa = \frac{1 \pm \sqrt{1 - 4(1)(-n+1)}}{2} = \frac{1 \pm \sqrt{4n-3}}{2},$$

where $n \geqslant 3$. Hence the roots are

$$\kappa_1 = \frac{1+\sqrt{4n-3}}{2} \quad \text{and} \quad \kappa_2 = -\left(\frac{\sqrt{4n-3}-1}{2}\right).$$

Thus, $$E_{\gamma-\min}(K_{1,n-1}) = \sum_{i=1}^{n} |\kappa_i| = \frac{1+\sqrt{4n-3}+\sqrt{4n-3}-1}{2} = \sqrt{4n-3}$$

$$\text{i.e., } E(K_{1,n-1}) = 2\sqrt{n-1} \leqslant E_{\gamma-\min}(K_{1,n-1}) = \sqrt{4n-3}. \qquad \square$$

Corollary 4.2 *For any given thorn star $S_{k,t}$ for $k=1$, $S_{k,1}$ is a star with t vertices.*

Theorem 4.3 *For any given Complete Graph K_n with $n \geqslant 3$, the characteristic polynomial is given by $(\kappa-1)^{n-2}\left(\kappa^2-(n-1)\kappa-1\right)=0$ and $E_{\gamma-\min}(K_n) = \sqrt{n^2-2n+5}+(n-2)$.*

Proof Label the vertices $v_1, v_2, v_3, \ldots, v_n$ such that v_1 is the dominating set, hence in the domination matrix $a_{11}=1$ and all other $a_{ii}=0$, the characteristic polynomial is found using the method of Souriau (Faddeev & Frame) [21] which is also an modified method of Leverrier's method similar to Theorem 1. That is, the characteristic polynomial is given by

$$\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \ldots + q_{n-1}\kappa + q_n = 0.$$

It can be shown that the characteristic polynomial of complete graph is given by

$$(\kappa-1)^{n-2}\left(\kappa^2-(n-1)\kappa-1\right)=0.$$

On solving the equation we get

$$(\kappa-1)^{n-2}=0 \text{ or } \left(\kappa^2-(n-1)\kappa-1\right)=0.$$

Notice that $(\kappa-1)^{n-2}=0$ implies $\kappa=-1,-1,-1,\ldots,-1(n-2)$ times and

$$\kappa^2-(n-1)\kappa-1=0.$$
$$\kappa = \frac{n-1\pm\sqrt{(n-1)^2-4(1)(-1)}}{2} = \frac{n-1\pm\sqrt{n^2-2n+5}}{2}$$

where $n \geqslant 3$. Hence the roots are

$$\kappa_1 = \frac{n-1+\sqrt{n^2-2n+5}}{2} \quad \text{and} \quad \kappa_2 = -\left(\frac{\sqrt{n^2-2n+5}-(n-1)}{2}\right).$$

Thus, $$\begin{aligned} E_{\gamma-\min}(K_n) &= \sum_{i=1}^{n}|\kappa_i| \\ &= \frac{n-1+\sqrt{n^2-2n+5}+\sqrt{n^2-2n+5}-(n-1)}{2}+n-2 \\ E_{\gamma-\min}(K_n) &= \sqrt{n^2-2n+5}+(n-2). \end{aligned} \qquad \square$$

Case 2 $\gamma(G) = 2$.

During the study of chemical graphs and its Weiner number, the Yugoslavian Chemist Ivan Gutman introduced the concept of Thorn graphs. This idea was further extended to the broader concept of generalized thorny graphs by Danail Bonchev and Douglas J Klein of USA. This class of graphs gain importance in spectral theory as it represents the structural formula of aliphatic and aromatic hydrocarbons [3].

