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The Genus of the Folded Hypercube

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Abstract: The folded hypercube FQ_n is a variance of the hypercube network and is superior to Q_n in some properties[IEEE Trans. Parallel Distrib. Syst. 2 (1991) 31-42]. The genus of n-dimensional hypercube Q_n were given by G. Ringel. In this paper, the genus $\gamma(FQ_n)$ of FQ_n is discussed. That is, $\gamma(FQ_n) = (n-3)2^{n-3}+1$ if n is odd and $(n-3)2^{n-3}+1 \leq \gamma(FQ_n) \leq (n-2)2^{n-3}+1$ if n is even.

Key Words: n-Dimensional hypercube, folded hypercube, genus, surface, Smarandache λ^S-drawing,.

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§1. Introduction

Let $G = (V(G), E(G))$ be a *graph*, where $V(G)$ is a finite *vertex set* and $E(G)$ is the *edge set* which is the subset of $\{(u,v)|(u,v)$ is an unordered pair of $V(G)\}$. Two vertices u and v are *adjacent* if $(u,v) \in E(G)$. A *path*, written as $\langle v_0, v_1, v_2, \cdots, v_m\rangle$, is a sequence of adjacent vertices, in which all the vertices $v_0, v_1, v_2, \cdots, v_m$ are distinct except possibly $v_0 = v_m$, the path with $v_0 = v_m$ is a *cycle*. The *girth* of a graph G is the length of the shortest cycle of G.

If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E(G)$ of fewer then l edges, then G is called *l-edge-connected*. The greatest integer l such that G is l-edge-connected is the *edge-connectivity* $\lambda(G)$ of G.

A *surface* is a compact connected orientable 2-manifold which could be thought of as a sphere on which has been placed a number of handles. The number of handles is referred to as the *genus* of the surface. A drawing of graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache λ^S-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S if existing, is called an embedding of G on S.

A *region* of a graph G embedded on a surface is the connected sections of the surface bounded by a set of edges of G. This set of edges is called the *boundary* of the region, and

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the number of edges is the length of the region. We will use $(v_0, v_1, v_2, \cdots, v_m)$, called a *facial cycle*, to denote the region bounded by edges $(v_0, v_1), (v_1, v_2), \cdots$, and (v_m, v_0). so a facial cycle of a graph is a region of the graph. A region is a *k-cycle* if its length is k. A region is a *2-cell* if any simple closed curve within the region can be collapsed to a single point. An embedding of a graph G on a surface S is a *2-cell embedding* if all embedded regions are 2-cells.

An embedding of G into an oriented surface S induce a rotation system as follows: The *local rotation* at a vertex v is the cyclic permutation corresponding to the order in which the edge-ends are traversed in an orientation-preserving tour around v. A *rotation system* of the given embedding of G in S is the collection of local rotations at all vertices of G. It is proved [19] that every 2-cell embedding of a graph G in an orientable surface is uniquely determined, up to homeomorphism, by its rotation system.

Let G be a graph and π be an embedding of G, the corresponding rotation system is denoted by ρ_π. For any $v \in V$, the local rotation at v determined by ρ_π is denoted by $\rho_\pi(v)$. In the following, we consider 2-cell embedding of simple undirected graphs on orientable surfaces, the rotation at a vertex is clockwise. The readers are referred to [1] for undefined notations.

The *genus* $\gamma(G)$ of a graph G is meant the minimum genus of all possible surfaces on which G can be embedded with no edge crossings, similarly, the $\gamma_M(G)$ is the *maximum genus*. As a measure of the complexity of a network, the genus gives an indication of how efficiently the network can be laid out. The smaller the genus, the more efficient the layout. The planer graphs have genus zero since no handles are needed to prevent edge intersections.

Let G be a connected graph with a 2-cell embedding on an orientable surface of genus g, having m vertices, q edges and r regions, then the well known Euler's formula [16] is: $m-q+r = 2-2g$. For embedding, Duke's interpolation theorem [5] is that a connected graph G has a 2-cell embedding on surface S_k if and only if $\gamma(G) \le k \le \gamma_M(G)$, where k is the genus of surface S_k.

