On Rotating Frames and the Relativistic Contraction of the Radius (The Rotating Disc)

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Abstract

The relativistic problem of the rotating disc or rotating frame is studied. The solution given implies the contraction of the radius and the change of the value of depending on the type of observer. Two forms of rotation are considered. One is with constant angular velocity, independent of the radius, implying a horizon, the other is with exponentially decreasing angular velocity with respect to the radius and does not imply a horizon. In all cases the paths of signals emanating from the origin of the rotating frame advance helically in the positive and negative z direction, where they are concentrated, due to the contraction of the radius, and in some cases appear as jets.

1. Introduction

The rotating disc has been the subject of a multitude of papers since Ehrenfest [1] published what is today known as the "Ehrenfest Paradox". He noted that since the perimeter of a rotating disc would relativistically contract, while the radius remained the same, a deformation of the disc should take place. Many authors have since then contributed to the understanding of the problem by studying the geometry of the rotating disc. A historical review can be found in Rizzi and Ruggiero [2] and in Grøn [3]. The approach in the present study is closer to the idea that there is a contraction of the radius of the rotating disc. Similar ideas have been explored by Ashworth, Davies and Jennison [4], [5] and by Grünbaum and Janis [6], [7]. In particular, Ashworth, Davies and Jennison show that the radius of the rotating disc contracts according to an observer that is rotating with the disc at a distance $r > 0$ from the center. Grünbaum and Janis argue that an observer that is not rotating with the disc will see the radius of the disc contract by the same relativistic factor as the perimeter. The latter approach is closer to ours although we do not find the same results because we allow the value of to change for the non-rotating observer, when he makes measurements regarding the rotating frame.

The paper is organized as follows: In section 2 we state our basic assumptions that will help us derive the transformations in the following sections. In sections 3 we present our notation and known relativistic results regarding the rotating frame. In section 4 we formulate and solve the problem of the contraction of the radius of the disc and the transformation of the value of for the non rotating observer. In section 5 we present a summary of the results up to that point. In section 6 we consider the space or disc deformation as seen by the non-rotating observer and distinguish between two kinds of non- rotating observers: one within the horizon of radius *c/w* and one outside. In section 7 we discuss the results. In section 8 we generalize to rotating frames in 3 dimensions and

plot the signals emanating from the origin of the rotating frame in the radial direction, as seen by the non-rotating observers. The signals gradually bend sideways until they reach a 90º degree angle with respect to the radius, while they advance in the positive and negative z axis direction. In section 9 we examine rotation with slippage so that the angular velocity is assumed to decrease exponentially as the radius increases and as z increases. We examine both the close by non-rotating observer and the far away non-rotating observer. The signals are in this case not limited to a horizon but gradually bend sideways until they reach a maximum deflection from the radial direction and then turn asymptotically back to the radial as the radius increases, while they advance in the positive and negative z direction. In section 10 we present our conclusions.

2. Assumptions on the rotation of two concentric discs

In the following we will make three assumptions. Assumptions 2 and 3 are standard in relativity theory. Only Assumption 1 is new.

1. Discussion to justify Assumption 1

Suppose an observer O_1 is at any place on disc (disc 1) and a second observer O_2 that sits at any place of another disc (disc 2) parallel and coaxial to disc 1. $O₁$ sticks a pencil through his disc parallel to the axis of rotation with its point touching the other disc (disc 2). O_2 does the same thing with the tip of his pencil touching disc 1. As there is relative rotation of one disc with respect to the other, each observer will watch the perimeter of a circle being drawn on his own and the other disc. Each will see that the duration of time, according to his own clock, for a complete circle to be drawn on his own disc will be the same with the duration it takes for him to draw a complete circle on the other disc. This observation holds for half a circle or any fraction of a circle. In short, we say that the two observers will agree on epicenter angles measured as fractions of a circle (not radians). If we denote Θ the magnitude of an angle as fraction of a circle and the magnitude of the reach a maximum deletoion from the radial direction and then turn asymptoically back to the radial as the radials increases, while they advance in the positive and negative z direction. In section 10 we present our conclus we are not sure the observers agree on f .

Imagine now that the same two observers stand at the center O of their disc (one on top of the other) and let the discs rotate with respect to each other. If the first observer announces that according to his measurements the second disc is rotating with frequency ϵ , since the situation is exactly symmetrical we will expect the second observer to make the same announcement regarding the first disc. But frequency is defined as revolutions per unit time or per unit time. Once they agree on and ϵ , they have to agree on time rate. In fact, they may even use the same clock, since they are collocated. We may summarize in the following,

Assumption 1

(a)Two observers sitting at two parallel concentric discs rotating with respect to each other around an axis vertical at their center, will agree that the magnitudes of epicenter angles traveled by the other disc, measured as fractions of a circle, are equal.

(b) If they also stand at the common center of their discs, they will further agree that time rates are equal.

Remark: The situation with the two observers at the center of their discs (one on top of the other) is symmetrical and there is no point in arguing who is rotating and who is

stationary. However, if they had a way to measure the centrifugal acceleration off their center, they would probably find that it is different. In that respect we may rightfully call the frame with zero centrifugal force the "preferred frame". In what follows we will assume that one observer, usually to be denoted as O' (or O_2), sits at the center of a

preferred frame K' (or K_2), which coincides with the laboratory frame unless otherwise specified.

2. Discussion to justify Assumption 2

Many experiments have verified the constancy of the speed of light, which is the basic assumption for special relativity theory. The speed of light is not the same for observers under acceleration or, according to the Principle of Equivalence, gravity.

Assumption 2

Light speed does not depend on the speed of the emitting source and its speed is constant for all observers that are not under the influence of acceleration.

3. *Discussion to justify Assumption 3*

An observer on a frame will agree with another observer on the same frame on the measurements of lengths. This is expected if measurements are made not by signals but by actually placing the measuring stick on the length to be measured and counting how many times it fits to it. We state then, 3. Docessimot to Justay Amangnon 3
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Assumption 3

Observers on the same frame will agree on measurements of length.

3. Time Rate and the Contraction of the Perimeter on a Rotating Disc

As we talk interchangeably about rotating discs and frames, we need to clarify that when we talk of a disc, we imagine it placed at the x-y pane of the respective frame with center at the origin.

Let two frames K_1 and K_2 with cylindrical coordinates, common origin O and common axis of rotation Z. Let observer O_1 sit at the origin of K_1 and O_2 at the origin of K_2 . Let

 K_2 be a non rotating frame - laboratory frame- and let O_2 observe frame K_1 rotate. Let a

third observer O_3 on K_1 sit at a distance from the center. When there is no danger of

confusion we will rename the three observers using $O = O_1$, $O' = O_2$, $\tilde{O} = O_3$ and the

shortly afterwards for economy:

A quantity Q_{ij} is defined as the quantity measured by observer *i* given it is stationary in the frame of observer *j*

For example, ΔL_{21} is the length measured by observer 2 for a line segment on the perimeter of the disc that is stationary in the frame of observer 1. Also, Δt_{21} is the time interval that observer 2 sees that a clock stationary and with observer 1 shows for the duration of an event.

Note that by Assumption 3, ΔL_i is a constant for all *i* and will be denoted as ΔL_{start} since all observers agree on the same segment when stationary in their frame. The same is not true for Δt_{ii} . In particular, we expect $\Delta t_{11} = \Delta t_{22} = \Delta t_{stat} \neq \Delta t_{33}$, and $\Delta t_{12} = \Delta t_{21} = \Delta t_{stat}$ Note that by Assumption 3, ΔL_{ii} is a constant for all *i* and will be denoted as ΔL_{sat} since
all observers agree on the same segment when stationary in their frame. The same is not
true for Δt_{ii} . In particular, because observers 1 and 2 have the same clock , while observer 3 is under the influence of centrifugal acceleration, which we suspect that will affect Δt_{33} in some unknown as yet way. Therefore, the clock of observer 1 and 2 represents stationary clock time intervals Note that by Assumption 3, ΔI_u is a constant for all *i* and will be denoted as ΔI_{star} since
all observers agree on the same segment when stationary in their frame. The same is not
true for ΔI_u . In particular, frame and time intervals measured by observer 1 looking at the clock of observer 3, agree with what observer 3 sees looking at his own clock. Distances in the radial direction are measured as r_{ii} , while the angular velocity of a disc is measured by w_{ii} . Specifically, Note that by Assumption 3, ΔL_{ii} is a constant for all *i* and will be denoted as ΔL_{sas} since
all observers agree on the same segment when stationary in their frame. The same is not
true for Δt_{ii} . In particular Note that by Assumption 3, ΔL_a is a constant for all *i* and will be denoted as ΔL_{aa} , since
all observers agree on the same segment when stationary in their frame. The same is not
true for M_a , In particular, we e Note that by Assumption 3, ΔL_x is a constant for all *i* and will be denoted as ΔL_{xx} since
all observers agree on the same segment when stationary in their frame. The same is not
true for ΔT_x . In particular, we on the same segment when stationary in their frame. The same is not
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observer 2, using his own clock does not depend on the second subscript since both observers 1 and 3 are stationary on the rotating disc. 2 and the more as seen be the non rotating observer 2 regardless of the

2 both observers 1 and 3 are stationary on the rotating disc. Similarly,

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Most of the authors (see for example Møller [8] pp.222-250 on rotating disc) start from the transformation between a rotating frame K_1 and a non rotating frame K_2 and the The metric for the rotating frame is second subscript since both observers 1 and 3 are stationary on the rotating disc. Similarly, $w_{21} = w_{23}$ because the angular velocity of the rotating disc as seen by the non rotating observer 2, using his own clock doe by 222-250 on rotating distribution on rotating frame K_2 a
 $r_1 = r_2$, $z_1 = z_2$, $r_1 = r_2 -$
 $d_{n_1} dt_1 - (c^2 - w^2 r_1^2) dt_1^2$

entials are null and equat
 $d^2 dt_2^2 = (c^2 - w^2 r_1^2) dt_1^2$ from *w* of the rotating disc as seen by the rotating disc as seen by the rotating disc as seen by the rotating disc.
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dirical coordinates

$$
ds^{2} = dr_{1}^{2} + r_{1}^{2} d_{n_{1}}^{2} + dz_{1}^{2} + 2wr_{1}^{2} d_{n_{1}} dt_{1} - (c^{2} - w^{2} r_{1}^{2}) dt_{1}^{2}
$$
 (1)

For a non-moving point in space, the space differentials are null and equating the metrics that the clock of the rotating system runs slower,

$$
dt_1 = \frac{dt_2}{\sqrt{1 - \frac{w^2 r_1^2}{c^2}}}
$$
 (2)

The line element $d\uparrow^2$ is given by $d\uparrow^2 = x_1 dx^2 dx^1$ where x^2 takes the values $\{r_1, r_1, z\}$ and $X_{\text{z}1} = 0$ except for $X_{\text{r}r} = 1$, $X_{\text{r}r} = \frac{r_1^2}{r_1^2 r_2^2}$, $X_{\text{z}r} = 1$. Hence, we find $X_{1} = \frac{r_1^2}{r_1^2}$, $X_{1} = 1$. Hence, we find,

The metric for the rotating frame is
\n
$$
ds^2 = dr_1^2 + r_1^2 d_{\pi_1}^2 + 2k r_1^2 d_{\pi_1} dt_1 - (c^2 - w^2 r_1^2) dt_1^2
$$
\n(1)
\nFor a non-moving point in space, the space differentials are null and equating the metrics
\nof the rotating and non-rotating frames we find $c^2 dt_2^2 = (c^2 - w^2 r_1^2) dt_1^2$ from which we find
\nthat the clock of the rotating system runs slower,
\n
$$
dt_1 = \frac{dt_2}{\sqrt{1 - \frac{w^2 r_1^2}{c^2}}}
$$
\n(2)
\nThe line element dt^2 is given by $dt^2 = x_{r1} dx^2 dx^T$ where x^2 takes the values $\{r_1, r_1, z\}$ and
\n $x_{r1} = 0$ except for $x_{rr} = 1$, $x_{rr} = \frac{r_1^2}{1 - \frac{w^2 r_1^2}{c^2}}$, $x_{zz} = 1$. Hence, we find,
\n
$$
d\tau^2 = dr_1^2 + \frac{r_1^2}{1 - \frac{w^2 r_1^2}{c^2}} d_{\pi_1}^2 + dz^2
$$
\n(3)
\nFor a line segment dL_t along the perimeter this formula implies that
\n
$$
dL_2 = \frac{dL_1}{\sqrt{1 - \frac{w^2 r_1^2}{c^2}}}
$$
\n(4)
\nWhere ($dL_1 = r_1 d_{\pi_1}$ and $dL_2 = r_2 d_{\pi_2}$).
\nWhereas the result (2) is within our expectations from the special theory of relativity, the
\nresult (4) is contrary to the expected result. Since the normal Lorentz contraction would
\ngive a contraction of the perimeter instead of a lengthening as we have found above. The

For a line segment dL along the perimeter this formula implies that

$$
dL_2 = \frac{dL_1}{\sqrt{1 - \frac{w^2 r_1^2}{c^2}}}
$$
(4)

Whereas the result (2) is within our expectations from the special theory of relativity, the result (4) is contrary to the expected result. Since the normal Lorentz contraction would give a contraction of the perimeter instead of a lengthening as we have found above. The result (4) is counterintuitive for another reason also: If we increase the radius, while decreasing the angular velocity so that that tangential velocity is constant ($wr_1 = \hat{ }$) then the perimeter tends to a straight line and the transformation should approach the Lorentz length contraction. Instead, according to (4) since $wr_1 = \hat{ }$ is constant the lengthening factor remains constant and there is no way of approaching the Lorentz contraction. As the method with which the result is obtained is correct, the only suspect is the form of transformation assumed. We are motivated, therefore, to search for another transformation that will also satisfy the contraction of the perimeter according to the Lorentz length contraction of special relativity. lg to (4) since $wr_1 = \hat{ }$ is constant the
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that that tangential velocity is constant ($w_{\overline{I}} = \hat{ }$) the
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that that tangential velocity is constant $(wr_1 = \hat{ })$ then

