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Star Chromatic and Defining Number of Graphs

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Abstract: Let u and v be adjacent vertices in G. If we assign colors to $N[v]$ and $N[u]$ such that the assignment colors to $N[v]$ are different with the assignment colors to $N[u]$, then this colorings is said to be vertex star colorings. In this paper we initiate the study of the star chromatic number and star defining number.

Key Words: Star coloring, star chromatic number, star defining number, Smarandachely Λ-coloring.

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§1. Introduction

In the whole paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. We use [9] for terminology and notation which are not defined here.

Let Λ be a subgraph of a graph G. A Smarandachely Λ-coloring $\varphi_\Lambda|_{V(G)} : \mathscr{C} \to V(G)$ of a graph G by colors in $\mathscr{C}$ is a mapping $\varphi_\Lambda : \mathscr{C} \to V(G) \cup E(G)$ such that $\varphi(u) \neq \varphi(v)$ if u and v are vertices of a subgraph isomorphic to Λ in G. Particularly, if $\Lambda = G$, such a coloring is called a k-coloring of G. A graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ is the least k such that G is k-colorable. Let $\chi(G) \leq k \leq |V(G)|$. A set $S \subseteq V(G)$ with an assignment of colors to them is called a defining set of the vertex coloring of G if there exists a unique extension of S to a k-coloring of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by $d(G,k)$, for more see [1, 3, 4, 5, 6, 7].

In this note we introduce vertex star coloring of graphs as follows:

If u and v are arbitrary adjacent vertices in G, then the set of colors that we assign to $N[v]$ is different with the set of colors that assign to $N[u]$. We call this vertex coloring as *vertex star coloring*. It is obvious that vertex star coloring does not include the family of graphs with

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following property:

$\exists u, v \in V(G)$ with $N[v] = N[u]$, for which $uv \in E(G)$.

The chromatic number and defining number of vertex star coloring are called *the star chromatic number* (χ^*) and *star defining number* (d^*), respectively.

We make the following observations:

Observation 1 *For every connected graph G of order $n \geqslant 3$, $\chi^*(G) \geq 3$.*

Observation 2 *If $\chi^*(G) = 3$, then $|f(N[v])| = 2, |f(N[u])| = 3$ for every two adjacent vertices $u,\ v \in V(G)$ for which f is a star coloring function.*

Our purpose in this paper is to initiate the study of the star chromatic number and the star defining number (d^*) of cycles, paths and complete bipartite, hyper cube and Cartesian product $P_n \times P_m$ graphs.

§2. Star Chromatic Numbers

In this section the star chromatic number of cycle, path, complete bipartite and Cartesian product $P_n \times P_m$ graphs are studied.

First, we present a general result as follows:

Proposition 3 *Let G be a graph. Then $\chi^*(G) > \chi(G)$.*

Proof On the one hand, $\chi^*(G) \geq \chi(G)$. On the other hand, it is enough to show that $\chi^*(G) \neq \chi(G)$. Suppose to the contrary. First, we increasingly order vertices of G and color the vertex with the least index by 1. Now, we color the remaining vertices by this manner, i.e: for the next uncolored vertex, we assign an unused color on its neighbors or a new color if be necessary (Greedy algorithm). Hence, a vertex color by $\chi(G)$ such that its neighbors colored by $\{1, 2, \cdots, \chi(G) - 1\}$. And a vertex color by $\chi(G) - 1$ such that its neighbors colored by $\{1, 2, \cdots, \chi(G) - 2\}$. Without loss of generality, we may assume that u and v are two vertices which colored by $\chi(G) - 1$ and $\chi(G)$. It follows that the set $\{1, 2, \ldots, \chi(G)\}$ is the used colors on u and its neighbors, and on the vertex v and its neighbors, a contradiction. □

Proposition 4 (*i*) *$\chi^*(C_n) = 3$ where $n = 4m$.*

(*ii*) *$\chi^*(C_n) = 4$ where $n = 4m + 2$.*

Proof (*i*) Consider the star coloring function f as follows:

$$f(v_i) = \begin{cases} 2 & i \text{ is odd}, \\ 1 & i = 4t + 2, \\ 3 & i = 4t. \end{cases}$$

It implies that $\chi^*(G) \leq 3$. Hence, by Proposition 3 the desired result follows.