Graph embeddings have been studied by many authors over years. Especially the study of the maximum and minimum orientable genus $\gamma_M(G)$ and $\gamma(G)$ of a graph G, they have been proved polynomial [7] and NP-complete [22], respectively. The embedding properties of a graph and some results about surfaces are extensively treated in the books [3,4,8,19]. More results about genera and embedding genus distributions are referred to see [9-11,13-15,17-18,20,23-25,27] etc.. Although there are much results about maximal genera, but minimum genera for most kinds of graphs are not known. The folded hypercube FQ_n is a variance of the hypercube network and is superior to Q_n in some properties such as diameters [6]. The genus $\gamma(Q_n)$ of n-dimensional hypercube Q_n were given by G. Ringel [21], the genus of n-cube is discussed by Beineke and Harary [2].

In this paper, the genus $\gamma(FQ_n)$ of FQ_n is discussed. That is, $\gamma(FQ_n) = (n-3)2^{n-3}+1$ for n is odd and $(n-3)2^{n-3}+1 \le \gamma(FQ_n) \le (n-2)2^{n-3}+1$ for n is even.

§2. Main Results

The n-dimensional hypercube, denoted by Q_n, is a bipartite graph with 2^n vertices, its any vertex v is denoted by an n-bit binary string $v = x_n x_{n-1} \cdots x_2 x_1$ or $(x_n x_{n-1} \cdots x_2 x_1)$, where

$x_i \in \{0,1\}$ for all i, $1 \le i \le n$. Two vertices of Q_n are adjacent if and only if their binary strings differ in exactly one bit position. So Q_n is an n-regular graph.

If $x = x_n x_{n-1} \cdots x_2 x_1$ and $y = y_n y_{n-1} \cdots y_2 y_1$ are two vertices in Q_n such that $y_i = 1 - x_i$ for $1 \le i \le n$, then we denote $y = \overline{x}$, and we say that x and $\overline{x}$ have complementary addresses. As a variance of the Q_n, the n-dimensional folded hypercube, denoted by FQ_n, proposed first by El-Amawy and Latifi[?], is defined as follows: FQ_n is an $(n+1)$-regular graph, its vertex set is exactly $V(Q_n)$, and its edge set is $E(Q_n) \bigcup E_0$, where $E_0 = \{x\overline{x} | x \in V(Q_n)\}$. In other words, FQ_n is a graph obtained from Q_n by adding edges, called complementary edges, between any pair of vertices with complementary addresses. FQ_2 and FQ_3 are shown in Fig.1.

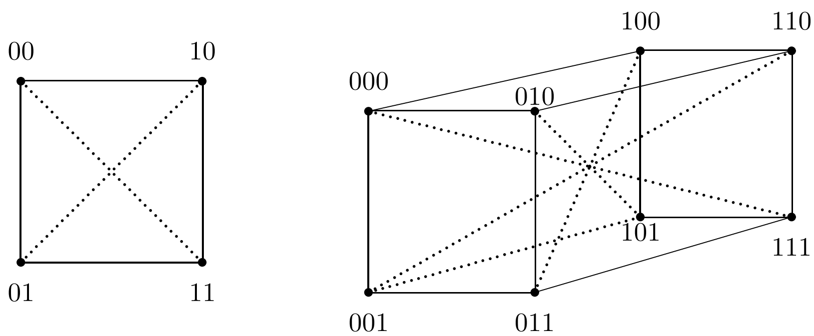

Fig.1 FQ_2 and FQ_3

Lemma 2.1([2, 21]) *Let Q_n be an n-hypercube, then $\gamma(Q_n) = (n-4)2^{n-3} + 1$.*

Lemma 2.1([6]) *The edge-connectivity of n-folded hypercube $\lambda(FQ_n) \ge n+1$.*

Lemma 2.3(Jungerman [12], Xuong [25]) *If G is a 4-edge-connected graph with m vertices and q edges, then $\gamma_M(G) = \lfloor \dfrac{q-m+1}{2} \rfloor$.*