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ding to (4) since $wr_1 = \hat{ }$ is constant the lengt *station* is obtained is correct, the only suppose is to toway of approaching the Lotentz contract sust is obtained is correct, the only suspect is the motivated, therefore, to search for another tration of the perimeter *w* is correct, the only suspect is the *r* force, to search for another transform deter according to the Lorentz lengthat satisfies in our notation the fol-
that satisfies in our notation the fol-
tr to observer O_2 :
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e motivated, therefore, to search for another tr

Our quest is, therefore, to find a transformation that satisfies in our notation the following,

(a) The rate of the clock of O_3 will appear slower to observer O_2 :

transformation that satisfies in our notation the following,
\nll appear slower to observer
$$
O_2
$$
:
\n
$$
\Delta t_{23} = \frac{1}{\sqrt{1 - \frac{w_{21}^2 r_{21}^2}{c^2}}} \Delta t_{\text{stat}}
$$
\n(5)
\n₃ as we mentioned above)
\nneter are contracted,
\n
$$
\Delta L_{21} = \Delta L_{\text{stat}} \sqrt{1 - \frac{w_{21}^2 r_{21}^2}{c^2}}
$$
\n(6)
\nrate of the clock of O_3 will also appear slower to O_1 , who
\ny,

(b) Line segments along the perimeter are contracted,

$$
\Delta L_{21} = \Delta L_{\text{stat}} \sqrt{1 - \frac{w_{21}^2 r_{21}^2}{c^2}} \tag{6}
$$

Observe that (5) implies that the rate of the clock of O_3 will also appear slower to O_1 , who has the same clock as O_2 . Namely,

$$
\Delta t_{23} = \frac{1}{\sqrt{1 - \frac{w_{21}^2 r_{21}^2}{c^2}}} \Delta t_{\text{stat}}
$$
(5)
\n= r_{23} as we mentioned above)
\nrrimeter are contracted,
\n
$$
\Delta L_{21} = \Delta L_{\text{stat}} \sqrt{1 - \frac{w_{21}^2 r_{21}^2}{c^2}}
$$
(6)
\nne rate of the clock of O_3 will also appear slower to O_1 , who
\nnelly,
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$$
\Delta t_{13} = \Delta t_{33} = \frac{1}{\sqrt{1 - \frac{w_{21}^2 r_{21}^2}{c^2}}} \Delta t_{\text{stat}}
$$
(7)
\nnotion between O_1 and O_3 , O_1 will think that is due to the
\n3 feels. Requirements (a), (b) along with the assumptions 1,

But since there is no relative motion between O_1 and O_3 , O_1 will think that is due to the centrifugal acceleration that O_3 feels. Requirements (a), (b) along with the assumptions 1, 2, 3 imply a transformation (contraction) on the radial distance as we will see below.

4. The Contraction of the Radius of the Rotating Disc

Refer again to observers O' (or O_2) on K' (or K_2), O (or O_1) and \tilde{O} (or O_3) on K (or K_1), with *K* rotating with respect to *K'* with frequency *v'* according to *O'* (and ϵ according to *O*). Suppose observer *O* has a rod that extends radially from the center O to some point A (see Figure 1). The rod is hollow mirrored inside and infinitesimally thin so that light traveling through it follows a straight line. The rod is just an artifact to help imagine things, a statement saying that *O*sends a light signal radially outward is enough. Observe that (5) implies that the rate of the clock of O_3 will also appear slower to O_1 , who
has the same clock as O_2 . Namely,
 $\Delta t_{13} = \frac{1}{\sqrt{1 - \frac{W_{22}^2 V_{13}^2}{v_{2}^2}}}$ (7)
But since there is no relative moti Observe that (5) implies that the of the clock of O_3 will also appear slower to O_4 , who has the same clock as O_2 . Namely,
 $\Delta t_{13} = \Delta t_{33} = \frac{1}{\sqrt{1 - \frac{W_2^2 P_3^2}{1}}}$, $\Delta t_{\alpha\alpha}$ (7). Thus the same clock as $O_$ agree on angular velocity measured in radians per unit time, because they will in general disagree on f and, therefore, we may say that for observer O the angular velocity is, *w* 2 *w 2 <i>w* 2 *w* $\Delta t_{13} = 2V_1 \times \text{max}_{13} - \frac{1}{\sqrt{1 - \frac{W_2^2 V_2^2}{r_1}}} \Delta t_{\text{net}}$ (7)
But since there is no relative motion between O_i and O_i , O_i will think that is due to the centrifugal acceleration th Observer *O* (who is not under acceleration because he sits at the origin although he is rotating with the disc) sends a light signal from O towards A through the rod. According to him the signal travels with velocity $\hat{c} = c$ (Assumption 2) the distance OA = *r* (see Figure 1). Until the signal reaches the end of the rod, the rod will have moved to position OB. Observer O' will see the signal travel a curved path (OCB^{\cdot}) with constant tangential velocity $\hat{i} = c$ (Assumption 2). At the perimeter the direction of the velocity of the signal, according to O' , will make an angle $\{\nabla$ with respect to the radius OB' which will have Observer *O* (who is not under acceleration because he sits at the origin although he is
rotating with the disc) sends a light signal from O towards A through the rod. According to
him the signal travels with velocity \hat

Figure 1 The path of the light signal originating at O is OCB' according to observer O' (the lab observer)

Let

 r : the radius as observer *O* measures it (stationary length) (OA=OB)

 r' : the radius according to observer O' (OB)

 \hat{r} : the radial component of the velocity \hat{r} according to observer *O'*

- \hat{C} : the component of \hat{C}' perpendicular to the radius according to observer *O'*
- $\{\n\t:$ the angle between \hat{r} and \hat{r}' (to be called *angle of deflection* from the radial direction)

We may write the following relationships for light signals letting $\hat{i} = c$ for : For a light signal \hat{c} *c* and then

$$
\hat{c}_r' = c \cos\left(1\tag{8}\right)
$$

$$
t' = c \sin \{ \tag{9}
$$

$$
\hat{U} = w'r'
$$
 (10)

where *w*' is the angular velocity in radians as observed by *O'* and therefore,
 $w' = 2fE$

$$
W = 2J \ \epsilon \tag{11}
$$

From (9) and (10)

intating at O is OCB' according to observer O' (the lab

\nis it (stationary length) (OA=OB)

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$$
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 according to observer O'

\nin a total value of *deflection* from the radial direction

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\nin the total value of *deflection* from the total direction

\nin the total value of *deflection* from the total direction

\nin the total value of *deflection* from the radius of *deflection* from the

And substituting in (8)

$$
\hat{r}'_r = c \sqrt{1 - \frac{w'^2 r'^2}{c^2}}
$$
\n(13)

\n(14)

\n(15)

\n(16)

\n(17)

\n(18)

\n(19)

\n(19)

\n(11)

\n(14)

In order to satisfy the condition for the contraction of the perimeter (see (6)) we further require that

$$
\int_{r}^{r} = c \sqrt{1 - \frac{w'^2 r'^2}{c^2}}
$$
\n
\nor the contraction of the perimeter (see (6)) we further\n
\n
$$
2f'r' = 2f r \sqrt{1 - \frac{w'^2 r'^2}{c^2}}
$$
\n(14)\n
\nto obtain\n
$$
\frac{w'}{2f\epsilon} = \frac{r}{r'} \sqrt{1 - \frac{w'^2 r'^2}{c^2}}
$$
\n(15)\n
\nbstitute in (15) and solve for w' to obtain\n
$$
w'^2 = \frac{w^2 r^2 c^2}{c^2 r'^2 + r^2 w^2 r'^2}
$$
\n(16)\n
\nthat
$$
\int_{r}^{r} \triangleq \frac{dr'}{dt'} = \frac{dr'}{dt}
$$
\nand we find\n
$$
\frac{dr'}{dt} = c \sqrt{1 - \frac{w^2 r^2}{c^2 + w^2 r^2}}
$$
\n(17)

Solve for f' and substitute in (11) to obtain

$$
\frac{w'}{2f\epsilon} = \frac{r}{r'}\sqrt{1 - \frac{{w'}^2{r'}^2}{c^2}}
$$
(15)

We already defined $w = 2f \epsilon$. Substitute in (15) and solve for *w'* to obtain

$$
w'^2 = \frac{w^2 r^2 c^2}{c^2 r'^2 + r^2 w^2 r'^2}
$$
 (16)

Now substitute w'^2 in (13) noting that $\hat{r}' = \frac{du}{dt} = \frac{dt}{dt}$ and we find In order to satisfy the condition for the contraction of

require that
 $2f'r' = 2f r \sqrt{1 - \frac{w'^2 r'}{c^2}}$

Solve for f' and substitute in (11) to obtain
 $\frac{w'}{2fE} = \frac{r}{r'} \sqrt{1 - \frac{w'^2}{c^2}}$

We already defined $w = 2fE$. Subs Solve for *f'* and substitute in (11) to obtain
 $\frac{w'}{2f\epsilon} = \frac{r}{r'} \sqrt{1 - \frac{w'^2r'^2}{c^2}}$

We already defined $w = 2f\epsilon$. Substitute in (15) and solve for w'
 $w'^2 = \frac{w^2r^2c^2}{c^2r'^2 + r^2w^2r'^2}$

Now substitute w'^2 in (

$$
c'_{r} = c \sqrt{1 - \frac{w'^{2}r'^{2}}{c^{2}}}
$$
(13)
\n
$$
c'_{r} = c \sqrt{1 - \frac{w'^{2}r'^{2}}{c^{2}}}
$$
(13)
\n
$$
2f'r' = 2f r \sqrt{1 - \frac{w'^{2}r'^{2}}{c^{2}}}
$$
(14)
\nd substitute in (11) to obtain
\n
$$
\frac{w'}{2f \epsilon} = \frac{r}{r'} \sqrt{1 - \frac{w'^{2}r'^{2}}{c^{2}}}
$$
(15)
\n
$$
\frac{w}{2f \epsilon} = \frac{r}{r'} \sqrt{1 - \frac{w'^{2}r'^{2}}{c^{2}}}
$$
(15)
\n
$$
w'^{2} = \frac{w^{2}r^{2}c^{2}}{c^{2}r'^{2} + r^{2}w^{2}r'^{2}}
$$
(16)
\n
$$
w'^{2}
$$
 in (13) noting that $\gamma' = \frac{d}{dt'} = \frac{dr'}{dt'}$ and we find
\n
$$
\frac{dr'}{dt} = c \sqrt{1 - \frac{w^{2}r^{2}}{c^{2} + w^{2}r^{2}}}
$$
(17)
\n7) becomes
\n
$$
\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w^{2}t^{2}}}
$$
(18)
\nwith respect to *t* we find
\n
$$
\frac{c}{\sqrt{1 + w^{2}t^{2}}}
$$
(18)
\nwith respect to *t* we find
\n
$$
\frac{c}{\sqrt{1 + w^{2}t^{2}}} = \frac{c}{wt} \ln(wt + \sqrt{1 + w^{2}t^{2}}) = \frac{r}{wt} \arcsinh(wt)
$$
(19)
\n
$$
\frac{w}{w}
$$
 in (11) can take the form
\n
$$
\sinh \frac{wr'}{c} = \frac{wr}{c}
$$
(20)
\nbecomes,
\n
$$
\frac{w^{4}r^{2}}{r^{2} + \sqrt{1 + w^{2}t^{2}}} = \frac{w^{4}t^{2}}{r^{2} + \sqrt{1 + w^{2}t^{2} + \sqrt{1 + w^{2}t^{2}}}} = \frac{w^{4}t^{2}}{r^{2}
$$

$$
\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w^2 t^2}}\tag{18}
$$

and integrating with respect to*t* we find

atisfy the condition for the contraction of the perimeter (see (6)) we further
\n
$$
2f'r' = 2fr\sqrt{1 - \frac{w'^2r'^2}{c^2}}
$$
\n(14)
\nand substitute in (11) to obtain
\n
$$
\frac{w'}{2f\epsilon} = \frac{r}{r'}\sqrt{1 - \frac{w'^2r'^2}{c^2}}
$$
\n(15)
\ndefined $w = 2f\epsilon$. Substitute in (15) and solve for w' to obtain
\n
$$
w'^2 = \frac{w^2r^2c^2}{c^2r'^2 + r^2w^2r'^2}
$$
\n(16)
\nute w'^2 in (13) noting that $\frac{ds'}{r'} = \frac{dr'}{dt'} = \frac{dr'}{dt'}$ and we find
\n
$$
\frac{dr'}{dt} = c\sqrt{1 - \frac{w^2r^2}{c^2 + w^2r^2}}
$$
\n(17)
\n(17) becomes
\n
$$
\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w^2t^2}}
$$
\n(18)
\n
$$
r' = \frac{c}{w}\ln(wt + \sqrt{1 + w^2t^2}) = \frac{ct}{wt}\ln(wt + \sqrt{1 + w^2t^2}) = \frac{r}{wt}\arcsinh(wt)
$$
\n(19)
\nconstant of integration is zero because we require that $r' = 0$ for $t = 0$.
\n*i*, since $r = ct$, (19) can take the form
\n
$$
\sinh \frac{wr'}{c} = \frac{wr}{c}
$$
\n(16) becomes.
\n(16) becomes.

where, the constant of integration is zero because we require that $r' = 0$ for $t = 0$.