(*ii*) Define the star coloring function f as follows:

$$f(v_i) = \begin{cases} 2 & \text{i is odd and } i \neq 4m+1, \\ 3 & i = 4t+2 \text{ and } i \leq 4m, \\ 1 & i = 4t, 4m+2, \\ 4 & i = 4m+1. \end{cases}$$

It follows that $\chi^*(G) \leq 4$. Now, we show that $\chi^*(G) \geq 4$. It is easy to check that for any four consecutive vertices in C_n, namely $v_i, v_{i+1}, v_{i+2}, v_{i+3}$, we have $f(v_i) \neq f(v_{i+3})$. Otherwise, a contradiction. Moreover, we must use 3 different colors on any four consecutive vertices. Using the star coloring function f in the proof of Part (i), which implies that the vertex v_{n-1} cannot be colored by 2. The set of the colors of v_{4m+1} and its neighbors will be the same as the ones of v_{4m+2} and its neighbors. Thus, it can be colored by 4. Hence the desired result follows. □

Now, we continue the study of the star chromatic numbers on odd cycle.

Proposition 5 *$\chi^*(C_n) = 4$ where $n(\neq 5, 7)$ is an odd integer.*

Proof For $n = 5$, the star coloring function of C_5 can be defined as follows: $f(v_1) = 1$, $f(v_2) = 3$, $f(v_3) = 2$, $f(v_4) = 4$, $f(v_5) = 5$.

For $n = 7$, the star coloring function of C_7 can be defined as follows: $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 1$, $f(v_4) = 3$, $f(v_5) = 4$, $f(v_6) = 3$, $f(v_7) = 5$.

Let $n - 1 = 6t + 4$. Consider the star coloring function f as follows:

$$f(v_i) = \begin{cases} 3 & i = 6t+2, t \geq 1 \text{ and } i = 1, 3, \\ 4 & i = 6t+4, \\ 2 & i = 6t, 2, \\ 1 & i = n \text{ and } i \text{ is odd and } i \neq 1, 3. \end{cases}$$

Let $n - 1 = 6t$. Consider the star coloring function f as follows:

$$f(v_i) = \begin{cases} 3 & i = 6t+2, n, \\ 4 & i = 6t+4, n-1, \\ 2 & i = 6t \text{ and } i = 1, n-3, \\ 1 & i \text{ is odd and } i \neq 1, n. \end{cases}$$

Let $n - 1 = 6t + 2, n > 9$. Consider the star coloring function f as follows:

$$f(v_i) = \begin{cases} 3 & i = 6t+2, t \geq 1 \text{ and } i = 1, 3 \\ 4 & i = 6t+4, n-1, \\ 2 & i = 6t \text{ and } i = 6t, 2, \\ 1 & i \text{ is odd and } i \neq 1, 3. \end{cases}$$

Hence, by Proposition 3 and the fact that $\chi(C_n) = 3$ for which n is an odd integer, we get that $\chi^*(G) = 4$. □

Proposition 6 *(i) $\chi^*(P_n) = 3$ where n is an odd integer.*

(ii) $\chi^*(P_n) = 4$ *where* $n \geqslant 4$ *is an even integer.*

Proof (*i*) Define the the star coloring function f as follows:

$$f(v_i) = \begin{cases} 2 & i = 2t, \\ 1 & i = 4t + 1, \\ 3 & i = 4t + 3. \end{cases}$$

This completes the proof.

(*ii*) Using a same fashion star coloring function f in Part (i), but $f(v_{n=2m}) = 4$. It follows that $\chi^*(P_{n=2m}) \leq 4$. Now, we consider two cases as follows.

Case 1 If $m = 2t$, then, according to the star coloring function f, let $f(v_{2m-1}) = 3$. It follows that the vertex v_{2m} cannot be colored by 2 or 3. Color the vertex v_{n-1} by 3, so the vertex v_n cannot be colored by 1, 2 and 3. Thus, it can be colored by 4. Hence the result holds.

Case 2 If $m = 2t + 1$, In the same manner in Case 1 settle this case. □

Proposition 7 $\chi^*(K_{m,n}) = 3$.