Lemma 2.4 *Let Q_n be an n-dimensional hypercube. Then there exists an embedding π_n of Q_n for $n \ge 3$ on the surface S of genus $(n-4)2^{n-3}+1$, such that each of the following three kinds of cycles for $x_i \in \{0,1\}, 3 \le i \le n$,*

$$((x_n \cdots x_3 10), (x_n \cdots x_3 00), (x_n \cdots x_3 01), (x_n \cdots x_3 11));$$

$$((x_n \cdots x_3 10), (x_n \cdots x_3 00), (\overline{x_n} x_{n-1} \cdots x_3 00), (\overline{x_n} x_{n-1} \cdots x_3 10)) \text{ and}$$

$$((x_n \cdots x_3 01), (x_n \cdots x_3 11), (\overline{x_n} x_{n-1} \cdots x_3 11), (\overline{x_n} x_{n-1} \cdots x_3 01))$$

is a facial 4-cycle of π_n.

Proof It is true for Q_3, shown in *Fig*.2. Assume it is true for $Q_{n-1}, n \ge 4$. There exists an embedding π_{n-1} of Q_{n-1} on the surface S' of genus $(n-5)2^{n-4}+1$, such that each of three kinds of cycles $((x_{n-1} \cdots x_3 10), (x_{n-1} \cdots x_3 00), (x_{n-1} \cdots x_3 01), (x_{n-1} \cdots x_3 11))$; $((x_{n-1} \cdots x_3 10), (x_{n-1} \cdots x_3 00), (\overline{x_{n-1}} x_{n-2} \cdots x_3 00), (\overline{x_{n-1}} x_{n-2} \cdots x_3 10))$ and $((x_{n-1} \cdots x_3 01), (x_{n-1} \cdots x_3 11), (\overline{x_{n-1}} x_{n-2} \cdots x_3 11), (\overline{x_{n-1}} x_{n-2} \cdots x_3 01))$ for $x_i \in \{0,1\}, 3 \le i \le n-1$, is a facial cycle on embedding π_{n-1} of Q_{n-1}. So the rotations of π_{n-1} are as follows:

$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 10) = (A'(x_{n-1}\cdots x_3 11)(x_{n-1}\cdots x_3 00)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 00) = (B'(x_{n-1}\cdots x_3 10)(x_{n-1}\cdots x_3 01)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 01) = (C'(x_{n-1}\cdots x_3 00)(x_{n-1}\cdots x_3 11)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 11) = (D'(x_{n-1}\cdots x_3 01)(x_{n-1}\cdots x_3 10))$$

because of $((x_{n-1}\cdots x_3 10),(x_{n-1}\cdots x_3 00),(x_{n-1}\cdots x_3 01),(x_{n-1}\cdots x_3 11))$ being facial cycles along counter-clockwise; or

$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 10) = (A'(x_{n-1}\cdots x_3 00)(x_{n-1}\cdots x_3 11)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 00) = (B'(x_{n-1}\cdots x_3 01)(x_{n-1}\cdots x_3 10)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 01) = (C'(x_{n-1}\cdots x_3 11)(x_{n-1}\cdots x_3 00)),$$
$$\rho_{\pi_{n-1}}(x_{n-1}\cdots x_3 11) = (D'(x_{n-1}\cdots x_3 10)(x_{n-1}\cdots x_3 01))$$

because of $((x_{n-1}\cdots x_3 11),(x_{n-1}\cdots x_3 01),(x_{n-1}\cdots x_3 00),(x_{n-1}\cdots x_3 10))$ being facial cycles along counter-clockwise, where A', B', C', D' are the ordered subsequences of vertices which incident with $(x_{n-1}\cdots x_3 10),(x_{n-1}\cdots x_3 00),(x_{n-1}\cdots x_3 01)$ and $(x_{n-1}\cdots x_3 11)$, respectively.