$$
\sinh \frac{wr'}{c} = \frac{wr}{c}
$$
 (20)

$$
\frac{w}{2f\epsilon} = \frac{r}{r'} \sqrt{1 - \frac{w}{c^2}}
$$
 (15)
We already defined $w = 2f\epsilon$. Substitute in (15) and solve for w' to obtain

$$
w'^2 = \frac{w^2 r^2 c^2}{c^2 r^2 + r^2 w^2 r'^2}
$$
 (16)
Now substitute w'^2 in (13) noting that $\frac{r'}{r} = \frac{d}{dt'} = \frac{dr'}{dt}$ and we find

$$
\frac{dr'}{dt} = c \sqrt{1 - \frac{w^2 r^2}{c^2 + w^2 r^2}}
$$
 (17)
Using $r = ct$, (17) becomes

$$
\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w^2 r^2}}
$$
 (18)
and integrating with respect to t we find

$$
r' = \frac{c}{w} \ln(wt + \sqrt{1 + w^2 t^2}) = \frac{ct}{wt} \ln(wt + \sqrt{1 + w^2 t^2}) = \frac{r}{wt} \arcsinh(wt)
$$
 (19)
where, the constant of integration is zero because we require that $r' = 0$ for $t = 0$.
Equivalently, since $r = ct$, (19) can take the form

$$
\sinh \frac{wr'}{c} = \frac{wr}{c}
$$
 (20)
Using (20), (16) becomes,

$$
w'^2 = \frac{w'^2 r^2}{(c^2 + r^2 w^2) \ln^2(\frac{wr}{c} + \sqrt{1 + \frac{w^2 r^2}{c^2}})} = \frac{w'^2}{(1 + w^2 t^2) \ln^2(wt + \sqrt{1 + w^2 t^2})} = \frac{w'^2 r^2}{(1 + w^2 t^2) \arcsinh^2 wt}
$$

(21)
A similar relation (21) holds also between f and f' because $w' = 2f\epsilon$ and $w = 2f\epsilon$ (see
(14) and (15))
Using (12) and (16) we find

$$
\cos \{\frac{c}{\sqrt{c^2 + w^2 r^2}} = \sqrt{1 - \frac{w'^2 r^2}{c^2}}
$$
 (22)
and we require that $w'r' \leq c$.
We have shown that O' (the laboratory observer) will see a contraction of

A similar relation (21) holds also between f and f' because $w' = 2f \in$ and $w = 2f \in$ (see (14) and (15))

Using (12) and (16) we find

$$
\cos\left\{\right. = \frac{c}{\sqrt{c^2 + w^2 r^2}} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}}\tag{22}
$$

rotating disc given by (19) or (20). Observers O' and O will not agree on the angular velocity, *w* and on the value of f . In fact, observer O' will perceive *w'* and f' as varying with the distance *r* . This situation arises from the fact that the contraction factor along the

perimeter is different from the contraction factor along the radius and the requirement that the speed of the signals is constant and agreed by all observers.

One remark about angular velocities is useful to clarify things. Observers O , O' and \tilde{O} will agree on epicenter angle $\Delta\Theta$, measured as fraction of a circle (see Assumption 1). But according to the definition of angular velocity we may write $w = \frac{2f\Delta\Theta}{g}$, $w' = \frac{2f'\Delta\Theta}{g}$, $t \Delta t'$ $=\frac{2f\Delta\Theta}{\Delta t}, \ \ w'=\frac{2f'\Delta\Theta}{\Delta t'},$ $t' = \frac{2f'\Delta\Theta}{\Delta t'},$, s different from the contraction factor along the radius and the requirement that

f the signals is constant and agreed by all observers.

stabout angular velocities is useful to clarify things. Observers *O*, *O'* and $\$

$$
\tilde{w} = \frac{2f\Delta\Theta}{\Delta\tilde{t}}
$$
, while for angles $_u = 2f\Theta$, $_u' = 2f\Theta$, $_u = 2f\Theta$, where the tildas refer to

observer $\tilde{O}(O_3)$, the primes to observer $O'(O_2)$, and the plane letters refer to observer O $(O₁)$. The correct notation using the subscript notation of section 3 is

perimeter is different from the contraction factor along the radius and the requirement that
the speed of the signals is constant and agreed by all observes. Soservers *O*, *O'* and
$$
\tilde{O}
$$

one remark about angular velocities is useful to clarify things. Observers *O*, *O'* and \tilde{O}
will agree on epicenter angle $\Delta\Theta$, measured as fraction of a circle (see Assumption 1).
But according to the definition of angular velocity we may write $w = \frac{2f\Delta\Theta}{\Delta t}$, $w' = \frac{2f'\Delta\Theta}{\Delta t'}$,
 $\tilde{w} = \frac{2f\Delta\Theta}{\Delta t} = w_1 = \frac{2f_{11}\Delta\Theta}{\Delta t_1}$, $w' = \frac{2f}{\Delta t} = \frac{2f_1\Delta\Theta}{\Delta t_2}$,
 $w = \frac{2f\Delta\Theta}{\Delta t} = w_{11} = \frac{2f_{11}\Delta\Theta}{\Delta t_{11}} = \frac{2f_{11}\Delta\Theta}{\Delta t_{\text{max}}}$, $w' = \frac{2f'\Delta\Theta}{\Delta t'} = w_{21} = \frac{2f_{21}\Delta\Theta}{\Delta t_{21}} = \frac{2f_{21}\Delta\Theta}{\Delta t_{22}}$.
 $\tilde{w} = \frac{2f\Delta\Theta}{\Delta t} = w_{31} = \frac{2f_{31}\Delta\Theta}{\Delta t_{33}} = \frac{2f_{31}\Delta\Theta}{\Delta t_{\text{max}}}$, $w' = \frac{2f'\Delta\Theta}{\Delta t'} = w_{21} = \frac{2f_{21}\Delta\Theta}{\Delta t_{21}} = \frac{2f_{21}\Delta\Theta}{\Delta t_{22}}$.
 $\tilde{w} = \frac{2f\Delta\Theta}{\Delta t} = w_{32} = \frac{2f_{33}\Delta\Theta}{\Delta t_{33}} = \frac{2f_{33}\Delta\Theta}{\Delta t_{\text{max}}} = \frac{2f_{31}\Delta\Theta}{\Delta t_{\text{max}}}$.\n\n $\tilde{w} = \frac{2f_{10}\Delta\Theta}{\Delta t}$, $w' = \frac{2f_{11}\Delta\Theta}{\Delta t}$, $w' = \frac{2f_{12}\Delta\Theta}{\Delta t_{21}} = \frac{f_{21}\Delta\Theta}{\Delta t_{22}}$.
\n<

observers agree both on radial and on perimeter lengths when stationary on their frame (Assumption 3). We conclude, therefore, that

$$
\frac{w'}{w} = \frac{w_{21}}{w_{11}} = \frac{f_{21}}{f_{11}} = \frac{f'}{f}
$$
 (23)

$$
\frac{w}{\tilde{w}} = \frac{w_{11}}{w_{33}} = \frac{\Delta t_{33}}{\Delta t_{11}} = \frac{1}{\cos \{}
$$
 (24)

and

$$
\frac{w'}{\tilde{w}} = \frac{w_{21}}{w_{33}} = \frac{f_{21}\Delta t_{33}}{f_{33}\Delta t_{21}} = \frac{f_{21}}{f_{33}\cos\xi} = \frac{f'}{f\cos\xi}
$$
(25)

5. Summary of Results

The angle of deflection $\{$ and in particular cos $\{$ takes many equivalent forms that are presented here for ease of calculations

$$
\frac{d}{dt} = w_{33} = \frac{2f_{33}\Delta\Theta}{\Delta t_{33}} = \frac{2f_{33}\Delta\Theta}{\Delta t_{max}} \cos\left(\frac{2f_{33}\Delta\Theta}{2\Delta t_{max}}\right) = 2f_{33} = f \text{ (or } f = f \text{), because}
$$
\n
$$
\frac{d}{dt} = \frac{w_{33}}{M_{max}} = \frac{2f_{33}\Delta\Theta}{\Delta t_{max}} \text{ the reference, that}
$$
\n
$$
\frac{w'}{w} = \frac{w_{21}}{w_{11}} = \frac{f_{21}}{f_{11}} = \frac{f'}{f} \tag{23}
$$
\n
$$
\frac{w}{w} = \frac{w_{11}}{w_{33}} = \frac{f_{21}\Delta t_{33}}{\Delta t_{11}} = \frac{f}{\cos\xi} \tag{24}
$$
\n
$$
\frac{w'}{w} = \frac{w_{21}}{w_{33}} = \frac{f_{21}\Delta t_{33}}{f_{33}\Delta t_{21}} = \frac{f_{21}}{f_{33}\cos\xi} = \frac{f'}{f \cos\xi} \tag{25}
$$
\n
$$
\frac{d}{dt} = \frac{1}{\cos\xi} \text{ and in particular } \cos\xi \text{ takes many equivalent forms that are}
$$
\n
$$
\sqrt{1 - \frac{w'^2 r'^2}{c^2}} = \sqrt{1 - \frac{w^2 r^2}{c^2 + w^2 r^2}} = \frac{c}{\sqrt{c^2 + w^2 r^2}} = \frac{1}{\sqrt{1 + \frac{w^2 r^2}{c^2}}} = \frac{w'r'}{wr} = \cos\xi \tag{26}
$$
\n
$$
\frac{w'r'}{c} = \sqrt{\frac{w^2 r^2}{c^2 + w^2 r^2}} = \sin\xi \tag{27}
$$
\n
$$
\tan\xi = \frac{w}{c} = w = s = \sinh\frac{wr'}{c} \tag{28}
$$
\n
$$
\frac{ds}{dt} = \frac{w}{c} = w = s = \sinh\frac{wr'}{c} \tag{28}
$$
\n
$$
\frac{ds}{dt} = \frac{w}{c} = \frac{w}{c} = \frac{w}{c} = \frac{w}{c} = \frac{w}{c} = \frac{w}{c}
$$
\

$$
\frac{w'r'}{c} = \sqrt{\frac{w^2r^2}{c^2 + w^2r^2}} = \sin\left\{\tag{27}
$$

$$
\tan\left\{\right. = \frac{wr}{c} = wt = \sqrt{r} = \sinh\frac{wr'}{c}
$$
\n⁽²⁸⁾

Where μ is the angle of the circle traveled by the signal until it reaches the distance r from the center (see Figure 1).

The transformation among the observers O , O' and \tilde{O} is summarized below in Table1(a), 1(b),1(c). However, our interest in this study will be focused on the relation between observer O and O' .

Table $1(a)$	Transformations between Observers $O(O_1)$ and $O'(O_2)$
Quantitities Time interval	Transformations
	$\Delta t = \Delta t' = \Delta t_{stat}$, $(\Delta t_{stat} = \Delta t_{21} = \Delta t_{12} = \Delta t_{11} = \Delta t_{22})$
Length segment on perimeter	$\Delta L' = \Delta L \cos \{ , (\Delta L_{21} = \Delta L_{stat} \cos \{) \}$
Radius	$r' = \frac{c}{r} \arcsinh \frac{wr}{r}$ w
Angular velocity	$w' = w^2 \frac{r \cos \{r\}}{r}$
	$c \operatorname{arcsinh}(\frac{wr}{c})$
Pi and angles	$\frac{w'}{w} = \frac{r'}{r} = \frac{f'}{f} = \frac{r}{r'} \cos \left($
	Table 1(b) Transformations between Observers $O(O_1)$ and $\tilde{O}(O_3)$
Quantitities Time interval	Transformations
	$\Delta \tilde{t} = \frac{\Delta t}{\cos \xi} \, , \, (\, \Delta t_{_{13}} = \Delta t_{_{33}} = \frac{\Delta t_{_{stat}}}{\cos \xi} \,)$
Length segment on perimeter	$\Delta \tilde{L} = \Delta L$, $(\Delta L_{13} = \Delta L_{31} = \Delta L_{stat})$
Radius	
	$r = \tilde{r}$ $\frac{\overline{\tilde{w}}}{\overline{w}} = \frac{\Delta t_{11}}{\Delta t_{21}} = \cos \left\{ \frac{\overline{w}}{\sqrt{w}} \right\}$
Angular velocity	Δt_{33} W
Pi and angles	
	$f=f,\overline{f}=\overline{f}$
	Table 1(c) Transformations between Observers O' (O_2) and $\tilde{O}(O_3)$
Quantitities Time interval	Transformations $\Delta t'$ $\Delta t_{\textit{stat}}$.

