A POSITION INDICATOR WITH APPLICATIONS IN THE FIELD OF DESIGNING FORMS WITH ARTIFICIAL INTELLIGENCE

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The indicators used so far within the Theory of Extension, see papers [1], [2] and [4], can be synthetically expressed by the notion of "position indicators". More exactly, these indicators can be grouped in two main sub-categories: point-set position indicators and point-two sets position indicators. The secondary goal of this paper is to define these classifications, while the primary goal is that to extend the two notions to the most general notion of set-set position indicator.

Key words: Hausdorff measure, Extension theory, position indicators, computer vision, artificial intelligence.

1. Introduction

The first part of this paper aims at defining the notion of point-set position indicator and that of point-two sets position indicator, at discussing the main examples of such indicators and their relevance for the applicative field.

Point-set Position Indicators

For any point $x \in \mathbb{R}^n$ and any set $A \subset \mathbb{R}^n$, formula $\delta(x, A) = \inf \{d(x, a) | a \in A\}$ defines (in classical mathematics) the distance from point x to set A. Based on the properties characteristic for the notion of distance, every time we have $\delta(x, A) > 0$ we can conclude that point x lies outside set $\overline{A}$ (closure of set A in relation to the usually topology of the space $\mathbb{R}^n$) at distance $\delta(x, A)$ from the nearest point of set A. On this account expression $\delta(x, A)$ somehow takes the role of position indicator of x towards A, but not entirely, because in case $\delta(x, A) = 0$ the sole information provided is that $x \in A$, with no further indication

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of how far or how close to frontier ∂A of set A this point lies. In many concrete situations the knowledge of this details can be more useful. For example, when A symbolizes the 2-dimensional representation of a risk zone for human (a very deep lake, a region contaminated with toxic substances etc.) and x symbolizes the position vector of a person situated in the interior of that zone, it would be useful for that person to know how close or far the exit ways are. It thus, becomes necessary the extension of the classical notion of distance by taking into account some indicators that deal with more requirements. This aspect has been pointed out by other researchers in their papers, as well, see, for example [4]. The most general point-set position indicator aiming at fulfilling the requirements formulated above is given by [2] and takes the form

$$\mathsf{s}(x,A)=\begin{cases}\delta(x,A), & x\in \eth A\\ -\delta(x,\eth A), & x\in A\end{cases}, \tag{1}$$

where $\eth A$ represents the absolute complement of A, i.e. $\eth A=\mathbb{R}^n\setminus A$. Given the way it has been defined, indicator $\mathsf{s}(x,A)$ has the following properties: $x\in\eth\overline{A}$ $\Leftrightarrow$ $\mathsf{s}(x,\ A)>0$; $x\in\overset{o}{A}$[4] $\Leftrightarrow$ $\mathsf{s}(x,\ A)<0$; $x\in\partial A$[5] $\Leftrightarrow$ $\mathsf{s}(x,\ A)=0$, respectively.

Observations: 1) The indicator defined in expression (1) does not represent a distance; it expresses the distance from x to A only when point x is exterior to set A.
2) Other examples of point-set position indicators can be found in [1] and [4].

Point-two Sets Position Indicators

Paper [2], mentioned above, provides us with an example of a point-two sets indicator, as well. This takes the expression

$$\mathsf{S}\,(x,A,B)=\frac{\mathsf{s}(x,A)}{\mathsf{s}(x,B)-\mathsf{s}(x,A)}, \tag{2}$$

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[4] $\overset{o}{A}$ represent the interior of the set A in the usually topology of the space $\mathbb{R}^n$.

[5] ∂A represent the boundaries of sets A, namely $\partial A=\overline{A}\setminus\overset{o}{A}$.

where $x \in \mathbb{R}^n$, A and B are two sets from $\mathbb{R}^n$ with the property $\overline{A} \subset \overset{o}{B}$, and s is indicator (1). This indicator has the following properties: $\mathsf{S}(x, A, B) < -1 \Leftrightarrow x \in \eth\overline{B}$; $-1 \leq \mathsf{S}(x, A, B) < 0 \Leftrightarrow x \in \overline{B} \setminus \overline{A}$; $\mathsf{S}(x, A, B) \geq 0 \Leftrightarrow x \in \overline{A}$.

Observations: 1) The properties presented earlier justify the designation of "point – two sets position indicator" given to indicator (2).
2) Other examples of point – two sets position indicators can be found in [1, 4].

2. Set-set Position Indicators

This paragraph focuses on the presentation of new results aiming at further developing and improving the existent theory of Extension. The frame we shall refer to in our discussion is that of any metric space (X, d). The mathematical apparatus we would like to advance further on requires to take into consideration the notion of "Hausdorff measure". To ensure a better understanding of the concepts presented, we have synthetized the minimum of knowledge required in this regard in the appendix.