By Euler's formula, the boundary of every region in π_{n-1} of Q_{n-1} on S' is a 4-cycle. Let Q_{n-1} embed on another copy surface S'' of genus $(n-5)2^{n-4}+1$ such that the embedding of Q_{n-1} on S'' is a "mirror image" of the embedding of Q_{n-1} on S'. As a subgraph of Q_n, the vertices in embedding of Q_{n-1} on S' and on S'' are labeled by $(0x_{n-1}\cdots x_3x_2x_1)$ and $(1x_{n-1}\cdots x_3x_2x_1)$ respectively, where $x_i \in \{0,1\}, 1 \le i \le n-1$. For simplification, we also use the signals of A', B', C' and D' in the following.

Based on π_{n-1}, the rotation system of π_n is given as follows:

$$\rho_{\pi_n}(x_n\cdots x_3 10) = ((\overline{x_n}x_{n-1}\cdots x_3 10)A'(x_n\cdots x_3 11)(x_n\cdots x_3 00)),$$
$$\rho_{\pi_n}(x_n\cdots x_3 00) = (B'(\overline{x_n}x_{n-1}\cdots x_3 00)(x_n\cdots x_3 10)(x_n\cdots x_3 01)),$$
$$\rho_{\pi_n}(x_n\cdots x_3 01) = ((\overline{x_n}x_{n-1}\cdots x_3 01)C'(x_n\cdots x_3 00)(x_n\cdots x_3 11)),$$
$$\rho_{\pi_n}(x_n\cdots x_3 11) = (D'(\overline{x_n}x_{n-1}\cdots x_3 11)(x_n\cdots x_3 01)(x_n\cdots x_3 10));\text{ or}$$

$$\rho_{\pi_n}(\overline{x_n}x_{n-1}\cdots x_3 10) = (A'(x_n\cdots x_3 10)(\overline{x_n}x_{n-1}\cdots x_3 00)(\overline{x_n}x_{n-1}\cdots x_3 11)),$$
$$\rho_{\pi_n}(\overline{x_n}x_{n-1}\cdots x_3 00) = ((x_n\cdots x_3 00)B'(\overline{x_n}x_{n-1}\cdots x_3 01)(\overline{x_n}x_{n-1}\cdots x_3 10)),$$
$$\rho_{\pi_n}(\overline{x_n}x_{n-1}\cdots x_3 01) = (C'(x_n\cdots x_3 01)(\overline{x_n}x_{n-1}\cdots x_3 11)(\overline{x_n}x_{n-1}\cdots x_3 00)),$$
$$\rho_{\pi_n}(\overline{x_n}x_{n-1}\cdots x_3 11) = ((x_n\cdots x_3 11)D'(\overline{x_n}x_{n-1}\cdots x_3 10)(\overline{x_n}x_{n-1}\cdots x_3 01)),$$

where $\overline{x_i} = 1 - x_i$.

By using the method of researching regions of embedding from rotation system in [19], the following four kinds of facial cycles on S' or S''

$$((00x_{n-2}\cdots x_3 10),(00x_{n-2}\cdots x_3 00),(01x_{n-2}\cdots x_3 00),(01x_{n-2}\cdots x_3 10));$$
$$((11x_{n-2}\cdots x_3 10),(11x_{n-2}\cdots x_3 00),(10x_{n-2}\cdots x_3 00),(10x_{n-2}\cdots x_3 10));$$
$$((00x_{n-2}\cdots x_3 11),(00x_{n-2}\cdots x_3 01),(01x_{n-2}\cdots x_3 01),(01x_{n-2}\cdots x_3 11));$$
$$((11x_{n-2}\cdots x_3 11),(11x_{n-2}\cdots x_3 01),(10x_{n-2}\cdots x_3 01),(10x_{n-2}\cdots x_3 11));$$

are replaced in π_n by the following eight facial 4-cycles:

$$((00x_{n-2}\cdots x_3 10),(00x_{n-2}\cdots x_3 00),(10x_{n-2}\cdots x_3 00),(10x_{n-2}\cdots x_3 10));$$