Definition 4.4(Thorn Rod) *A Thorn rod is a graph $P_{p,t}$ which includes a linear chain (termed as a rod) of p vertices and degree t terminal vertices at each of the two rod ends.*

Definition 4.5(Thorn Star) *A Thorn Stars are the graphs obtained from a k arm star by attaching $t-1$ terminal vertices to each of the star arms and are denoted as $S_{k,t}$.*

Definition 4.6(Thorn Ring) *A t Thorny Ring has a simple cycle as the parent, and $t-2$ thorns at each cycle vertex.*

C_n^+ consists of $2n$ vertices where n vertices on the cycle are degree three and remaining n vertices are pendant vertices.

$C_n^\vee$ consists of $n(t-1)$ vertices of which n vertices are in cycle each of degree t and $n(t-2)$ pendant vertices.

Theorem 4.7 *For any given thorn rod $P_{2,t}$, the characteristic polynomial is given by $\kappa^{2t-4}(\kappa^2 - (t-1))(\kappa^2 - 2\kappa - (t-1))$, where n being the order of G given by $n = 2t$ and $E_{\gamma-\min}(P_{2,t}) = 2\sqrt{t-1} + 2\sqrt{t}$.*

Proof The characteristic polynomial can be found using the method of Souriau (Faddeev & Frame) [21] which is also an modified method of Leverrier's method. Instead we generalize the result obtained for few thorn rods. For $t = 1$, $P_{2,1}$ is a path with 2 vertices, $t = 2$, $P_{2,2}$ is a path with 4 vertices. The characteristic polynomial of $P_{2,t}$ for $t > 2$ is given by, for $P_{2,3}$, $\kappa^2(\kappa^2 - 2)(\kappa^2 - 2\kappa - 2)$, $P_{2,4}$, $\kappa^4(\kappa^2 - 3)(\kappa^2 - 2\kappa - 3)$, $P_{2,5}$, $\kappa^6(\kappa^2 - 4)(\kappa^2 - 2\kappa - 4)$, $P_{2,6}$, $\kappa^8(\kappa^2 - 5)(\kappa^2 - 2\kappa - 5)$, hence for $P_{2,t}$ the characteristic polynomial is given by, $\kappa^{2k-4}(\kappa^2 - (t-1))(\kappa^2 - 2\kappa - (t-1))$, $n = 2t$. Solving the two quadratic equations and summing their absolute eigenvalues we obtain $E_{\gamma-\min}(P_{2,t}) = 2\sqrt{t-1} + 2\sqrt{t}$. □

Theorem 4.8 *For any given thorn rod $P_{3,t}$, the characteristic polynomial is given by $\kappa^{2t-3}(\kappa^2 - \kappa - (t-1))(\kappa^2 - \kappa - (t+1))$, $n = 2t+1$ and $E_{\gamma-\min}(P_{3,t}) = \sqrt{4t-3} + \sqrt{4t+5}$.*

Proof The proof is similar to the above theorem. For $t = 1$, $P_{3,1}$ is a path with 3 vertices, $t = 2P_{3,2}$ is a path with 5 vertices. For $t > 2$ the characteristic polynomial is given by, for $P_{3,3}$, $\kappa^3(\kappa^2 - \kappa - 2)(\kappa^2 - \kappa - 4)$, $P_{3,4}$, $\kappa^5(\kappa^2 - \kappa - 3)(\kappa^2 - \kappa - 5)$, $P_{3,5}$, $\kappa^7(\kappa^2 - \kappa - 4)(\kappa^2 - \kappa - 6)$, hence for $P_{3,t}$, $\kappa^{2k-3}(\kappa^2 - \kappa - (t-1))(\kappa^2 - \kappa - (t+1))$, $n = 2t+1$. The corresponding minimal domination energy is $\sqrt{4t-3} + \sqrt{4t+5}$. □

Theorem 4.9 *For any given thorn rod $P_{4,t}$, the characteristic polynomial is given by $\kappa^{2t-4}(\kappa^3 - (t+1)\kappa - (t-1))(\kappa^3 - 2\kappa^2 - (t-1)\kappa + (t-1))$, $n = 2t+2$.*