Table 1(a) Transformations between Observers $O(O_1)$ and $O'(O_2)$

Table 1(b) Transformations between Observers O (O_1) and \tilde{O} (O_3)

Angular velocity	$w' = w^2 \frac{r \cos \{f}{r}}{r}$	
	$c \operatorname{arcsinh}(\frac{wr}{c})$	
Pi and angles	$\frac{w'}{w} = \frac{r'}{r} = \frac{f'}{f} = \frac{r}{r'} \cos \left($	
	Table 1(b) Transformations between Observers $O(O_1)$ and $\tilde{O}(O_2)$	
Quantitities	Transformations	
Time interval	$\Delta \tilde{t} = \frac{\Delta t}{\cos \left\{ \right. \right.}, \left(\Delta t_{13} = \Delta t_{33} = \frac{\Delta t_{stat}}{\cos \left\{ \right. \right.}} \right)$	
Length segment on perimeter	$\Delta \tilde{L} = \Delta L$, $(\Delta L_{13} = \Delta L_{31} = \Delta L_{stat})$	
Radius	$r = \tilde{r}$	
Angular velocity	$\frac{\tilde{w}}{w} = \frac{\Delta t_{11}}{w} = \cos \left($	
	$\Delta t^{}_{33}$	
Pi and angles	$f=f, r_{\frac{1}{2}}$	
	Table 1(c) Transformations between Observers $O'(O_2)$ and $\tilde{O}(O_3)$	
Quantitities	Transformations	
Time interval	$\Delta \tilde{t} = \frac{\Delta t'}{\cos \xi}, (\Delta t_{23} = \Delta t_{33} = \frac{\Delta t_{stat}}{\cos \xi})$	
Length segment on perimeter	$\Delta L' = \Delta \tilde{L} \cos \{ , (\Delta L_{23} = \Delta L_{\text{stat}} \cos \{) \}$	
Radius	$r' = \frac{c}{\tilde{w}} \cos \{ \arcsinh \frac{\tilde{w}\tilde{r}}{c \cos \{ \}} \}$	
Apoular vologity	\sim 2 \sim	

6. Warp or Ripples?

From (14) and using (20) we see that $f' = f \frac{c}{\log c} \cos \{$. This implies that $f' < f$ $\frac{m}{2}$ *wr* $\frac{c}{c}$ cos { This implies that f' wr ²⁰⁵ (\therefore 1115 improvement $f(x)$ *c*

 $f' = f \frac{\frac{wr}{c}}{\arcsinh \frac{wr}{c}} \cos \{\text{ } \}$. This implies that $f' < f$
holds. The decrease of *f* implies a warping of the
ure 2(a) and Figure 2(b)). In the case of warping
ius as the segment OA with length *r*, observer O' except for $wr = 0$ for which equality holds. The decrease of f implies a warping of the disc or the creation of ripples (see Figure 2(a) and Figure 2(b)). In the case of warping (Figure 2(a)) observer O sees the radius as the segment OA with length r , observer O' sees the curved segment OB with length r', which for him looks as straight and PB is the theoretical straight line (projection of r' on Euclidean space) with length r'' . However, we may exclude warping, because of the following argument: Consider three parallel concentric discs. Let the middle one rotate with frequency ϵ and with respect to the other two that are stationary. If indeed warping occurred the middle disc would intersect one of the other two discs, since we are allowed to bring them arbitrarily close to the middle disc. This seems unphysical. We are, therefore, inclined to exclude warping as a possibility and to consider ripples instead. 6. Warp or Ripples?

From (14) and using (20) we see that $f' = f \frac{wr}{c} \csc(\frac{r}{c})$. This implies that $f' < f$

arcsinh $\frac{wr}{c}$

arcsinh $\frac{w}{c}$

except for $wr = 0$ for which equality holds. The decrease of f implies

The ripples formed on the disc (Figure 2(b)) make the radius be looked in two different ways. One is the radius touching the surface of ripples (*r*) (the surface radius) and another is the theoretical straight line disregarding ripples (*r*ⁿ) (the straight radius or the projection of r' on the flat plane of rotation). The latter one satisfies the equation

$$
r'' = r \cos\left\{\tag{29}
$$

Figure 2 (a) The rotating disc is warped. The radius on the surface of the warped disc is *r* . The straight line (Euclidean) radius is r'' . The stationary radius is r . (b) The rotating disc forms ripples. The length of the radius on the rippled surface is *r* . The straight line (Euclidean) radius of the disc is r'' . The stationary radius is r .

 $r'' \rightarrow -$, $w' \rightarrow 0$, $\cos \{\rightarrow 0.$ T *w* mother is the theoretical straight line disregarding ripples (r^*) (the straight radius or the
original of r' on the flat plane of rotation). The latter one satisfies the equation
 $r^* = 2f'r' = 2f r \cos\{$ and therefore r unother is the theoretical straight line disregarding ripples (r^*) (the straight radius or the projection of r^* on the flat plane of rotation). The latter one satisfies the equation $2f r^* = 2f r \times 2f r \cos\{$ and theref Euclidean surface) radius r'' tends to $\frac{c}{r}$. *w* Physically, r'' is the radius that an observer O'' on the laboratory frame will observe, who

is located at a distance greater than $\frac{c}{c}$ from the axis of rotation of the di *w* from the axis of rotation of the disc. If O'', enters into the region of distance less than $\frac{c}{c}$ from the axis of rotation, then hi *w* from the axis of rotation, then his geometry ceases

to be Euclidean. He becomes observer of type O' . He is now on the rippled surface (which he perceives as flat) his pi is now f' and the radius of the disc is now given by r' , which is not limited by any boundary. Since rotation of the disc does not affect the time rate of their clocks, because they do not participate in the rotation, we expect that the time rate for to be Euclidean. He becomes observer of type O' . He is now on the rippled surface (which
he perceives as flat) his pi is now f' and the radius of the disc is now given by r' , which
is not limited by any boundary. Si they cannot exchange light signals. If we denote by double primes the quantities observed by O'' , the above implies that 2 is now on the rippled surf the disc is now given by

disc does not affect the time

ation, we expect that the t

pual or that $\Delta t' = \Delta t'' = \Delta t_{s}$

buble primes the quantitie
 $\frac{1}{2r'^2}$

sees the lengths of observ erver of type O'. He is now on th
 f' and the radius of the disc is no

ince rotation of the disc does not a

participate in the rotation, we exp

undary to remain equal or that Δt
 t" = $\Delta t' = \Delta \tilde{t}$
 $t'' = \Delta t' = \$ He is now on the rippled surface (v. of the disc is now given by r', w.

e disc does not affect the time rate

otation, we expect that the time rate

equal or that $\Delta t' = \Delta t'' = \Delta t_{stat}$, alth

double primes the quantities ob server of type O'. He is now on the rippled surface (which
 w f' and the radius of the disc is now given by r', which

Since rotation of the disc does not affect the time rate of

participate in the rotation, we expect . He is now on the rippled surface (which
us of the disc is now given by r', which
the disc does not affect the time rate of
protation, we expect that the time rate for
in equal or that $\Delta t' = \Delta t'' = \Delta t_{\text{stat}}$, although
by *r r* \int *r c l <i>n c n c n c n c n c n c <i>n c n c n c n c n c <i>n c n c n c n c <i>n c* erver of type O'. He is now on the rippled surface (which

f' and the radius of the disc is now given by r', which

ince rotation of the disc does not affect the time rate of

participate in the rotation, we expect that t f type O'. He is now on the rippled surface (which

d the radius of the disc is now given by r', which

tation of the disc does not affect the time rate of

pate in the rotation, we expect that the time rate for

y to rem to be Fuclidean. He becomes observer of type O'. He is now on the rippled surface (which
he perceives as flat) his pi is now f' and the radius of the disc is now given by r', which
is not limited by any boundary. Since ro to be Euclidean. He becomes observer of type O'. He is now on the rippled surface (which
the perceives as flat) his pi is now f' and the radius of the disc is now given by r' , which
is not limited by any boundary. Sin he perceives as flat) his pi is now f' and the radius of the disc is now given by r', which
is not limited by any boundary. Since rotation of the disc does not affect the time rate of
their clocks, because they do not par *w* between the disc does not affect the time reate of the disc does not diffect the time reate of O' and O' after O' renses the boundary to remain equal or that $\Delta t' = \Delta t' = \Delta t_{\text{max}}$, although they cannot exchange *heir* clocks, because they do not participate in the rotation, we expect that the time rate for O' and O'' after O'' crosses the boundary to remain equal or that $\Delta t' = \Delta t'' = \Delta t'' = \Delta t_{\text{max}}$, although by O'' , the abov

$$
\Delta t'' = \Delta t' = \Delta \tilde{t} \frac{1}{\sqrt{1 - \frac{w'^2 r'^2}{c^2}}}
$$
(30)

Regarding lengths (perimeter and radial) observer O'' sees the lengths of observer O smaller by a factor of $\cos\{\theta}$ because of (29). Hence,

$$
\Delta L'' = \frac{r''}{r} \Delta L = \Delta L \cos \{ \tag{31}
$$

Further, epicenter angles are not affected,

$$
\Delta \Theta = \Delta \Theta'' \tag{32}
$$

because $v'' = \frac{\Delta \Theta''}{\Delta t_{stat}}$

r

cted,
 $\Delta\Theta = \Delta\Theta''$ (32)

ncy of revolution measurements, that is $v = v' = v''$

Since observer O'' lives in Euclidean space, his pi

since observer O'' lives in Euclidean space, his pi

ormal pi or $f = f''$. It follows that Ignt signals. It we define by dodne primes the quantities observed

lies that
 $\Delta t'' = \Delta t' = \Delta \tilde{t} \frac{1}{\sqrt{1 - \frac{W^2 r'^2}{c^2}}}$ (30)

Frimeter and radial) observer O^r sees the lengths of observer O

cos { because of (29) = $\Delta t' = \Delta \tilde{t} \frac{1}{\sqrt{1 - \frac{W^2 r'^2}{c^2}}}$ (30)

dial) observer *O'* sees the lengths of observer *O*

of (29). Hence,
 $r = \frac{r''}{r} \Delta L = \Delta L \cos \{$ (31)

creed,
 $\Delta \Theta = \Delta \Theta''$ (32)

creed,
 $\Delta \Theta = \Delta \Theta''$ (32)

Since observer *O'* the boundary to remain equal or that $\Delta t' = \Delta t' = \Delta t_{\text{max}}$, although
signals. If we denote by double primes the quantities observed
at
 $\Delta t'' = \Delta t' = \lambda \overline{t} \frac{1}{\sqrt{1 - \frac{w'^2 r'^2}{c^2}}}$ (30)
er and radial) observer O^* sees t $v'' = \frac{\Delta \Theta''}{\Delta t_{stat}}$
 $v''' = \frac{\Delta \Theta''}{\Delta t_{stat}}$
 Δt_{stat}
 $\sum v'' = 2f \in E = w$. From thes
 $v''r'' = wr''$. Substituting in
 $\cos \{ = \sqrt{1 - \frac{1}{2r^{n^2}}}$. Finally, substituting egarding lengths (perimeter and radial) on

naller by a factor of cos { because of (2
 $\Delta L'' = \frac{r''}{r}$

urther, epicenter angles are not affected,

ence, there is agreement on frequency of

scause $v'' = \frac{\Delta \Theta'''}{\Delta t_{sat}}$

lso $\Delta t' = \Delta t' = \Delta t$

garding lengths (perimeter and radial) observer *O'* sees the lengths of observer *O*

aller by a factor of cos{ because of (29). Hence,
 $\Delta t'' = \frac{r'}{r} \Delta L = \Delta L \cos \left($ (31)

ther, epicenter angles are not a ver O'' lives in Euclidean s
 $f = f''$. It follows that

one easily deduces using (
 $\frac{w^2 r^{n_2}}{c^2} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}}$

of the perimeter for O'' is

we obtain,
 $\frac{2r^{n_2}}{c^2}$. Hence,
 $L = \Delta L \cos \{\right$
 $= \Delta \Theta''$

evolution measurements, that is $v = v' =$

bserver O'' lives in Euclidean space, his

io or $f = f''$. It follows that

ind,
 $\sqrt{1 - \frac{w^2 r''^2}{c^2}} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}}$

tion of the perimeter *r* = $\frac{r^n}{r}$ Δ*L* = Δ*L* cos {

cted,
 $\Delta\Theta = \Delta\Theta^n$

ncy of revolution measurements, that

Since observer *O''* lives in Euclidea

ormal pi or *f* = *f''*. It follows that

been values usin

6) we find,
 $\frac{r^2 r^{n2}}{$ al) observer O^* sees the lengths of observer O
 $= \frac{r^2}{r} \Delta L = \Delta L \cos \{$
 $= \frac{V}{r} \Delta L = \Delta L \cos \{$

(31)

ted,
 $\Delta \Theta = \Delta \Theta^*$

(32)

y of revolution measurements, that is $v = v' = v''$

ince observer O^* lives in Euclidean cted,
 $\Delta\Theta = \Delta\Theta''$

mcy of revolution measurements, that

Since observer *O''* lives in Euclidea

ormal pi or $f = f''$. It follows that

bbservations one easily deduces usir

(6) we find,
 $\frac{r^2 r^{n^2}}{c^2} = \sqrt{1 - \frac{w^2 r^{n$ = $\Delta \Theta^{\nu}$

evolution measurements, that is $v = v'$

bserver O'' lives in Euclidean space, h

i or $f = f''$. It follows that

ions one easily deduces using (26) tha

ind,
 $\sqrt{1 - \frac{w^2 r^{n2}}{c^2}} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}}$

tion er O" lives in Euclidean space, his pi
 $c = f''$. It follows that

one easily deduces using (26) that
 $\frac{v^2 r''^2}{c^2} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}}$ (3

of the perimeter for O" is given by

we obtain,
 $\frac{r r''^2}{c^2}$ (3
 $\frac{c}{c^2$ Also from (14) and (29) $f'r' = f r''$. Since observer O'' lives in Euclidean space, his pi

denoted as f'' must be equal to the normal pi or $f = f''$. It follows that
 $w' = 2f'\mathbf{C}'' = 2f\mathbf{C} = w$, *From these observations one* and pi or $f = f''$. It follows that

mal pi or $f = f''$. It follows that

we find,
 $\frac{r^2}{2} = \sqrt{1 - \frac{w^2 r'^2}{c^2}} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}}$ (33)

that we find,
 $\frac{r^2}{2} = \sqrt{1 - \frac{w^2 r'^2}{c^2}} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}}$ (33)

thraction

$$
\cos\left\{ = \sqrt{1 - \frac{w^{n^2}r^{n^2}}{c^2}} = \sqrt{1 - \frac{w^2r^{n^2}}{c^2}} = \sqrt{1 - \frac{w^{n^2}r^{n^2}}{c^2}} \tag{33}
$$

This was expected since the Lorentz contraction of the perimeter for O'' is given by c^2 and c^2 and d,
 $1 - \frac{w^2 r^{n2}}{c^2} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}}$

ion of the perimeter for O" is .