$((00x_{n-2}\cdots x_3 00), (01x_{n-2}\cdots x_3 00), (11x_{n-2}\cdots x_3 00), (10x_{n-2}\cdots x_3 00))$;

$((01x_{n-2}\cdots x_3 00), (01x_{n-2}\cdots x_3 10), (11x_{n-2}\cdots x_3 10), (11x_{n-2}\cdots x_3 00))$;

$((01x_{n-2}\cdots x_3 10), (00x_{n-2}\cdots x_3 10), (10x_{n-2}\cdots x_3 10), (11x_{n-2}\cdots x_3 10))$;

$((00x_{n-2}\cdots x_3 11), (00x_{n-2}\cdots x_3 01), (10x_{n-2}\cdots x_3 01), (10x_{n-2}\cdots x_3 11))$;

$((00x_{n-2}\cdots x_3 01), (01x_{n-2}\cdots x_3 01), (11x_{n-2}\cdots x_3 01), (10x_{n-2}\cdots x_3 01))$;

$((01x_{n-2}\cdots x_3 01), (01x_{n-2}\cdots x_3 11), (11x_{n-2}\cdots x_3 11), (11x_{n-2}\cdots x_3 01))$;

$((01x_{n-2}\cdots x_3 11), (00x_{n-2}\cdots x_3 11), (10x_{n-2}\cdots x_3 11), (11x_{n-2}\cdots x_3 11))$

and the other regions are not changed. As a result, each region of π_n is a 4-cycle. By the Euler's formula, the genus of embedding π_n of Q_n is exactly $2((n-5)2^{n-4}+1)+2^{n-3}-1 = (n-4)2^{n-3}+1$.

Further more, it could be found that the following three kinds of 4-cycles

$((x_n\cdots x_3 10), (x_n\cdots x_3 00), (x_n\cdots x_3 01), (x_n\cdots x_3 11))$;

$((x_n\cdots x_3 10), (x_n\cdots x_3 00), (\overline{x_n}x_{n-1}\cdots x_3 00), (\overline{x_n}x_{n-1}\cdots x_3 10))$ and

$((x_n\cdots x_3 01), (x_n\cdots x_3 11), (\overline{x_n}x_{n-1}\cdots x_3 11), (\overline{x_n}x_{n-1}\cdots x_3 01))$

for $x_i \in \{0,1\}$ and $3 \le i \le n$ are facial 4-cycles on π_n. □

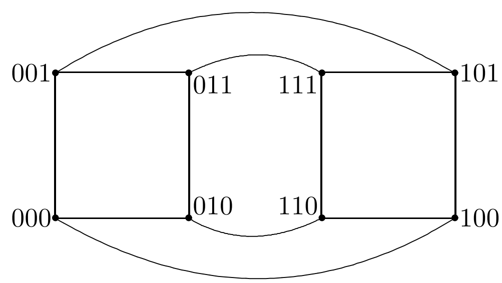

Fig.2 Q_3

Lemma 2.5([26])

(1) *FQ_n is a bipartite graph if and only if n is odd.*

(2) *If n is even, then the length of any shortest odd cycle in FQ_n is $n+1$.*

Theorem 2.6 *The genus of $FQ_n (n \ge 3)$ is given as $\gamma(FQ_n) = (n-3)2^{n-3}+1$ for n is odd and $(n-3)2^{n-3}+1 \le \gamma(FQ_n) \le (n-2)2^{n-3}+1$ for n is even.*

Proof FQ_n is embedded on the surface of genus $\gamma(FQ_n)$ with m vertices, q edges and r regions, where $m = 2^n$ and $q = (n+1)2^{n-1}$. From Lemma 2.5, the girth of FQ_n is 4 for $n \ge 3$. By Euler's formula, $4r \le 2q$, $m - q + r = 2 - 2\gamma(FQ_n) \le m - \frac{q}{2}$, so $2\gamma(FQ_n) - 2 \ge \frac{q}{2} - m$. That implies $\gamma(FQ_n) \ge (n-3)2^{n-3}+1$.