Proof For $t=1$, $P_{4,1}$ is a path with 4 vertices, $t=2$, $P_{4,2}$ is a path with 6 vertices. For $t>2$ the characteristic polynomial is given by, for $P_{4,3}$, $\kappa^2(\kappa^3-4\kappa-2)(\kappa^3-2\kappa^2-2\kappa+2)$, $P_{4,4}$, $\kappa^4(\kappa^3-5\kappa-3)(\kappa^3-2\kappa^2-3\kappa+3)$, $P_{4,5}$, $\kappa^6(\kappa^3-6\kappa-4)(\kappa^3-2\kappa^2-4\kappa+4)$, hence for $P_{4,t}$, $\kappa^{2k-4}(\kappa^3-(t+1)\kappa-(t-1))(\kappa^3-2\kappa^2-(t-1)\kappa+(t-1))$, $n=2t+2$. Solving a cubic equation is quite difficult. □

Corollary 4.10 *For any given thorn star $S_{k,t}$ for $k=2$, $S_{2,t}$ is a $P_{3,t}$.*

Case 3 $\gamma(G)=3$.

Theorem 4.11 *For any given thorn star $S_{3,t}$, the characteristic polynomial is given by, $\kappa^{3t-5}(\kappa^2-\kappa-(t+2))(\kappa^2-\kappa-(t-1))^2$, $n=3t+1$ and $E_{\gamma-\min}(S_{3,t})=\sqrt{4t+9}+2\sqrt{4t-3}$.*

Proof Thorny star $S_{3,t}$ has 3 arm and $t-1$ terminal vertices, hence $\gamma(G)=3$. The characteristic polynomial of $S_{3,2}$, $\kappa^1(\kappa^2-\kappa-4)(\kappa^2-\kappa-1)^2$, $S_{3,3}$, $\kappa^4(\kappa^2-\kappa-5)(\kappa^2-\kappa-2)^2$, $S_{3,4}$, $\kappa^7(\kappa^2-\kappa-6)(\kappa^2-\kappa-3)^2$, hence for $S_{3,t}$, $\kappa^{3k-5}(\kappa^2-\kappa-(t+2))(\kappa^2-\kappa-(t-1))^2$, $n=3t+1$. The corresponding minimal domination energy is $\sqrt{4t+9}+2\sqrt{4t-3}$. □

Theorem 4.12 *For a given thorn ring C_3^t, with three vertices on the cycle of degree t and $n(t-1)$ vertices, the characteristic polynomial is given by, $\kappa^{3t-9}(\kappa^2-(t-2))^2(\kappa^2-3\kappa-(t-2))$ and $E_{\gamma-\min}(C_3^t)=4\sqrt{t-2}+\sqrt{4t+1}$.*

Proof Thorn ring C_n^t, has $n(t-1)$ vertices, n vertices on the cycle and n (t-2) pendant vertices, t is the degree of each vertex on the cycle. The characteristic polynomial of C_3^3, $\kappa^0(\kappa^2-1)^2(\kappa^2-3\kappa-1)$, C_3^4, $\kappa^3(\kappa^2-2)^2(\kappa^2-3\kappa-2)$, C_3^5, $\kappa^6(\kappa^2-3)^2(\kappa^2-3\kappa-3)$, C_3^6, $\kappa^9(\kappa^2-4)^2(\kappa^2-3\kappa-4)$. Hence for C_3^t, $\kappa^{3t-9}(\kappa^2-(t-2))^2(\kappa^2-3\kappa-(t-2))$. The corresponding minimal domination energy is $4\sqrt{t-2}+\sqrt{4t+1}$. □

Case 4 $\gamma(G)=4$.