29) we obtain,
 $\frac{w^2 r^{n2}}{c^2}$
 $\frac{c}{\sqrt{c^2 + w^2 r^2}}$
 $\frac{c}{\sqrt{c^2 + w^2 r^2}}$ This was expected since the Lorentz contraction of the perimeter for O" is given $\sqrt{1 - \frac{w^{n^2}r^{n^2}}{c^2}}$. Finally, substituting (33) into (29) we obtain,
 $r^n = r \sqrt{1 - \frac{w^2r^{n^2}}{c^2}}$

And solving we find
 $r^n = r \frac{c}{\$ lim priori priori in the contraction of the peripheter suring (26) that
 $\frac{w^{r^2}r^{r^2}}{c^2} = \sqrt{1 - \frac{w^2r^{r^2}}{c^2}} = \sqrt{1 - \frac{w'^2r^{r^2}}{c^2}} = \sqrt{1 - \frac{w'^2r^{r^2}}{c^2}} = \sqrt{1 - \frac{w'^2r^{r^2}}{c^2}} = \sqrt{1 - \frac{w'^2r^{r^2}}{c^2}}$ (33)

rtz and $\frac{1}{\sqrt{2r^2 r^2}} = \sqrt{1 - \frac{w^2 r^2}{c^2}} = \sqrt{1 - \frac{w^2 r^2}{c^2}} = \sqrt{1 - \frac{w^2 r^2}{c^2}}$ (33)

Lorentz contraction of the perimeter for *O''* is given by

ituting (33) into (29) we obtain,
 $r'' = r \sqrt{1 - \frac{w^2 r'^2}{c^2}}$ (34)
 $r'' = r$ $rac{ln(20) \text{ we find}}{1 - \frac{w'^2 r'^2}{c^2}} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}}$

rentz contraction of the perimeter for *G*

ng (33) into (29) we obtain,
 $r'' = r \sqrt{1 - \frac{w^2 r''^2}{c^2}}$
 $r'' = r \frac{c}{\sqrt{c^2 + w^2 r^2}}$

by (26) $\cos \left(\frac{c$ $=\sqrt{1-\frac{w''^2r''^2}{c^2}} = \sqrt{1-\frac{w'^2r''^2}{c^2}} = \sqrt{1-\frac{w'^2r'^2}{c^2}}$ (33)

Lorentz contraction of the perimeter for *O''* is given by

uting (33) into (29) we obtain,
 $r'' = r\sqrt{1-\frac{w^2r''^2}{c^2}}$ (34)
 $r'' = r\frac{c}{\sqrt{c^2+w^2r^2}}$ (35 see observations one easily deduces using (26) that
 $\frac{(26) \text{ when } (16) \text{ when } (16) \text{ when } (16) \text{ when } (10) \text{ when } (10$

$$
r'' = r \sqrt{1 - \frac{w^2 r''^2}{c^2}}
$$
 (34)

And solving we find

$$
r'' = r \frac{c}{\sqrt{c^2 + w^2 r^2}}
$$
 (35)

 $\cos\left(1-\frac{c}{\sqrt{c^2-c^2}}\right)$ $+w^2r^2$

7. Discussion of Results

And solving we find
\n
$$
r'' = r \frac{c}{\sqrt{c^2 + w^2 r^2}}
$$
\n(35)
\nWhich was also expected since by (26) $\cos \left(\frac{c}{\sqrt{c^2 + w^2 r^2}} \right)$
\n7. Discussion of Results
\nFirst we note that from (26), (29) and because $w'r' = w''r'' = wr''$
\n
$$
\lim_{w \to \infty} w'r' = \lim_{w \to \infty} w''r'' = vr''
$$
\n
$$
\lim_{w \to \infty} w'r' = \lim_{w \to \infty} w''r'' = cv''
$$
\n(36)
\n
$$
\lim_{w \to \infty} r'' = \lim_{w \to \infty} \left(\frac{r}{\sqrt{1 + w^2 t^2}} \right) = 0
$$
\n(37)
\nAnd similarly,

$$
\lim_{w \to \infty} r'' = \lim_{w \to \infty} r' = \lim_{w \to \infty} \left(\frac{r}{\sqrt{1 + w^2 t^2}} \right) = 0
$$
\n(37)

And similarly,

$$
\lim_{w \to 0} r'' = \lim_{w \to 0} r' = \lim_{w \to 0} (\frac{r}{\sqrt{1 + w^2 t^2}}) = r
$$
(38)

$$
\lim_{r \to \infty} w' = 0
$$
(39)

$$
\lim_{w \to \infty} w' = \infty
$$
(40)

$$
w'r' \to c
$$
 and $\cos f \to 0$ and $\hat{e}' \to 0$ and $\hat{e}' \to c$. This says

Also

$$
\lim_{r \to \infty} w' = 0 \tag{39}
$$

And

$$
\lim_{w \to \infty} w' = \infty \tag{40}
$$

- $r' = \lim_{w \to 0} (\frac{r}{\sqrt{1 + w^2 t^2}}) = r$ (38
 $\lim_{r \to \infty} w' = 0$ (39
 $\lim_{w \to \infty} w' = \infty$ (40
 $\lim_{w \to \infty} w' = \infty$ and $\int_{r}^{r} \to 0$ and $\int_{r}^{r} \to c$. This say

the rotating disc becomes big, the rays at the transportial velocit $\lim_{t \to 0} \left(\frac{r}{\sqrt{1 + w^2 t^2}} \right) = r$ (38)
 $t' = 0$ (39)
 $t' = \infty$ (40)
 $s \left(\rightarrow 0 \text{ and } \hat{r} \right) \rightarrow 0 \text{ and } \hat{r} \right) \rightarrow c$. This says

obtating disc becomes big, the rays at the

order the discrimential velocity approaches the
 $\lim_{w\to 0} r^n = \lim_{w\to 0} r' = \lim_{w\to 0} \frac{r}{\sqrt{1 + w^2 t^2}} = r$ (38)

0

0
 $\lim_{w\to \infty} w' = 0$ (39)

1

1

1. As $wr \to \infty$; $wr'' = w'r' \to c$, and $\cos \left(\to 0 \right)$ and $\hat{\ } , \to 0$ and $\hat{\ } , \to c$. This says

that as the tangential velocity 1. As $wr \rightarrow \infty$; $wr'' = w'r' \rightarrow c$, and $cos\{-100\}$ and $\hat{i} \rightarrow 0$ and $\hat{i} \rightarrow c$. This says that as the tangential velocity of the rotating disc becomes big, the rays at the circumference are almost tangential and their tangential velocity approaches the speed of light, while their radial velocity tends to zero. In other words, light signals emanating from the center, O, will bend and turn around in circles expanding very slowly as they will be bend almost entirely tangentially.
- $\lim_{w \to 0} y'' = \lim_{w \to 0} y' = \lim_{w \to 0} (\frac{r}{\sqrt{1 + w^2 r^2}}) = r$ (38)
 $\lim_{x \to \infty} w' = 0$ (39)
 $\lim_{r \to \infty} w' = w' = 0$ (40)

1. As $wr \to \infty$; $wr'' = w'r' \to c$, and $\cos(\frac{r}{2})$ o and $\frac{r'}{2} \to 0$. This says

that as the tangential veloci $r' \rightarrow 0$, $r'' \rightarrow 0$, $\hat{r}' \rightarrow 0$ and $\hat{r}' \rightarrow c$. In this case, when the angular velocity (*w*) becomes big, while the rest radius (*r*) remains finite, the radius (*r'* and *r"*) for the lab observers O' and O" shrinks to become very small (but $wr'' = w'r' \rightarrow c$), the lim $n'' = \lim_{\epsilon \to 0} r' = \lim_{\epsilon \to 0} (r' + \frac{1}{w^2})^2 = r$ (38)
 $\lim_{\epsilon \to 0} w' = 0$ (39)

As $wr \to \infty$; $wr'' = w'r' \to c$, and $\cos(-\to 0)$ and $\hat{\ }' \to 0$ and $\hat{\ }' \to c$. This says

that as the tangential velocity of the rotating disc bec lim $r'' = \lim_{u \to 0} r' = \lim_{u \to 0} \left(\frac{r}{1 + u^2 r^2} \right) = r$ (38)
 $\lim_{u \to 0} w' = 0$ (39)
 $\lim_{u \to \infty} w' = 0$ (40)
 $\lim_{u \to \infty} w' = \infty$ (40)

As $wr \to \infty$; $wr'' = w'r' \to c$, and $\cos(-\to 0)$ and $\int_{-\infty}^{\infty} \to c$. This says

that as the ta tangential velocity \hat{i} \rightarrow *c* and the radial velocity of the light signals tends to zero $\binom{?}{r} \rightarrow 0$) 3. If *^w* is finite but *^r* then *^r* , *^c* (39)
 \rightarrow 0 and $\hat{i}' \rightarrow 0$ and $\hat{i}' \rightarrow c$. This says

engit discomes big, the rays at the

engit discomes big, the rays at the

ends to zero. In other words, light signals

and turn around in circles expanding very

y tan (40)

As $wr \rightarrow \infty$; $wr'' = w'r' \rightarrow c$, and $\cos(-\rightarrow 0$ and $\pi' \rightarrow 0$ and $\pi' \rightarrow c$. This says

that as the trangential velocity of the rotating disc becomes big, the rays at the

circumfrence are almost tangential and their tangrat enconnected are assumed and their angle unto their angle that the signal vector
speed of light, while their radial velocity tends to zero. In other words, light signals
showly as they will be beend almost entirely tangent Let all due transferral velocity tends to zero. In other words, light signals
velocity tends to zero. In other words, light signals
ill bend and turn around in circles expanding very
st entirely tangentially.
mad hence t le r (and hence t) remains finite, $w \rightarrow \infty$, $\cos \left(\rightarrow 0 \right)$

ad $\hat{ }_{\perp} \rightarrow c$. In this case, when the angular velocity (w)

r adius (r) remains finite, the radius (r' and r'') for the

rrinks to become very small
- $r'' \rightarrow -$, $w' \rightarrow 0$, $\cos \{\rightarrow 0.0$ *w* O' will see the surface radius (r') tend very slowly (logarithmically) to infinity *w* . The light signals bend and go around in tighter and tighter circles with tangential velocity that tends to $c(\hat{i} \rightarrow c)$, while the radial velocity drops to zero. and very slowly (logarithmic

e the radius (r") tend to $\frac{c}{w}$.

ther circles with tangential view

is no rotation $w = 0$ we end
 $= \hat{ }_r$ as expected.

iral out in tighter and tighter
 $\frac{1}{(c^2)^2}$ = cos {

ate as *(r)* remains finite, the radius (*r'* a
become very small (but $wr'' = w'$
r, O, bend to turn in circles (cos {
e radial velocity of the light signal
 $\rightarrow \infty$, $r'' \rightarrow \frac{c}{w}$, $w' \rightarrow 0$, $\cos \{\rightarrow 0$
tend very slowly (logarithmica $\int_{0}^{1} \rightarrow c$. In this case, when the angular velocity (*w*)
tilus (*r*) remains finite, the radius (*r'* and *r''*) for the
ts to become very small (but *wr'* = *w'* → c), the
exenter, O, bend to turn in circles (\sim 0) lius (*r*) remains finite, the radius (*r'* and *r''*) for the

sks to become very small (but $wr'' = w'r' \rightarrow c'$), the

senter, O, bend to turn in circles ($\infty s(-\rightarrow 0)$, with

dd the radial velocity of the light signals tends *r*^{r} $\rightarrow \frac{c}{w}$, *w'* \rightarrow 0, cos { \rightarrow 0. O

1 very slowly (logarithmically) to

he radius (*r''*) tend to $\frac{c}{w}$. The lighter circles with tangential velocity

y drops to zero.

no rotation $w = 0$ we end back i $\Rightarrow \infty$, $r'' \rightarrow \frac{c}{w}$, $w' \rightarrow 0$, $\cos \left(\rightarrow 0$. Observer
tend very slowly (logarithmically) to infinity
see the radius (r'') tend to $\frac{c}{w}$. The light signals
tighter circles with tangential velocity that tends
locity d $\Rightarrow \infty$, $r'' \rightarrow \frac{c}{w}$, $w' \rightarrow 0$, $\cos \left(\rightarrow 0 \right)$. Observer
tend very slowly (logarithmically) to infinity
see the radius (r'') tend to $\frac{c}{w}$. The light signals
tighter circles with tangential velocity that tends
locit
- 4. It is straightforward that when there is no rotation $w = 0$ we end back in frame K
- 5. To show that the light signals will spiral out in tighter and tighter circles we may examine

$$
\frac{dr'}{dr} = \frac{1}{\left(1 + \frac{w^2 r^2}{c^2}\right)^{\frac{1}{2}}} = \cos\left\{ \tag{41} \right\}
$$

which is increasing with diminishing rate as r increases. Similar observations hold for r'' where

$$
= \Delta \tilde{L}, \hat{r}' = \hat{r}, \text{ as expected.}
$$
\n
$$
= \Delta \tilde{L}, \hat{r}' = \hat{r}, \text{ as expected.}
$$
\n
$$
= \frac{1}{\left(1 + \frac{w^2 r^2}{c^2}\right)^{\frac{1}{2}}} = \cos\left\{\n\begin{array}{c}\n(41) \\
\left(1 + \frac{w^2 r^2}{c^2}\right)^{\frac{1}{2}}\n\end{array}\n\right\}
$$
\nmissing rate as r increases. Similar observations hold

\n
$$
\frac{dr''}{dr} = \frac{1}{\left(1 + \frac{w^2 r^2}{c^2}\right)^{\frac{3}{2}}}
$$
\n
$$
(42)
$$

6. Angle $\mu = \angle AOB$ in Figure 1 is traversed by the signal whilst it travels the length *r*, and is given by $_{n} = wt$ where $t = \frac{r}{t}$ or $_{n} = \frac{wr}{t} = \tan \{ \frac{1}{t} = \sinh \frac{wr'}{t} \}$ versed by the signal whilst it travels the length
 $=\frac{r}{c}$ or $_{n} = \frac{wr}{c} = \tan \left\{ = \sinh \frac{wr'}{c} \right\}$. Angle $_{n}$ can

sult of many revolutions.

imeter towards the center, then its path will be

s OB' in Figure 1.