Proof Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be partite sets of $K_{m,n}$. On the one hand, we may define the star coloring function f as follows: $f(v_i) = 1 (1 \leq i \leq m)$, $f(u_j) = 2$ $(1 \leq j \leq n-1)$, $f(u_n) = 3$. Thus $\chi^*(K_{m,n}) \leq 3$. On the other hand, if we use two colors on vertices of complete bipartite graphs, we imply that $N[u] = N[v]$ for every vertex $u \in X$ and $v \in Y$. So $\chi^*(K_{m,n}) \geq 3$. Hence the result holds. □

Theorem 8 $\chi^*(P_n \times P_m) = 3$.

Proof Let v_{ij} be the vertex in ith row and jth column. Define the star coloring function c^* as follows:

$$c^*(v_{ij}) = \begin{cases} 2 & j \equiv 2 \ (mod\ 4) \text{ and } i \text{ is odd or } \ j \equiv 3 \ (mod\ 4) \text{ and } i \text{ is even}, \\ 3 & j \equiv 0 \ (mod\ 4) \text{ and } i \text{ is odd } or \ j \equiv 1 \ (mod\ 4) \text{ and } i \text{ is even}, \\ 1 & o.w. \end{cases}$$

Hence the result holds. □

The following observation has straightforward proof.

Observation 9 $\chi^*(Q_k) = 3$.

§3. Star Defining Numbers

Proposition 10 $d^*(C_n, \chi^*) = 2$ *where* $n = 4m$.

Proof Let $S = \{v_1, v_3\}$ and define the star coloring function f on S as follows: $f(v_1) = 1$, $f(v_3) = 3$. It is easy to check that the remaining vertices are forced to get one color which implies that $d^*(C_{n=4m}, \chi^*) \leq 2$.

On the other side, it is well-known that $d^*(C_{n=4k}, \chi^*) \geq \chi^*(G) - 1 = 2$. This completes the proof. □

Now, the star defining numbers of odd paths are studied.

Proposition 11 *(i)* $d^*(P_n, \chi^*) \leq m - 1$ *where* $n = 2m$.
(ii) $d^*(P_n, \chi^*) = 2$ *where* $n = 2m + 1$.

Proof *(i)* We define $S = \{v_i | i = 3t + 1 \text{ and } t(> 0)\ t \text{ is even}\} \cup \{v_i | i = 3t,\ t = 1 \text{ and } t(\geqslant 3) \text{ is odd}\} \cup \{v_i | i = 3t + 2 \text{ and } t \text{ is odd}\}$ with

$$f(v_i) = \begin{cases} 2 & i = 3t \text{ and } t = 1 \text{ and } t \geq 3 \text{ and } t \text{ is odd,} \\ 4 & i = 3t + 1 \text{ and } t > 0 \text{ and } t \text{ is even,} \\ 3 & i = 3t + 2 \text{ and } t \text{ is odd.} \end{cases}$$

(ii) Define $S = \{v_1, v_2\}$ with $f(v_1) = 1$, $f(v_2) = 2$. The rest of vertices orderly get colors from $v_3,\ v_4,\ \cdots,\ v_{2n+1}$. We know that for every graph G, $d^*(G, \chi^*) \geq \chi^* - 1$. Therefore $d^*(P_n, \chi^*) = 2$ where $n = 2m + 1$. □

Proposition 12 $d^*(K_{1,n}, \chi^*) = n$.

Proof Let $X = \{x_1\}$ and $Y = \{y_1, \cdots, y_n\}$ be partite sets of $K_{1,n}$. Define $S = Y$ with $f(y_i) = 2$ $(1 \leq i \leq n - 1)$, $f(y_n) = 3$. So $f(x_1) = 1$. Thus, $d^*(K_{1,n}, \chi^*) \leq n$.

Now, we show that $d^*(K_{1,n}, \chi^*) \geq n$. It is easy to check that if we use two colors on $n - 1$ vertices of Y, thus one can obtain two different colorings. Hence, $d^*(K_{1,n}, \chi^*) = n$. □

Proposition 13 $d^*(K_{m,n}, \chi^*) = m$ *where* $1 < m \leq n$.

Proof Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be partite sets of $K_{m,n}$. We define $S = \{x_1, x_2, \ldots, x_m\}$ with $f(x_i) = 2$ $(1 \leq i \leq m - 1)$, $f(x_m) = 3$ and get the result $f(y_j) = 1$ $(1 \leq j \leq n)$.