Let A and B be two non-empty sets from X. About set A we additionally assume that it admits a Hausdorff measure of dimension $r \geq 0$, $\mathsf{H}^r(A)$ finite and nonzero. Under these conditions, by using indicator s defined by the generalized relation (1) from the Euclidean metric space $\mathbb{R}^n$ for the actual metric space (X, d), we are able to consider the expression

$$\mathsf{S}(A, B) = \frac{\mathsf{H}^r\left(\left\{a \in A \mid \mathsf{s}(a, B) \leq 0\right\}\right)}{\mathsf{H}^r(A)}, \tag{3}$$

which accurately defines the indicator we wanted to introduce.

This indicator fulfills several mathematical properties important to the applicative field:

Proposition 1: $\mathsf{S}(A, B) = 0 \Leftrightarrow A \cap B = \varnothing$ H^r- almost everywhere (or differently expressed, $\mathsf{H}^r(A \cap B) = 0$).

Demonstration: $\mathsf{S}(A, B) = 0 \Leftrightarrow \mathsf{H}^r\left(\left\{a \in A \mid \mathsf{s}(a, B) \leq 0\right\}\right) = 0$. Since $\mathsf{s}(a, B) \leq 0$ is equivalent to relation $a \in \overline{B}$, we deduce that $\mathsf{H}^r(A \cap B) = 0$.

Proposition 2: $\mathsf{S}(A,\ B) > 0 \Rightarrow A \cap \overline{B} \neq \varnothing$.

Demonstration: $\mathsf{S}(A,\ B) > 0 \Leftrightarrow\ \mathsf{H}^r\left(\{a \in A \mid \mathsf{s}(a,\ B) \le 0\}\right) > 0$. From relation $\mathsf{H}^r\left(\{a \in A \mid \mathsf{s}(a,\ B) \le 0\}\right) > 0$ we deduce that there is $a \in A$ so that $\mathsf{s}(a,\ B) \le 0$. But $\mathsf{s}(a,\ B) \le 0$ implies that $a \in \overline{B}$, so $A \cap \overline{B} \neq \varnothing$.

Proposition 3: If besides the initial hypothesis made over set *A* and *B*, we assume additionally that set $\overline{B}$ is measurable[6] with respect to Hausdorff measure H^r (regarded as an outer measure on $\mathsf{P}(X)$, the family of all subsets of *X*), then the relation $\mathsf{S}(A,\ B) = 1$ is equivalently with $A \subseteq \overline{B}$ H^r - almost everywhere.

Demonstration: $\mathsf{S}(A,\ B) = 1 \Leftrightarrow \mathsf{H}^r\left(\{a \in A \mid \mathsf{s}(a,\ B) \le 0\}\right) = \mathsf{H}^r(A)$. The way how indicator s has been defined implies that $\{a \in A \mid \mathsf{s}(a,\ B) \le 0\} = A \cap \overline{B}$. Because the set $\overline{B}$ is measurable we have $\mathsf{H}^r(A)$ $= \mathsf{H}^r\left(A \cap \overline{B}\right) + \mathsf{H}^r\left(A \cap \eth\overline{B}\right)$. Bout $\mathsf{H}^r\left(A \cap \overline{B}\right) = \mathsf{H}^r\left(\{a \in A \mid \mathsf{s}(a,\ B) \le 0\}\right)$ $= \mathsf{H}^r(A) \Rightarrow \mathsf{H}^r\left(A \cap \eth\overline{B}\right) = 0$, namely $A \subseteq \overline{B}$ H^r - almost everywhere.

Corollary: Let A and B be two closed nonempty sets from X for which there exists a Hausdorff measure of dimension $r \ge 0$ such that $\mathsf{H}^r(A)$ and $\mathsf{H}^r(B)$ are finite and nonzero. If the sets A and B are H^r- measurable and if $\mathsf{S}(A,B) = 1$ and $\mathsf{S}(B,A) = 1$, then $A = B$ H^r- almost everywhere, and reciprocally.

Demonstration: From proposition 3 it results that $\mathsf{S}(A,\ B) = 1 \Leftrightarrow A \subseteq B$ H^r- almost everywhere and $\mathsf{S}(B,\ A) = 1 \Leftrightarrow B \subseteq A$ H^r- almost everywhere. Then $A = B$ H^r- almost everywhere.

Observations: 1) From the definition of indicator S, (relation (3)) it can be easily deduced that $0 \le \mathsf{S}(A,B) \le 1$, for any pair of non-empty subsets A and B of

[6] By definition we say that set $\overline{B}$ is H^r measurable if for any $T \subseteq X$ the relation $\mathsf{H}^r(T) =$ $\mathsf{H}^r\left(T \cap \overline{B}\right) + \mathsf{H}^r\left(T \cap \eth\overline{B}\right)$ takes place.

space X for which set A admits a Hausdorff measure H^r of dimension $r \geq 0$, so that $\mathsf{H}^r(A) \neq 0$ and $\mathsf{H}^r(A) < \infty$.