1 loo e signal whilst it travels the length
 $\frac{vr}{c} = \tan \left\{ 1 - \frac{wr'}{c} \right\}$. Angle , can

revolutions.

ds the center, then its path will be

ure 1.

de following Figure 3 by the signal whilst it travels the length
 $\frac{w}{c} = \frac{wr}{c} = \tan \left(\frac{1}{c} \right) = \sinh \frac{wr'}{c}$. Angle , can

many revolutions.

towards the center, then its path will be

in Figure 1.

ike the following Figure 3

become very big and even be the result of many revolutions.

- 7. If the signal originates from the perimeter towards the center, then its path will be symmetric with respect to the radius OB' in Figure 1.
- 8. A plot of *r'* for increasing time will look like the following Figure 3

Figure 3 The path of a light signal originating from the center of a rotating frame as seen by the non rotating observer O' . Numbers on the axes are nonessential since scaling changes with w .

8. Generalization to three Dimensions

An observer O at the center O of a rotating frame K is rotating with the frame. He carries a rod (similar to the one we used in the 2 dimensional case above) pointing radially but with an angle \leftarrow with respect to the z axis. As we said in the 2-dimensional case, the rod is simply an artifact to help us imagine the situation. It suffices to say that observer *O* sends a signal from the origin with an angle \lt to the z axis. The situation is depicted in Figure 4

Figure 4 The signals originate from O and move along the rod OA for observer *O*, who rotates with the frame. The non-rotating observer O' sitting at O will see the signal travel a helical path OCB' while the rod travels from position OA to OB until the signal traverses the rod. The projection of the velocity vector B'F of the signal on the plane of rotation is B'F', as observer *O* traversed by the rod is $G\hat{O}H_{\pi}$ ^{*} Figure 4 The signals originate from O and move along the rod OA for observer O, who rotates
with the frame. The non-rotating observer O' sitting at O will see the signal travel a helical path
of S' while the rod ravels fr 2 2 **Example 12**
 c axis

ignals originate from O and move along the rod

The non-rotating observer O' sitting at O will set

rod travels from position OA to OB until the signal

velocity vector BF of the signal on the plan

8.1 Non-rotating nearby observer *O*

Suppose a signal with velocity c originates from O with angle \leftarrow with respect to the axis of rotation as seen by observer *O* and is directed towards A through the rod OA (see

observer *O* is $c \sin \leftarrow c \frac{m}{\sqrt{m}}$ and the *z* component is $c \cos \leftarrow c$ $\epsilon = c \frac{d}{\sqrt{m^2 + z^2}}$ and the z component is $c \cos \epsilon$. Sup $+z^2$ and the *z* component is *c* cos . Suppose now that *O*

rotates with the rod OA with frequency ϵ' as seen by another observer O' that sits on top of O but does not rotate with O. Let also ϵ be the frequency that O thinks his frame, K, **Figure 4** The signals originate from O and move along the rod OA for observer *O*, who rotates with the first come the rotating observer *O* sitting at O will see the signal travel a heired a performance of the role of t meaning to define the angular velocity of the frame *K* as $w = 2f \epsilon$. Observer *O'* will see the signal travel the helical path OCB' in the same time that it takes to traverse the rod for observer *O*, while the rod moves from position OA to OB. For him light travels along the helical path with the same velocity *c* and the *z* component equals that of observer *O* (*c* cos \leftarrow). The velocity vector for observer *O'* is B[']F and it makes an angle \leftarrow with the zaxis. The projection B'F' of the velocity vector B'F (tangential to the helical path for observer O') on the plane of rotation is denoted as \hat{p}_{proj} and

$$
\gamma_{\text{proj}} = c \sin \leftarrow \tag{43}
$$

y vector B^T (tangential to the helical path for

s denoted as γ_{proj} and
 $\gamma'_{proj} = c \sin \leftarrow$ (43)

cylindrical coordinates) BT' for Observer O' is { .

tion of the velocity vector of the signal from the

city in the radia The angle between it and the radial (in cylindrical coordinates) $B E'$ for Observer O' is $\{$. This angle is called *the angle of deflection* of the velocity vector of the signal from the radial direction. Let us denote the velocity in the radial direction (in cylindrical coordinates) as observer *O'* sees it by \hat{C}' . Then $\hat{C}' = \hat{C}_{proj} \cos \{$ and therefore, costrainty vector BT (tangential to the helical path for

is denoted as γ_{proj}^r and
 $\gamma_{proj}^r = c \sin \kappa$ (43)

in cylindrical coordinates) BE' for Observer O' is { .
 ection of the velocity vector of the signal from the
 city vector BF (tangential to the helical path for

i is denoted as \int_{proj}^{r} and
 $\int_{proj}^{r} = c \sin \xi$

(in cylindrical coordinates) BE' for Observer O' is {
 lection of the velocity vector of the signal from the

elocity vector B^T (tangential to the helical path for

denoted as \int_{proj}^{∞} and
 $\int_{proj}^{\infty} = c \sin \cos \left(\frac{43}{\cos \theta}\right)$

cylindrical coordinates) BT for Observer O' is { .
 on of the velocity vector of the signal from the

rity **extor B F** (tangential to the helical path for

noted as $\int_{\text{proj and}}^{\infty}$ and
 $\int_{\text{F}} = c \sin \left(\frac{43}{\text{m}} \right)$

indicial coordinates) B E³ for Observer *O'* is {.

of the velocity vector of the signal from the

in the elocity vector BT (tangential to the helical path for

ion is denoted as \int_{proj}^{∞} and
 $\int_{proj}^{\infty} = c \sin x$ (43)

al (in cylindrical coordinates) BT for Observer O' is { .

leflection of the velocity vector of the signal ocity vector B^T (tangential to the helical path for

on is denoted as ²_{*frog*} and

1 (in cylindrical coordinates) BT' for Observer *O'* is { .
 Hection of the velocity vector of the signal from the

velocity in axis. The projection BF' of the velocity vector BF (tangential to the belical path for observer O') on the plane of rotation is denoted as $\frac{V_{\text{avg}}}{V_{\text{avg}}} = \text{csin}$ (43)
The angle between it and the radial (in cylindr *w* continues) BE' for Observer O'

drical coordinates) BE' for Observer O'

f the velocity vector of the signal from the radial direction (in cylindrical

len $\gamma' = \gamma_{proj} \cos \{\}$ and therefore,

in < cos {

for observer O' *com* of the velocity vector of the signal from the
 con of the velocity vector of the signal from the
 c', Then $\int_{-\infty}^{x} = \int_{\text{proj}}^{\infty} \cos \{$ and therefore,
 c sin { = $\int_{-\infty}^{x}$ cos { (44)

grad for observer *O n* = c sin < (43)

indrical coordinates) BE' for Observer O' is { .

(43)

indrical coordinates) BE' for Observer O' is { .

if the velocity vector of the signal from the

in the radial direction (in cylindrical

Then $\$ = csin < (43)

endrical coordinates) BE' for Observer O' is { .

of the velocity vector of the signal from the

n the radial direction (in cylindrical

Fhen $\degree t = \degree_{proj} \cos \{$ and therefore,

sin < cos { (44)

1 for observer This angle is called *the angle of deflection* of the velocity vector of the signal

ardial direction. Let us denote the velocity in the radial direction (in cylind

coordinates) as observer *O'* sees it by \int_a^b . Then

$$
\hat{c}' = c \sin \varsigma \cos \left\{ \tag{44}
$$

The tangential component of the light signal for observer O' on the plane of rotation will be perpendicular to \hat{a} and will be given by

$$
c\sin\leftarrow\sin\left\{\right. = w'...'\right.\tag{45}
$$

Also, by the definition of angular velocity,

$$
\hat{\boldsymbol{\zeta}}' = \boldsymbol{w}' \dots' \tag{46}
$$

As usual, the primed quantities \int_{proj} , \int' , \int' , \ldots' , w' are as observer *O'* perceives them. For the same reasons (Lorentz contraction of perimeter) as in the two dimensional case we require that (14) holds. Namely, [
erver O' on the plane of row and the plane of row interesting the plane of row of the plane of row are as observer O' perceived the plane since $\frac{2 \cdot \frac{r^2}{c^2}}{c^2}$ { = w'...'

w'...'

...', w' are as observer O' pe

perimeter) as in the two dim
 $\sqrt{1 - \frac{w'^2 ... r^2}{c^2}}$

ve find
 $\frac{2...^2c^2}{a^2}$
 $+ ...^2w^2...^2$

ume as (16), as expected. 2 2 2 2 2 The tangential component of the light signal for observer O' on the plane of rotation will
be perpendicular to $\frac{8}{1}$ and will be given by
exists ($\frac{1}{2}$ w'...' (45)
Also, by the definition of angular velocity,
As u (44)

tation will

(45)

(46)

(88)

res them.

mal case we

(47)

(48)

(49)

, ...', \hat{C} , \hat{C}

(50) The tangential component of the light signal for ob

be perpendicular to \tilde{C} and will be given by
 \tilde{C} is \tilde{C} and will be given by
 \tilde{C} is \tilde{C} and will be given by
 \tilde{C} is \tilde{C} and $\tilde{C$ in { = w'...' (45)

= w'...' (46)

= w'...' (46)

', ...', w' are as observer O' perceives them.

of perimeter) as in the two dimensional case we
 $\frac{1}{\sqrt{1-\frac{w'^2...r^2}{c^2}}}$ (47)

we find
 $\frac{w^2...^2c^2}{(1-\frac{w'^2...^2c^2}{c^2$ \therefore = w...
 \therefore ', '', ...', w' are as observer *O'* perceives them.

on of perimeter) as in the two dimensional case we
 $2f \sqrt{1 - \frac{w'^2 - v^2}{c^2}}$ (47)
 $f'v$ we find
 $\frac{w^2 - v^2}{c^2 - w^2 + ...^2 w^2 - v^2}$ (48)

the same a

$$
2f'...' = 2f...\sqrt{1 - \frac{w'^2...'}{c^2}}
$$
 (47)

$$
w'^2 = \frac{w^2 \dots^2 c^2}{c^2 \dots^2 + \dots^2 w^2 \dots^2}
$$
 (48)

Finally,

$$
\hat{a}' = \frac{d \dots'}{dt} \tag{49}
$$

as (16), as expected.
tions in five unknowns: {,
 $\frac{1}{2}$, $\frac{2}{\sin^2 2}$ traction of perimeter) as in the two dimensional c
 $f'_{\cdots} = 2f_{\cdots} \sqrt{1 - \frac{w'^2 \cdots'^2}{c^2}}$
 $w' = 2f'v$ we find
 $w'^2 = \frac{w^2 \cdots^2 c^2}{c^2 \cdots'^2 + \cdots^2 w^2 \cdots'^2}$

48) is the same as (16), as expected.
 $\int_0^1 = \frac{d \cdots'}{dt}$

9 For the same reasons (Lorentz contraction of perimeter) as in the two dimensional case we
require that (14) holds. Namely,
 $2f'_{\cdots} - 2f_{\cdots} \sqrt{1 - \frac{w'^2 \cdots^2}{c^2}}$ (47)
Solving for f' and substituting in $w'^2 = \frac{w^2 \cdots^$ 2 2 g in $w' = 2f'v$ we find
 $w'^2 = \frac{w^2...^2c^2}{c^2...^2 + ...^2w^2...^2}$

hat (48) is the same as (16), as expected.
 $\frac{d}{dt} = \frac{d}{dt}$

(3), (49) are five equations in five unknowns: {, w',
 $\sin \left\{ \frac{w'...'}{c \sin \left(\frac{w'^2...^2}{c^2 \sin^2 \left$ 'v we find
 $w^2 - c^2$
 $\frac{w^2 - c^2}{a^2 + \frac{w^2 w^2 - c^2}{b^2}}$ (48)

e same as (16), as expected.
 $= \frac{d...'}{dt}$ (49)

ve equations in five unknowns: {, w', ...', ^', ^', ^'
 $= \frac{w'...'}{c \sin \theta}$ (50)
 $\sqrt{1 - \frac{w'^2 ...^2}{c^2 \sin^2 \theta}}$ *v* we find
 $w^2 ...^2c^2$
 $\frac{r^2 + ...^2w^2...^2}{r^2 + ...^2w^2...^2}$ (48)

same as (16), as expected.
 $= \frac{d...'}{dt}$ (49)
 $= \frac{w'...'}{c \sin \left(\frac{w'^2}{c^2 \sin^2 \left(\frac{w'^2}{c^2 \sin^2 \left(\frac{w'^2}{c^2 \sin^2 \left(\frac{w'^2}{c^2 \sin^2 \left(\frac{w'^2}{c^2 \sin^2 \left(\frac{w'^2}{c^2 \sin^$ $\begin{aligned}\n\frac{d^2 u}{dx^2} &= 2f \cdot \sqrt{1 - \frac{w^2 - \frac{u^2}{c^2}}{c^2}}$ (47)
 $v^2 = 2f'v \text{ we find} \\
\frac{w^2 - \frac{2}{c^2}}{c^2 - \frac{u^2 + \frac{u^2}{c^2}v^2 - \frac{u^2}{c^2}}{dt}}$ (48)

8) is the same as (16), as expected.
 $\frac{d^2 u}{dx^2} = \frac{d^2 u'}{dt}$ (49)

(a9)
 Solving for f' and substituting in $w' = 2f'v$ we find
 $w'^2 = \frac{w^2...^2c^2}{c^2...^2 + ...^2w^2...^2}$

where $w = 2f v$ and we note that (48) is the same as (16), as expected.