Theorem 4.13 *For any given thorn star $S_{4,t}$, the characteristic polynomial is given by, $\kappa^{4t-7}(\kappa^2-\kappa-(t+3))(\kappa^2-\kappa-(t-1))^3$, $n=4t+1$ and $E_{\gamma-\min}(S_{4,t})=\sqrt{4t+13}+3\sqrt{4t-3}$.*

Proof Thorny Star $S_{4,t}$ has 4 arm and $t-1$ terminal vertices, hence $\gamma(G)=4$. The characteristic polynomial of $S_{4,2}$, $\kappa^1(\kappa^2-\kappa-5)(\kappa^2-\kappa-1)^3$, $S_{4,3}$, $\kappa^5(\kappa^2-\kappa-6)(\kappa^2-\kappa-2)^3$, $S_{4,4}$, $\kappa^9(\kappa^2-\kappa-7)(\kappa^2-\kappa-3)^3$, hence for $S_{4,t}$, $\kappa^{4k-7}(\kappa^2-\kappa-(t+3))(\kappa^2-\kappa-(t-1))^3$, $n=4t+1$. The corresponding minimal domination energy is $\sqrt{4t+13}+3\sqrt{4t-3}$. □

Theorem 4.14 *For any given thorn rod $P_{5,t}$, the characteristic polynomial is given by, $\kappa^{2t-3}(\kappa^2-\kappa-t)(\kappa^4-2\kappa^3-(t+1)\kappa^2+(t+2)\kappa+(2t-2))$, $n=2t+3$.*

Proof For $P_{5,t}$, $\gamma(G)=3$, for $t=1$, $P_{5,1}$ is a path with 5 vertices, $t=2$, $P_{5,2}$ is a path with 7 vertices. For $t>2$ the characteristic polynomial is given by, for $P_{5,3}$, $\kappa^3(\kappa^2-\kappa-3)(\kappa^4-2\kappa^3-4\kappa^2+5\kappa+4)$, $P_{5,4}$, $\kappa^5(\kappa^2-\kappa-4)(\kappa^4-2\kappa^3-5\kappa^2+6\kappa+6)$, $P_{5,5}$, $\kappa^7(\kappa^2-\kappa-5)(\kappa^4-2\kappa^3-6\kappa^2+7\kappa+8)$, $P_{5,6}$, $\kappa^9(\kappa^2-\kappa-6)(\kappa^4-2\kappa^3-7\kappa^2+8\kappa+10)$, hence for $P_{5,t}$, $\kappa^{2k-3}(\kappa^2-\kappa-t)(\kappa^4-2\kappa^3-(t+1)\kappa^2+(t+2)\kappa+(2t-2))$, n = 2t + 3. These result can be extended to $P_{7,t}$ which has a unique minimal dominating set where as $P_{6,t}$ has two minimal dominating set. □

Theorem 4.15 *For any given thorn ring C_4^t, with four vertices on the cycle of degree t and $n(t-1)$ vertices, the characteristic polynomial is given by,* $\kappa^{4t-12}(\kappa^2+\kappa-(t-2))(\kappa^2-\kappa-(t-2))^2(\kappa^2-3\kappa-(t-2))$ *and* $E_{\gamma-\min}(C_4^t)=\sqrt{4t+1}+3\sqrt{4t-7}$.

Proof The characteristic polynomial of C_4^3, $\kappa^0(\kappa^2+\kappa-1)(\kappa^2-\kappa-1)^2(\kappa^2-3\kappa-1)$, C_4^4, $\kappa^4(\kappa^2+\kappa-2)(\kappa^2-\kappa-2)^2(\kappa^2-3\kappa-2)$, C_4^5, $\kappa^8(\kappa^2+\kappa-3)(\kappa^2-\kappa-3)^2(\kappa^2-3\kappa-3)$, hence for C_4^t $\kappa^{4t-12}(\kappa^2+\kappa-(t-2))(\kappa^2-\kappa-(t-2))^2(\kappa^2-3\kappa-(t-2))$. The corresponding minimal domination energy is $\sqrt{4t+1}+3\sqrt{4t-7}$. □

§5. Open Problems

1. Domination energy for other standard graphs can be explored.
2. The relation between these parameters can be extended to other classes of graphs and other types of domination.
3. Application of Set and domination energy has to Explored.

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