A new proof of Menelaus's Theorem of Hyperbolic Quadrilaterals in the Poincaré Model of Hyperbolic Geometry

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Abstract. In this study, we present a proof of the Menelaus theorem for quadrilaterals in hyperbolic geometry, and a proof for the transversal theorem for triangles

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1. Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models [3] etc. Following [6] and [7] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model. Menelaus of Alexandria was a Greek mathematician and astronomer, the first to recognize geodesics on a curved surface as natural analogs of straight lines. The well-known Menelaus theorem states that if l is a line not through any vertex of a triangle ABC such that l meets BC in D, CA in E, and AB in F, then $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$ [2]. Here, in this study, we give hyperbolic version of Menelaus theorem for quadrilaterals in the Poincaré disc model. Also, we will give a reciprocal hyperbolic version of this theorem. In [1] has been given proof of this theorem, but to use Klein's model of hyperbolic geometry.

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Definition 1. *The hyperbolic distance function in D is defined by the equation*

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

For further details we refer to the recent book of A.Ungar [7].

Theorem 2. (***The Menelaus's Theorem for Hyperbolic Gyrotriangle***). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and with gyroangles α, β, and γ at the vertices A, B, and C. If l is a gyroline not through any vertex of an gyrotriangle ABC such that l meets BC in D, CA in E, and AB in F, then*

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1.$$

where $v_\gamma = \frac{v}{1 - \frac{v^2}{s^2}}$ [6].

2. Main results

In this section, we prove Menelaus's theorem for hyperbolic quadrilateral.

Theorem 3. (***The Menelaus's Theorem for Gyroquadrilateral***). *If l is a gyroline not through any vertex of a gyroquadrilateral $ABCD$ such that l meets AB in X, BC in Y, CD in Z, and DA in W, then*

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BY)_\gamma}{(CY)_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1. \tag{1}$$

Proof. Let T be the intersection point of the gyroline DB and the gyroline XYZ (See Figure 1). If we use a theorem 2 in the gyrotriangles ABD and BCD respectively, then

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BT)_\gamma}{(DT)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1 \tag{2}$$

and

$$\frac{(DT)_\gamma}{(BT)_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(BY)_\gamma}{(CY)_\gamma} = 1. \tag{3}$$

Multiplying relations (2) and (3) member with member, we obtain

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BY)_\gamma}{(CY)_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1.$$

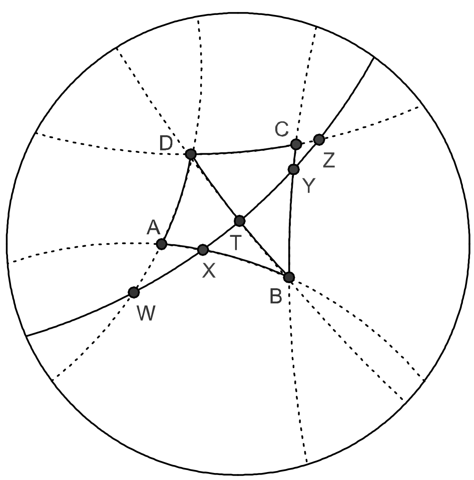

Figure 1

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Naturally, one may wonder whether the converse of Menelaus theorem for hyperbolic quadrilateral exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 4. (***Converse of Menelaus's Theorem for Gyroquadrilateral***). *Let $ABCD$ be a gyroquadrilateral. Let the points X, Y, Z, and W be located on the gyrolines AB, BC, CD, and DA respectively. If three of four gyropoints X, Y, Z, W are collinear and*

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BY)_\gamma}{(CY)_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1,$$

then all four gyropoints are collinear.

Proof. Let the points W, X, Z are collinear, and gyroline WXZ cuts gyroline BC, at Y' say. Using the already proven equality (1), we obtain

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BY')_\gamma}{(CY')_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1,$$

then we get

$$\frac{(BY)_\gamma}{(CY)_\gamma} = \frac{(BY')_\gamma}{(CY')_\gamma}. \tag{4}$$

This equation holds for $Y = Y'$. Indeed, if we take $x := |\ominus B \oplus Y'|$ and $b := |\ominus B \oplus C|$, then we get $b \ominus x = |\ominus Y' \oplus C|$. For $x \in (-1, 1)$ define

$$f(x) = \frac{x}{1 - x^2} : \frac{b \ominus x}{1 - (b \ominus x)^2}. \tag{5}$$

Because $b \ominus x = \frac{b-x}{1-bx}$, then $f(x) = \frac{x(1-b^2)}{(b-x)(1-bx)}$. Since the following equality holds

$$f(x) - f(y) = \frac{b(1 - b^2)(1 - xy)}{(b - x)(1 - bx)(b - y)(1 - by)}(x - y), \tag{6}$$

we get $f(x)$ is an injective function and this implies $Y = Y'$, so W, X, Z, and Y are collinear. ■

We have thus obtained in (1) the following

Theorem 5. (***Transversal theorem for gyrotriangles***). *Let D be on gyroside BC, and l is a gyroline not through any vertex of a gyrotriangle ABC such that l meets AB in M, AC in N, and AD in P, then*

$$\frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CA)_\gamma}{(NA)_\gamma} \cdot \frac{(NP)_\gamma}{(MP)_\gamma} \cdot \frac{(MA)_\gamma}{(BA)_\gamma} = 1. \tag{7}$$

Proof. If we use a theorem 2 for gyroquadrilateral $BCNM$ and collinear gyropoints D, A, P, and A (See Figure 2), we obtain the conclusion.

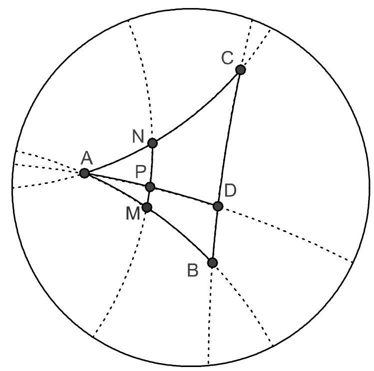

Figure 2

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The Einstein relativistic velocity model is another model of hyperbolic geometry. Many of the theorems of Euclidean geometry are relatively similar form in the Poincaré model, Menelaus's theorem for hyperbolic gyroquadrilateral and the transversal theorem for gyrotriangle are an examples in this respect. In the Euclidean limit of large s, $s \to \infty$, gamma factor v_γ reduces to v, so that the gyroinequalities (1) and (7) reduces to the

$$\frac{AX}{BX} \cdot \frac{BY}{CY} \cdot \frac{CZ}{DZ} \cdot \frac{DW}{AW} = 1$$

and

$$\frac{BD}{CD} \cdot \frac{CA}{NA} \cdot \frac{NP}{MP} \cdot \frac{MA}{BA} = 1,$$

in Euclidean geometry. We observe that the previous equalities are identical with the equalities of theorems of euclidian geometry.

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