Finally,

Equations (44), (45), (46), (48), (49) are $c^2 \dots^{r^2} + \dots^{2} w^2 \dots^{r^2}$

(8) is the same as (16), as expected.
 $\therefore \frac{d}{dr} = \frac{d \dots^{r}}{dt}$

(a) are five equations in five unknowns: {,
 $\sin \left\{ = \frac{w' \dots^{r}}{c \sin \left(\frac{w'^2 \dots r^2}{c^2 \sin^2 \left(\frac{w}{c}\right)^2}\right)}\right\}$
 $\cos \left\{ = \sqrt{1 - \frac$ 1 sin *e*²_{*xx*}²*x*_{*x*}*x*²*x*^{*x*}_{*x*}²*z*^{*x*}_{*x*}^{*x*}*y*²*x*^{*x*}_{*x*}^{*x*}*y*^{*x*}*x*^{*x*}*y*^{*x*}*x*^{*x*}*y*^{*x*}*x*^{*x*}*y*^{*x*}*x*^{*x*}*y*^{*x*}*x*^{*x*}*y*^{*x*}*x*^{*x*}*y*^{*x*}*x*^{*x*}*y*^{*x*}*x*^{*x*}*y* s the same as (16), as expected.
 $\frac{d}{dt} = \frac{d}{dt}$ (49)
 $\frac{d}{dt}$ re five equations in five unknowns: {, w', ...', ",', ",'
 \therefore
 $\sin\left(-\frac{w^2 - v^2}{c^2 \sin^2 x}\right)$ (50)
 $\frac{1}{\sqrt{c^2 + v^2}} = \sqrt{\frac{1 - \frac{w^2 - v^2}{c^2 \sin^2 x}}{1 - \frac{$ $rac{m^2w^2m'^2}{dt}$

ie as (16), as expected.
 $rac{n'}{dt}$ (49)

uations in five unknowns: {, w', ...', ^_', ^_', ^_'
 $\frac{v'...'}{sin x}$ (50)
 $rac{w^2...^2}{w^2...^2}$ (51)

3)
 $rac{w^2...^2}{...^2w^2 sin^2 x}$ (52)

ie the above relation as,
 $rac{x$ $\frac{1}{c^2} = \frac{w^2 \dots^2 c^2}{c^2 \dots^2 + \dots^2 w^2 \dots^2}$ (48)

is the same as (16), as expected.
 $\frac{1}{c^2} = \frac{d \dots'}{dt}$ (49)

are five equations in five unknowns: {, w', ...', ',',',','

sin { $\frac{1}{c^2 \sin^2 \left(\frac{w^2 \dots v^2}{c^2 \sin^2 \left(\frac$ $w'^2 = \frac{w'^2 - w^2}{c^2 - u^2 + ...^2 w^2 - u^2}$ (48)

tt (48) is the same as (16), as expected.
 $\frac{w'}{dt} = \frac{d}{dt}$ (49)

(49) are five equations in five unknowns: {, w', ...', '..', '.'
 $\sin \left(\frac{w' - v}{\cosh x} \right)$
 $\cos \left(\frac{1}{2} \sqrt{1 - \$ (48)
 $= \frac{1}{c^2...^2 + ...^2w^2...^2}$ (48)

is the same as (16), as expected.
 $\int_{c}^{c} = \frac{d...'}{dt}$ (49)

are five equations in five unknowns: {, w', ...', ',',',','
 \int_{c}^{c}
 $\sin \left(\frac{1}{c} \frac{w'...'}{c^2 \sin^2 \sqrt{\left(\frac{w'^2}{c^2} \right)^2$

$$
\sin\left\{\frac{w'...'}{c\sin\left(\frac{x}{c}\right)}\right\}\tag{50}
$$

and hence,

$$
\sin \left\{ = \frac{w'...'}{c \sin \left(\frac{w'^2...'}{c^2 \sin^2 \left(\frac
$$

$$
\cos\left\{ = \sqrt{1 - \frac{w^2 \dots^2}{(c^2 + \dots^2 w^2)\sin^2 \varsigma}}\right\}
$$
(52)

$$
\cos\left\{\right. = \sqrt{\frac{1 - w^2 t^2 \cos^2\left(\frac{1}{2}\right)^2}{1 + w^2 t^2 \sin^2\left(\frac{1}{2}\right)^2}} = \sqrt{\frac{c^2 - w^2 z^2}{c^2 + w^2 \cdot z^2}}
$$
\n(53)

 $z \leq$ – \int (or *w'*...' \leq *c* sin <) *w*

with the condition that $1 - w^2 t^2 \cos^2 \leftarrow 20$ (or $z \leq \frac{c}{w}$) (or $w' ...' \leq c \sin \leftarrow 2$)

(Note that $\cos \leftarrow 10$ either when $z = \frac{c}{w}$ or when $w ...$ goes to ∞)

Substituting in (44)
 $\int_0^1 \frac{1 - w^2 t^2 \cos^2 \leftarrow 1}{1 + w^2 t^2$ with the condition that $1 - w^2 t^2 \cos^2 \le 20$ (or $z \le \frac{c}{w}$) (or $w' ...' \le c \sin \le 1$)

(Note that $\cos \left(\frac{c}{w} \right)$ either when $z = \frac{c}{w}$ or when $w...$ goes to ∞)

Substituting in (44)
 $\int_0^{\infty} z - c \sin \le \cos \left(\frac{1 - w^2 t^2$ $z =$ - or when w... goes to ∞) *w* $=$ or when *w* goes to ∞) Substituting in (44) with the condition that $1 - w^2 t^2 \cos^2 \leftarrow 20$ (or $z \leq \frac{c}{w}$) (or $w' ... \leq c$;

(Note that $\cos \leftarrow 10$ either when $z = \frac{c}{w}$ or when $w ...$ goes to ∞)

Substituting in (44)
 $\int_{-a}^{a} = c \sin \leftarrow \cos \leftarrow 10 \Rightarrow c \sin \leftarrow \sqrt{\frac{1 - w^2 t$

$$
1 - w^{2}t^{2} \cos^{2} \le 20 \text{ (or } z \le \frac{c}{w}) \text{ (or } w'...' \le c \sin \le 20
$$
\n
$$
\text{ner when } z = \frac{c}{w} \text{ or when } w... \text{ goes to } \infty
$$
\n
$$
\int_{0}^{\infty} \frac{1 - w^{2}t^{2} \cos^{2} \le 20}{1 + w^{2}t^{2} \sin^{2} \le 200 \text{ s}^{2} \text{ s}^{2}} = \frac{d...'}{dt},
$$
\n
$$
= \frac{d...'}{dt},
$$
\n
$$
y' = c \sin \le \left(\frac{1 - w^{2}t^{2} \cos^{2} \le 200 \text{ s}^{2} \text{ s}^{2}}{1 + w^{2}t^{2} \cos^{2} \le 200 \text{ s}^{2} \text{ s}^{2}} \right)
$$
\n
$$
(55)
$$

and since $\hat{a}' = \frac{d \dots'}{dt} \frac{dt}{dt'} = \frac{d \dots'}{dt},$

condition that
$$
1 - w^2 t^2 \cos^2 \left(\frac{1}{2} \right)
$$
 (or $x \leq \frac{c}{w}$) (or $w' \leq c \sin \left(\frac{1}{2} \right)$

\ncos $\left\{ = 0$ either when $z = \frac{c}{w}$ or when $w \leq \csc \left(\frac{1}{2} \right) \left(\frac{1}{2} \right)^2 \left(\frac{1}{2} \right$

It is convenient to represent the integral in the RHS of (55) as a function of \leftarrow and t . So we define,

$$
u' = c \sin \left(\int_{0}^{\frac{\pi}{2}} \sqrt{\frac{1 - w t \cos^2 x}{1 + w^2 t^2 \sin^2 x}} dt \right)
$$
\n
$$
u'' = c \sin \left(\int_{0}^{\frac{\pi}{2}} \sqrt{\frac{1 - w^2 t^2 \cos^2 x}{1 + w^2 t^2 \sin^2 x}} dt \right)
$$
\n
$$
I(\cos t) = \int_{0}^{t} \sqrt{\frac{1 - w^2 t^2 \cos^2 x}{1 + w^2 t^2 \sin^2 x}} dt \qquad (56)
$$
\n
$$
u'' = c \sin \left(\frac{\pi}{2} \right)
$$
\n
$$
u'' = c \sin \left(\frac{\pi}{2} \right)
$$
\n
$$
u'' = \frac{c}{w} \arcsin \frac{w}{c}
$$
\n
$$
u'' = \frac{w}{c}
$$
\n
$$
u''' = \frac{w}{\sin \left(\sqrt{c^2 + w^2 w^2} \right)} = \frac{w}{\sqrt{1 + w^2 t^2 \sin^2 x}}
$$
\n
$$
u''' = \frac{1}{c^2 \sin^2 x} = \sqrt{\frac{c^2 - w^2 z^2}{1 + w^2 t^2 \sin^2 x}} = \sqrt{\frac{1 - w^2 t^2 \cos^2 x}{1 + w^2 t^2 \sin^2 x}}
$$
\n
$$
u''' = \frac{1}{\cos^2 x} = \frac{1}{\sin \left(\frac{\pi}{2} \right)^2} = \frac{1}{\sqrt{1 + w^2 t^2 \sin^2 x}}
$$
\n
$$
u''' = \frac{1}{\cos^2 x} = \frac{1}{\sin \left(\frac{\pi}{2} \right)^2} = \frac{1}{\sqrt{1 + w^2 t^2 \sin^2 x}}
$$
\n
$$
u''' = \frac{1}{\cos^2 x} = \frac{1}{\sin \left(\frac{\pi}{2} \right)^2} = \frac{1}{\sqrt{1 + w^2 t^2 \sin^2 x}}
$$
\n
$$
u''' =
$$

Then we may rewrite (55) as

$$
\dots' = c \sin \left(I(\langle , t \rangle) \right) \tag{57}
$$

Note that for $\epsilon = \frac{3}{5}$ 2^{7} $\epsilon = \frac{f}{2}$, (57) becomes

Hence the two
$$
u = 0
$$
 either when $z = \frac{c}{w}$ or when w_{∞} goes to ∞)
\nSubstituting in (44)
\n
$$
u' = c \sin \cos \left(1 - \frac{1}{2} \arctan \left(1 + \frac{1}{2
$$

dimensional case (recall (19)).

Below we summarize the following equations that are useful for calculations,

$$
-\frac{1}{2} \cosh \left(\cos \left(-\frac{1}{2}\right)\right) + \frac{1}{2} \sin \left(-\frac{1}{2}\right) \cos \left(-\frac{1}{2}\right) + \frac{1}{2} \cos \left(-\frac{1}{2}\right) \cos \left(-\frac{1}{2}\right)
$$
\n
$$
-\frac{1}{2} \int \frac{dt}{dt} = \frac{d \ln t}{dt}, \qquad ...' = c \sin \left(-\frac{t}{2}\right) \sqrt{\frac{1 - w^2 t^2 \cos^2 \left(-\frac{1}{2}\right)t}{1 + w^2 t^2 \sin^2 \left(-\frac{1}{2}\right)t}} \qquad (55)
$$
\nto represent the integral in the RHS of (55) as a function of $\left(-\cos \left(-\frac{1}{2}\right)\right) \sin \left(-\frac{1}{2}\right) \sin \left(-\frac{1}{2}\right) \sin \left(-\frac{1}{2}\right) \cos \left(-\frac{$

$$
\cos\left\{\right. = \sqrt{1 - \frac{w'^2 \dots'^2}{c^2 \sin^2 \varsigma}} = \sqrt{\frac{c^2 - w^2 z^2}{c^2 + \dots^2 w^2}} = \sqrt{\frac{1 - w^2 t^2 \cos^2 \varsigma}{1 + w^2 t^2 \sin^2 \varsigma}}
$$
(59)

$$
\tan\left\{\right. = \frac{wt}{\sqrt{1 - w^2 t^2 \cos^2 \left(1 - w^2\right) \sin^2 \left(1 - w^2\right) \sin^2 \left(1 - w^2\right)}} = \frac{w' ...'}{\sqrt{c^2 \sin^2 \left(1 - w^2\right) \sin^2 \left(1 - w^2\right) \sin^2 \left(1 - w^2\right)}}\tag{60}
$$

$$
\frac{w'...'}{w...} = \frac{2f \in ...'}{2f \in ...} = \frac{f'...'}{f...} = \frac{c}{\sqrt{c^2 + w^2 ...^2}} = \cos\left\{\right\}_{z=0} = \cos\left\{\right\}_{x=\frac{f}{2}}
$$
(61)

$$
I(\langle ,t) = \int_{0}^{t} \cos \{ \, dt \tag{62}
$$

8.1.1 Plot of signals for observer *O*

x(x) $\frac{d}{dz} = c_0 \int_0^z \frac{dt}{\sqrt{1 + w^2 t^2}} = \frac{C}{w}$ arcsinh $w = \frac{C}{w}$ arcsinh $\frac{w}{c}$, which is what we found for the two dimensional case (recall (19).