Now, we show that $d^*(K_{m,n}, \chi^* = 3) \geq m$. Suppose that we color $m - 1$ vertices of X by two colors, then the remaining vertex of X can be colored by two different colors, a contradiction. Hence the result. □

Proposition 14 *If* $G = K_{m,n}$, $m \leq n$ *and* $m > 1$ *then*

$$d^*(K_{m,n}, c \geq \chi^* + 1) = \begin{cases} m & c \leq m, \\ m + n & c > max\{m, n\}, \\ n & m < c \leq n. \end{cases}$$

Proof The same used manner in Propositions 12 and 13 settles the stated result. □

Proposition 15 *(i)* $d^*(P_3 \times P_3) = d^*(P_3 \times P_4) = d^*(P_3 \times P_5) = 2$.
(ii) $d^*(P_2 \times P_3) = d^*(P_2 \times P_4) = d^*(P_2 \times P_5) = 2$.

Proof We know that $d^*(P_n \times P_m) \geq \chi^*(P_n \times P_m) - 1 = 3 - 1 = 2$. It is enough to present a star defining set of size 2 for each of these graphs. Define the star defining sets of $P_2 \times P_3, P_2 \times P_4, P_2 \times P_5, P_3 \times P_3, P_3 \times P_4, P_3 \times P_5$, as follows:

$$\begin{bmatrix} * & 2 & * \\ 3 & * & * \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ 2 & * & 3 & * \end{bmatrix}, \begin{bmatrix} * & * & * & * & * \\ * & * & 2 & * & 3 \end{bmatrix},$$

$$\begin{bmatrix} * & 2 & * \\ 3 & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & 3 & * & 2 \\ * & * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * & * & * \\ 3 & * & 2 & * & * \\ * & * & * & * & * \end{bmatrix}.$$

□

Theorem 16 *If n is an even integer and $n/2 \times \lfloor m/2 \rfloor \neq 1$, then $d^*(P_n \times P_m) \leq n/2 \times \lfloor m/2 \rfloor$.*

Proof In the following table, a star defining set of size $n/2 \times \lfloor m/2 \rfloor$ is presented.

$$\begin{bmatrix} * & 2 & * & 2 & * & \dots \\ * & * & * & * & * & \dots \\ * & 3 & * & 3 & * & \dots \\ * & * & * & * & * & \dots \\ * & 2 & * & 2 & * & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & a & * & a & * & \dots \\ * & * & * & * & * & \dots \end{bmatrix}$$

if $n = 4k + 2$, then $a = 2$, and if $n = 4k$, then $a = 3$. □

Conjecture 17 *If n is an even number and $n/2 \times \lfloor m/2 \rfloor \neq 1$, then $d^*(P_n \times P_m) = n/2 \times \lfloor m/2 \rfloor$.*

Theorem 18 *If $m(k+1) \geq 4$, then $d^*(P_{2k+1} \times P_{2m+1}, \chi^*) \leq m(k+1) - 2$.*

Proof In the following table, a star defining set of size $m(k+1) - 2$ is shown.

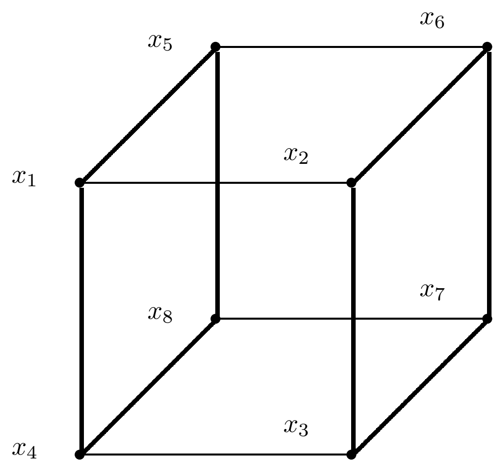
$$\begin{bmatrix} * & 2 & * & 2 & * & \dots & 2 & * & 2 & * \\ * & * & * & * & * & \dots & * & * & * & * \\ * & 3 & * & 3 & * & \dots & 3 & * & 3 & * \\ * & * & * & * & * & \dots & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & 3 & * & \dots & 3 & * & * & * \end{bmatrix}$$

So, the star defining number is less or equal to this value. □

Conjecture 19 *If $m(k+1) \geq 4$ and $k \leq m$, then $d^*(P_{2k+1} \times P_{2m+1}, \chi^*) = m(k+1) - 2$.*

Theorem 20 *If $k \geq 2$, then $d^*(Q_k, 3) = 2^{k-2} + 1$.*

Proof First, we show that $d^*(Q_k, \chi^*) \leq 2^{k-2} + 1$. It is well-known that each Q_k is 2^{k-3} copies of Q_3. We label the vertices of Q_3 as the following figure:



We define the star defining set as the following matrix for which ith row is dependent to the vertices of ith copy of Q_3 in Q_k. Note that at the defining set of Q_k, just one vertex gets color i and the remaining vertices get color j.