2) The properties presented earlier within propositions 1 - 3 aim at justifying the designation of "set-set position indicator" that indicator (3) receives.

3) Another example of set-set position indicator can be found in [3].

3. Applications

Indicator S defined by us in this paper can be used as an example with computer vision while developing software applications regarding the automatic inclusion of a certain object O into a target region R of a given video image (*VIm*). To realize this, we propose an algorithm which in broad terms has the following content: by means of a set of isometries I_i, $i \in I$ of the plan, we move object O to different regions and positions of image *VIm* by calculating the value of indicator $\mathsf{S}(\mathsf{I}_i(\mathsf{O}), \mathsf{R})$, each time. Finding that index $i_0 \in I$ for which $\mathsf{S}(\mathsf{I}_{i_0}(\mathsf{O}), \mathsf{R}) = 1$, is equivalent to finding the solution to the problem.

Observations: 1) In some cases solution I_{i_0} found by using the method presented above can not be fully satisfactory because relation $\mathsf{I}_{i_0}(\mathsf{O}) \subseteq \mathsf{R}$, concerned by $\mathsf{S}(\mathsf{I}_{i_0}(\mathsf{O}), \mathsf{R}) = 1$, (see proposition 3, aplicata pentru cazul in care multimile prin care sunt abstractizate obiectele O si R sunt presupuse a fi compacte[7]) is only guaranteed H^r- almost everywhere.

2) Just as the algorithm presented in [3], this algorithm can be easily adapted to solving any similar problem in a space with three dimensions, becoming, thus, more useful to the field of designing forms with artificial intelligence.

4. Appendix

Hausdorff measure

Let (X, d) be a metric space, Y a subset from X, and δ a strictly positive real number. A finite or countable collection of sets $\{U_1, U_2, ...\}$ from X with diameter $D(U_1) \leq \delta$, $D(U_2) \leq \delta$,..., for which $Y \subset U_1 \cup U_2 \cup \cdots$, is called δ-cover of set Y. By virtue of this notion, for any subset Y from X and for any two real numbers $r \geq 0$, and $\delta > 0$, we can define the indicator

[7] In cazul aplicatiei pe care o analizam aceste ipoteze sunt cat se poate de naturale.

$$\mathcal{H}_\delta^r(Y) = \inf_{\{U_1, U_2, \ldots\} \in \mathcal{C}_\delta(Y)} \left\{ D^r(U_1) + D^r(U_2) + \cdots \right\},$$

where $\mathcal{C}_\delta(Y) = \{\{U_1, U_2, \ldots\} \mid \{U_1, U_2, \ldots\} \text{ is a } \delta-\text{cover of } Y\}$. This indicator defines a decreasing function $\delta \to \mathcal{H}_\delta^r(Y)$. This property guaranties the existence of the limit

$$\mathcal{H}^r(Y) = \lim_{\delta \to 0} \mathcal{H}_\delta^r(Y),$$

which, by definition, is called the *r*-dimensional Hausdorff measure of Y.

Among the properties of the Hausdorff measure, $\mathcal{H}^r(\cdot)$, we mention:

1) $\mathcal{H}^r(\varnothing) = 0$;

2) $\mathcal{H}^r(Y_1) \leq \mathcal{H}^r(Y_2)$, if $Y_1 \subseteq Y_2$, $Y_1, Y_2 \in \mathcal{P}(X)$;

3) $\mathcal{H}^r\left(\bigcup_{n=1}^{\infty} Y_n\right) \leq \sum_{n=1}^{\infty} \mathcal{H}^r(Y_n)$, if $\{Y_n \mid n \in \mathbb{N}^*\} \subset \mathcal{P}(X)$, is any countable collection of sets.

B I B L I O G R P H Y

[1] *F. Smarandache*, "Generalizations of the Distance and Dependent Function in Extenics to 2D, 3D, and n-D", viXra.org, http://vixra.org/pdf/1206.0014v1.pdf, 2012, pp. 1-17.

[2] *O. I. Şandru*, *L. Vlădăreanu*, *P. Şchiopu*, *V. Vlădăreanu*, *A. Şandru,* "Multidimensional Extenics Theory", U.P.B. Sci. Bull., Series A, Vol. 75, Iss. 1, 2013, pp. 3-12.

[3] *O. I. Şandru*, *F. Smarandache*, "An indicator of inclusion with applications in computer vision", viXra.org, http://vixra.org/abs/1304.0133, 2013, pp. 1-3.

[4] *C. Wen*, "Extension Set and Non-Compatible Problems", Advances in Applied Mathematics and Mechanics in China, Peking: International Academic Publishers, 1990, pp. 1-21.