To finish the proving, we only need to give an embedding of FQ_n such that the genus of embedded surface is $(n-3)2^{n-3}+1$ if n is odd, and is $(n-2)2^{n-3}+1$ if n is even, respectively.

First, Q_n is embedded on the surface with rotation system σ which is the same as the embedding π_n in Lemma 2.4, then we have the following rotations:

$$\begin{aligned}
\rho_\sigma(x_n\cdots x_3 10) &= (A(x_n\cdots x_3 11)(x_n\cdots x_3 00)),\\
\rho_\sigma(x_n\cdots x_3 00) &= (B(x_n\cdots x_3 10)(x_n\cdots x_3 01)),\\
\rho_\sigma(x_n\cdots x_3 01) &= (C(x_n\cdots x_3 00)(x_n\cdots x_3 11)),\\
\rho_\sigma(x_n\cdots x_3 11) &= (D(x_n\cdots x_3 01)(x_n\cdots x_3 10)).
\end{aligned} \tag{2.1}$$

Or

$$\begin{aligned}
\rho_\sigma(x_n\cdots x_3 10) &= (A(x_n\cdots x_3 00)(x_n\cdots x_3 11)),\\
\rho_\sigma(x_n\cdots x_3 00) &= (B(x_n\cdots x_3 01)(x_n\cdots x_3 10)),\\
\rho_\sigma(x_n\cdots x_3 01) &= (C(x_n\cdots x_3 11)(x_n\cdots x_3 00)),\\
\rho_\sigma(x_n\cdots x_3 11) &= (D(x_n\cdots x_3 10)(x_n\cdots x_3 01)),
\end{aligned} \tag{2.2}$$

where A, B, C, D are the ordered sequences of vertices which is incident with $(x_n\cdots x_3 10)$, $(x_n\cdots x_3 00), (x_n\cdots x_3 01), (x_n\cdots x_3 11)$, respectively.

According to ρ_σ of formulae (2.1) and (2.2) respectively and the fact that graph FQ_n is obtained from Q_n by adding complementary edges, the rotation system, denoted by θ, of FQ_n is gotten from rotation system σ as followings:

$$\begin{aligned}
\rho_\theta(x_n\cdots x_3 10) &= (A(x_n\cdots x_3 11)(\overline{x_n}\cdots\overline{x_3}01)(x_n\cdots x_3 00)),\\
\rho_\theta(x_n\cdots x_3 00) &= (B(x_n\cdots x_3 10)(\overline{x_n}\cdots\overline{x_3}11)(x_n\cdots x_3 01)),\\
\rho_\theta(x_n\cdots x_3 01) &= (C(x_n\cdots x_3 00)(\overline{x_n}\cdots\overline{x_3}10)(x_n\cdots x_3 11)),\\
\rho_\theta(x_n\cdots x_3 11) &= (D(x_n\cdots x_3 01)(\overline{x_n}\cdots\overline{x_3}00)(x_n\cdots x_3 10)).
\end{aligned} \tag{2.3}$$

Or

$$\begin{aligned}
\rho_\theta(\overline{x_n}\cdots\overline{x_3}10) &= (A(\overline{x_n}\cdots\overline{x_3}00)(x_n\cdots x_3 01)(\overline{x_n}\cdots\overline{x_3}11)),\\
\rho_\theta(\overline{x_n}\cdots\overline{x_3}00) &= (B(\overline{x_n}\cdots\overline{x_3}01)(x_n\cdots x_3 11)(\overline{x_n}\cdots\overline{x_3}10)),\\
\rho_\theta(\overline{x_n}\cdots\overline{x_3}01) &= (C(\overline{x_n}\cdots\overline{x_3}11)(x_n\cdots x_3 10)(\overline{x_n}\cdots\overline{x_3}00)),\\
\rho_\theta(\overline{x_n}\cdots\overline{x_3}11) &= (D(\overline{x_n}\cdots\overline{x_3}10)(x_n\cdots x_3 00)(\overline{x_n}\cdots\overline{x_3}01)),
\end{aligned} \tag{2.4}$$

where $\overline{x_i} = 1 - x_i$.