For $Q_3 : \begin{bmatrix} i & * & j & * & * & .j & * & * \end{bmatrix}.$

For $Q_4 : \begin{bmatrix} i & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \end{bmatrix}.$

For $Q_5 : \begin{bmatrix} i & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \\ * & j & * & * & * & * & j & * \\ * & * & j & * & * & j & * & * \end{bmatrix}.$

For $Q_6 : \begin{bmatrix} i & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \\ * & j & * & * & * & * & j & * \\ * & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \\ * & * & j & * & * & j & * & * \\ * & * & j & * & * & j & * & * \\ * & j & * & * & * & * & j & * \end{bmatrix}.$

We know that Q_k is constructed by two copies of Q_{k-1}. Therefore, we may give a star defining set in general form for the graph as follows: We assign for the first copy as above. For the next copy; if in a row of the first copy we define $* \ * \ j \ * \ * \ j \ * \ *$, we may define in the symmetric row of the new copy as $* \ j \ * \ * \ * \ * \ j \ *$, and if in the first copy we define $* \ j \ * \ * \ * \ * \ j \ *$, we may define in the symmetric row of the next copy we define $* \ * \ j \ * \ * \ j \ * \ *$. Note that in the first row we have $i \ * \ j \ * \ * \ j \ * \ *$ but for the its symmetric row in the new copy we define as $* \ j \ * \ * \ * \ * \ j \ *$.

Now, we show that $d^*(Q_k, \chi^*) \geq 2^{k-2} + 1$. If $k = 2$, it is obvious. For completing of the proof, first we show that in each Q_3 of Q_k which colored by three colors i, j, k. Then we have just one way to color of each Q_3. Let $c(i)$ be the set of vertices with color i. It is easy to check that $|c(i)| = 1$, $|c(j)| = 1$ or $|c(k)| = 1$ is not possible. Because, we cannot find a proper star coloring for Q_k. Now, let $|c(i)| = 2$. We have two cases: (a): $|c(j)| = |c(k)| = 3$. By simple verification one can see that this cases also cannot be holden. (b): $|c(j)| = 2$ and $|c(k)| = 4$ (or symmetrically $|c(k)| = 2$ and $|c(j)| = 4$). Hence, we may color the graphs Q_3, Q_4, Q_5 and Q_6 as follows, respectively.

$$Q_3 : \begin{bmatrix} i & k & j & k & k & j & k & i \end{bmatrix}.$$

$$Q_4 : \begin{bmatrix} i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \end{bmatrix}.$$

$$Q_5 : \begin{bmatrix} i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \\ k & j & k & i & i & k & j & k \\ i & k & j & k & k & j & k & i \end{bmatrix}.$$

$$Q_6 : \begin{bmatrix} i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \\ k & j & k & i & i & k & j & k \\ i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \\ i & k & j & k & k & j & k & i \\ i & k & j & k & k & j & k & i \\ k & j & k & i & i & k & j & k \end{bmatrix}.$$

To color of the graph Q_k with $k \geqslant 5$, we should color it by the above method, otherwise we cannot find a proper star coloring for the graph. We may also replace color 2 with 3, and conversely to find a new proper star coloring of Q_k. Let S be a defining set of Q_k. It is so easy that $|S| \geqslant 3$ for Q_3. It is well-known that the graph Q_k with $k \geqslant 3$ containing of 2^{k-3} copies of Q_3. Simple verification shows that there exist no copy Q_3 of Q_k such that $S \cap V(Q_3) = 1$. Because, it is possible to assign at least two star coloring functions. It follows that $S \cap V(Q_3^i) \geqslant 2$ where $2 \leqslant i \leqslant 2^{k-3}$. Hence, the desired result follows. □

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