If n is odd, by the embedding σ of Q_n, the two kinds of 4-cycles

$$\begin{aligned}
&((x_n\cdots x_3 10), (x_n\cdots x_3 00), (x_n\cdots x_3 01), (x_n\cdots x_3 11));\\
&((\overline{x_n}\cdots\overline{x_3}10), (\overline{x_n}\cdots\overline{x_3}11), (\overline{x_n}\cdots\overline{x_3}01), (\overline{x_n}\cdots\overline{x_3}00))
\end{aligned} \tag{2.5}$$

are facial cycles of this embedding of Q_n on the clockwise direction (or counter-clockwise direction). From the definition of θ of FQ_n, the following four kinds of complementary edges are added in the facial cycles (2.5) shown in (a)(b) of Fig.3.

$$
\begin{aligned}
&((x_n\cdots x_3 10),(\overline{x_n}\cdots\overline{x_3}01));\ (x_n\cdots x_3 00),\overline{x_n}\cdots\overline{x_3}11));\\
&((x_n\cdots x_3 01),(\overline{x_n}\cdots\overline{x_3}10));\ (x_n\cdots x_3 11),\overline{x_n}\cdots\overline{x_3}00)).
\end{aligned}
\tag{2.6}
$$

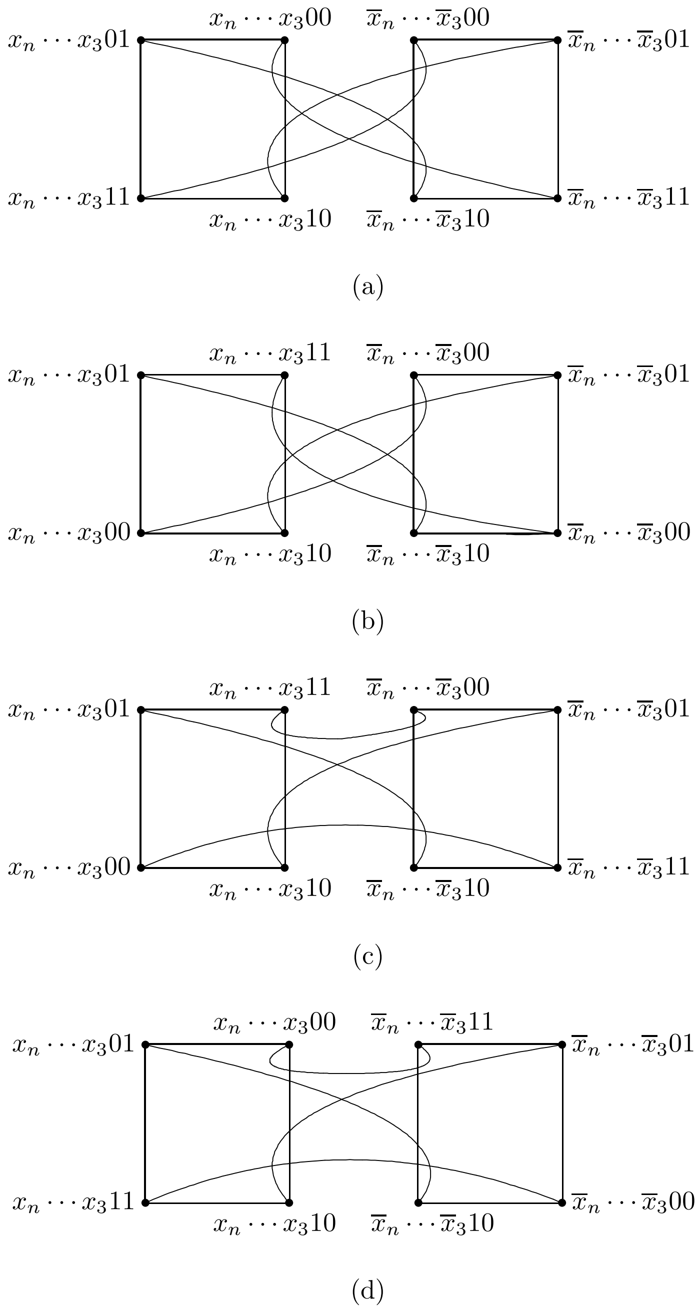

Fig.3 Two kinds of embedding depending on n being odd or even

As a result, the regions (2.5) of σ are replaced by the following four kinds of 4-regions in θ of FQ_n:

$$\begin{aligned}
&((x_n\cdots x_3 11),(x_n\cdots x_3 10),(\overline{x_n}\cdots\overline{x_3}01),(\overline{x_n}\cdots\overline{x_3}00));\\
&((x_n\cdots x_3 01),(x_n\cdots x_3 11),(\overline{x_n}\cdots\overline{x_3}00),(\overline{x_n}\cdots\overline{x_3}10));\\
&((x_n\cdots x_3 00),(x_n\cdots x_3 01),(\overline{x_n}\cdots\overline{x_3}10),(\overline{x_n}\cdots\overline{x_3}11));\\
&((x_n\cdots x_3 10),(x_n\cdots x_3 00),(\overline{x_n}\cdots\overline{x_3}11),(\overline{x_n}\cdots\overline{x_3}01)).
\end{aligned}\tag{2.7}$$

The other regions are not changed, thus all regions of embedding θ of FQ_n are all 4-cycles, and the number of regions is $2^{n-2}(n+1)$. Recalled that FQ_n have 2^n vertices, $2^{n-1}(n+1)$ edges. By Euler's formula, the total genus of θ of FQ_n for n being odd is $(n-3)2^{n-3}+1$.

If n is even, by the embedding σ of Q_n, the two kinds of 4-cycles

$$\begin{aligned}
&((x_n\cdots x_3 10),(x_n\cdots x_3 00),(x_n\cdots x_3 01),(x_n\cdots x_3 11));\\
&((\overline{x_n}\cdots\overline{x_3}10),(\overline{x_n}\cdots\overline{x_3}00),(\overline{x_n}\cdots\overline{x_3}01),(\overline{x_n}\cdots\overline{x_3}11))
\end{aligned}\tag{2.8}$$

are facial cycles of this embedding of Q_n on the clockwise direction (or counter-clockwise direction). From the definition of θ of FQ_n, By adding four kinds of complementary edges of (2.6) in facial cycles (2.8) shown in (c) and (d) of Fig.3, the regions in (2.8) of σ are replaced by the following two kinds of 8-cycles in θ of FQ_n:

$$\begin{aligned}
&((x_n\cdots x_3 10),(x_n\cdots x_3 11),(\overline{x_n}\cdots\overline{x_3}00),(\overline{x_n}\cdots\overline{x_3}10),\\
&(x_n\cdots x_3 01),(x_n\cdots x_3 00),(\overline{x_n}\cdots\overline{x_3}11),(\overline{x_n}\cdots\overline{x_3}01));\\
&((x_n\cdots x_3 10),(x_n\cdots x_3 00),(\overline{x_n}\cdots\overline{x_3}11),(\overline{x_n}\cdots\overline{x_3}10),\\
&(x_n\cdots x_3 01),(x_n\cdots x_3 11),(\overline{x_n}\cdots\overline{x_3}00),(\overline{x_n}\cdots\overline{x_3}01)).
\end{aligned}$$

As a result, the number of regions in θ is less 2^{n-1} than regions in σ. By Euler's formula $2^n - 2^{n-1}(n+1) + (2^{n-2}n - 2^{n-1}) = 2 - 2h$, the genus h of embedding θ of FQ_n for n being even is $(n-2)2^{n-3}+1$. □

From Lemmas 2.2 and 2.3, the following theorem is immediately obtained.

Theorem 2.7 *The maximum genus of FQ_n is given by $\gamma_M(FQ_n) = (n-1)2^{n-2}$ for $n \geq 3$. Furthermore, $\gamma(FQ_2) = 0, \gamma_M(FQ_2) = 1$.*

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