Below we summarize the following equations that are useful for ca number) and $w \neq 0$, and $0 \lt \lt \leq \frac{1}{2}$ we want to calculate $I(\lt, t)$ 2 and \overline{a} f are constant to colored to $I(\cdot, \cdot)$ Iteshim $w_1 = \frac{w}{w}$ atteshim $\frac{w}{c}$, which is what we found for the two
 $\frac{w}{\sin \left(\frac{w}{c} \right)^2} = \frac{w}{\sin \left(\sqrt{c^2 + ...^2 w^2} \right)^2} = \frac{wr}{\sqrt{1 + w^2 r^2 \sin^2 c}}$ (58)
 $\frac{1 - \frac{w^2 - w^2}{c^2 \sin^2 c}}{1 - \frac{w^2 - w^2}{c^2 \sin^2 c}} = \sqrt{\frac{e^2 -$ Make the substitution

$$
k = i \cot \leftarrow \tag{63}
$$

And

$$
x = iwt \sin \leftarrow \tag{64}
$$

$$
k = i \cot \leftarrow
$$
\n(63)
\n
$$
x = iwt \sin \leftarrow
$$
\n(64)
\n6) becomes
\n
$$
I(\leftarrow, t) = \frac{1}{i w \sin \leftarrow} \int_{0}^{x} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} dx
$$
\n(65)
\nis an incomplete Elliptic integral of the second kind denoted
\ncan be written as
\n
$$
I(\leftarrow, t) = \frac{1}{i w \sin \leftarrow} E(i \cot \leftarrow, i w t \sin \leftarrow)
$$
\n(66)
\nulations.
\nral of (55) for $0 \le t \le \frac{1}{w \cos \leftarrow}$ (the limit allowed by the
\n) we find the max value that ...' can take for each particular w

And
 $x = i \cot \leftarrow$ (63)

Where $z = \sqrt{-1}$ and then (56) becomes
 $I(\leftarrow, t) = \frac{1}{i w \sin \leftarrow} \int_{0}^{x} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} dx$ (65)

But the integral on the RHS is an incomplete Elliptic integral of the second kind denoted

as $E(k, x)$ But the integral on the RHS is an incomplete Elliptic integral of the second kind denoted

$$
I(\langle ,t) = \frac{1}{iw \sin \langle} E(i \cot \langle , iwt \sin \langle \rangle) \tag{66}
$$

which can be used for calculations.

And
 $x = iwt \sin \cos \theta$

Where $z = \sqrt{-1}$ and then (56) becomes
 $I(\sin \cos \theta) = \frac{1}{iw \sin \cos \theta} \int_{0}^{\pi} \sqrt{\frac{1}{1}}$

But the integral on the RHS is an incomplete Elliption

as $E(k, x)$. Therefore, (65) can be written as
 $I(\sin \theta) = \frac{1}{iw \sin$ If we take the definite integral of (55) for $0 \le t \le \frac{1}{\sqrt{1 - \frac{1}{n}}}$ (the limit allowed by the $\cos \leftarrow$ $t \leq$ $\frac{1}{t}$ (the limit a $W\cos\left(\frac{\sin\theta + \sin\theta + \cos\theta}{\sin\theta}\right)$ and

and \langle . Namely,

And
\n
$$
x = iwt \sin \leftarrow
$$
 (63)
\nWhere $z = \sqrt{-1}$ and then (56) becomes
\n $I(\leftarrow, t) = \frac{1}{iwt \sin \leftarrow} \int_{0}^{x} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} dx$ (65)
\nBut the integral on the RHS is an incomplete Elliptic integral of the second kind denoted
\nas $E(k, x)$. Therefore, (65) can be written as
\n $I(\leftarrow, t) = \frac{1}{iwt \sin \leftarrow} E(i \cot \leftarrow, iwt \sin \leftarrow)$ (66)
\nwhich can be used for calculations.
\nIf we take the definite integral of (55) for $0 \le t \le \frac{1}{w \cos \leftarrow}$ (the limit allowed by the
\ncondition $1 - w^2 t^2 \cos^2 \leftarrow 20$) we find the max value that ...' can take for each particular w
\nand \leftarrow . Namely,
\n $\frac{1}{w \cos \leftarrow} \int_{0}^{\frac{1}{w \cos \leftarrow}} \sqrt{\frac{1 - w^2 t^2 \cos^2 \leftarrow}{1 + w^2 t^2 \sin^2 \leftarrow dt}} dt = c \sin \leftarrow I(\leftarrow, \frac{1}{w \cos \leftarrow})$ (67)
\nThis says that for any fixed \leftarrow , the signals in the radial direction are bounded (see Figure
\n5). The signals bend and rotate until they reach $z = \frac{c}{w}$ at $t_m = \frac{1}{w \cos \leftarrow}$, the angle of
\ndeflection becomes 90° degrees, the radial velocity diminishes to zero and the radial
\ndistance of the signal becomes.

This says that for any fixed \langle , the signals in the radial direction are bounded (see Figure 5). The signals bend and rotate until they reach $z = \frac{c}{x}$ at $t_m = \frac{1}{x_m}$, the angle of

 $=\frac{c}{w}$ at $t_m = \frac{1}{w \cos \varsigma}$, the angle of $t_{\rm m} = \frac{1}{w \cos \varsigma}$, the angle of

deflection becomes 90º degrees, the radial velocity diminishes to zero and the radial

$$
\dots'_{m}(\varsigma) = c \sin \varsigma I_{m}(\varsigma)
$$
\nwhere, we denote $I_{m}(\varsigma) = I(\varsigma, \frac{1}{w \cos \varsigma})$

\n(68)

 $I(\varsigma, t) = \frac{1}{i\nu \sin \varsigma_0^2} \int_0^1 \frac{1 - k^2 x^2}{1 - x^2} dx$ (65)

But the integral on the RHS is an incomplete Elliptic integral of the second kind denoted

as $E(k, x)$. Therefore, (65) can be written as
 $I(\varsigma, t) = \frac{1}{i\nu \sin \varsigma_$ by (65) can be written as
 $I(\varsigma, t) = \frac{1}{i w \sin \varsigma} E(i \cot \varsigma, i w t \sin \varsigma)$

r calculations.

r calculations.

e integral of (55) for $0 \le t \le \frac{1}{w \cos \varsigma}$ (the lim
 $s^2 < 0$) we find the max value that ...' can take
 $s^2 < 0$) we ore, (65) can be written as
 $I(\langle \cdot, t \rangle) = \frac{1}{i w \sin \langle \cdot \rangle} E(i \cot \langle \cdot \rangle)$

for calculations.

inte integral of (55) for $0 \le t \le \frac{1}{w \cos \langle \cdot \rangle}$
 $\cos^2 \langle \cdot \rangle = 0$ we find the max value t
 $\frac{1}{\sqrt{1 + w^2 t^2} \sin^2 \langle \cdot \rangle}$
 $\sin \sqrt{1$ **EXECUTE:** $\frac{1}{w \sin x} = \frac{1}{0} \sqrt{1-x}$

(65) can be written as
 $I(\langle t \rangle) = \frac{1}{iw \sin \langle \rangle} E(i \cot \langle \langle \langle \rangle, i \rangle + i \sin \langle \rangle)$

calculations.
 \therefore integral of (55) for $0 \le t \le \frac{1}{w \cos \langle \rangle}$ (the limit and the max value that \therefore can However, for $\epsilon = \frac{f}{2}$ we come back to the two dimensional case and the signal is not bounded. It expands slowly all the time logarithmically and again the radial velocity tends to zero. If we take the definite integral of (55) for $0 \le t \le \frac{1}{w \cos \epsilon}$ (the limit allowed by the condition $1 - w^2t^2 \cos^2 \epsilon \ge 0$) we find the max value that ...' can take for each particular w and ϵ . Namely,
 $\frac{1}{\sqrt{6}} \left(\epsilon$ Further, because of $\cos \theta$ *w* c θ of θ ($\left\langle \frac{1}{w \cos \varsigma} \right\rangle$ (67)

1 are bounded (see Figure
 $\frac{1}{w \cos \varsigma}$, the angle of

to zero and the radial

(68)

see and the signal is not

in the radial velocity tends

ey are at $z_m = \frac{c}{w}$. After
 $\frac{c}{w}$, where $\frac{w}{2}$ $\left(\frac{w}{2} - e \sin \alpha\right)$ $\frac{1}{\sqrt{2}} \left(\frac{1 - w^2 t^2 \cos^2 \alpha}{1 + w^2 t^2 \sin^2 \alpha} dt - e \sin \alpha t (\alpha, \frac{1}{w \cos \alpha})\right)$ (67)

This says that for any fixed c, the signals in the radial direction are bounded (see Figure

5). The signals be

 $z_{\rm m} = \frac{c}{\sqrt{2}}$. After *w*

that, the signals are not observable by *O*

$$
w'_{m\cdots m} = c\sin\left(\frac{69}{2}\right)
$$

and hence at \mathcal{L}_{m} the angular velocity of the signals w'_{m} for observer *O'* is

$$
w'_{\mathbf{m}} = \frac{c}{w'_{\mathbf{m}}} \tag{70}
$$

In order for the normal Lorentz contraction of the perimeter to hold we must satisfy (47) and (48) which lead to

$$
\frac{f'}{f} = \frac{...}{...} \frac{c}{\sqrt{c^2 + w^2 ...^2}}
$$
\n
$$
\frac{1}{\sqrt{c^2 + w^2 ...^2}}}
$$
\n
$$
\frac{1}{\sqrt{c^2 + w^2 ...^2}}
$$
\n
$$
\frac{1}{\sqrt{c^2 + w^2 ...^2}}}
$$
\n(71)

The effect of *w* is to scale down distances as it increases. The faster the body rotates, the faster the signals bend making tighter revolutions closer to the body.

The signals are not limited in the radial direction but cover the whole space allowed by

 $|z| \leq \frac{c}{z}$, as angle < varies from 0 to 2*f w* \sim *w* $\$ \leq \leq \leq as angle \lt varies from 0 to 2f

In Figure 5 we plot the path of the signals in the region $0 \le z \le \frac{c}{c}$ for several signals with *w* (71)
The faster the body rotates, the
o the body.
The whole space allowed by
 $\le z \le \frac{c}{w}$ for several signals with
 ≤ 0 are symmetric with respect different values of $\left\langle \right\rangle$. The paths are for the region $-\frac{c}{\leq} \leq 0$ are symmetric with respect *w* ^{2_n, 2} (71)

reases. The faster the body rotates, the

reloser to the body.

ut cover the whole space allowed by

gion $0 \le z \le \frac{c}{w}$ for several signals with
 $-\frac{c}{w} \le z \le 0$ are symmetric with respect to the plane of rotation.

Figure 5 Plot of signals emanating from O for observer O' as they rotate with radius \ldots' while the axes are not essential since they depend on the value of *w*.

8.2 Introduction of precession

Up till now we talked of *K* as rotating frame. However, as we plan later to place a point mass at the origin and imagine the signals it sends out, it is useful to add precession to , *K* since rotating bodies also precess as they rotate unless they are symmetrical in their axes of inertia. It is also possible to add nutation as well (if the rotating body is subject to an external force). However, we will not consider nutation in this study as it does not add much to what we want to demonstrate. The effect of precession (and nutation) is the same as if observer *O* oscillates his rod, through which the light ray travels, up and down around an angle \leftarrow from the z axis. The addition of precession (and nutation) to the case of observer O' will make him see a wavy and curved path for the signal, instead of only curved. The precession thus also justifies our preference for "ripples" in the two

dimensional case that was studied above. Referring to Figure 6, a point body at O is rotating around the axis OC. The axes OC rotates around the z axis with the angular velocity of precession Ω . The angle of inclination of OC with the z axis is Γ . The projection of AC on AB which is drawn parallel to the x axis is AD. Hence, **AD** AC and the same studied above. Referring to Figure 6, a point body at O is cotating around the axis OC. The axes OC rotates around the z axis with the angular velocity of precession Ω . The angle of inclination of above. Referring to Figure 6, a point body at O is
axes OC rotates around the z axis with the angular
gle of inclination of OC with the z axis is '₀. The
drawn parallel to the x axis is AD. Hence,
 $\frac{AD}{OA} = \frac{AC}{OA} \cos \Omega t$ d above. Referring to Figure 6, a point body at O is
axes OC rotates around the *z* axis with the angular
gle of inclination of OC with the *z* axis is '₀. The
drawn parallel to the *x* axis is AD. Hence,
 $\frac{AD}{OA} = \frac{AC$ we. Referring to Figure 6, a point body at O is
OC rotates around the z axis with the angular
of inclination of OC with the z axis is '₀. The
wn parallel to the x axis is AD. Hence,
 $=\frac{AC}{OA}\cos \Omega t$ or tan' = tan'₀ cos dimensional case that was studied above. Referring to Figure 6, a point body at O is
rotating around the axis OC. The axes OC rotates around the z axis with the angular
velocity of precession Ω . The angle of inclinatio

 $AD = AC \cos \Omega t$. It follows that $\frac{AD}{OA} = \frac{AC}{OA} \cos \Omega t$ or $\tan' = \tan' \int_0^1 \cos \Omega t$ where

Figure 6 A point body at O rotates with angular velocity *w* around axis OC which itself rotates around the z axis with inclination \prime ₀ = $\angle AOC$ and angular velocity Ω . If we let \prime = $\angle AOD$ then $\tan' = \tan'$ $\cos \Omega t$.

 O' is given by y *w* around axis OC which its
ular velocity Ω . If we let
from the z axis when the have an angle $\leftarrow +$ to the
prmulation the problem for
 $\frac{2 \pi r^2}{c^2}$
 $w'...'$ with angular velocity w around axis OC which itself rot
 $y = \angle AOC$ and angular velocity Ω . If we let ' = $\angle A$

int body at angle < from the z axis when the body c

it precesses it will have an angle < +' to the z axis
 x
 a
 b angular velocity *w* around axis OC which itself rotates
 $\angle AOC$ and angular velocity Ω . If we let ' = $\angle AOD$

body at angle < from the z axis when the body does

precesses it will have an angle < +' to t *h* angular velocity w around axis OC which itself rotates $\angle AOC$ and angular velocity Ω . If we let $\ell = \angle AOD$

body at angle ϵ from the z axis when the body does

recesses it will have an angle $\epsilon + \ell$ to the z ax **g**

ggular velocity *w* around axis OC which itself rotates
 $20C$ and angular velocity Ω . If we let ' = $\angle AOD$

y at angle < from the z axis when the body does

sesses it will have an angle < +' to the z axis.

<. In *z* and a metric is the set of the *were were the insular velocity w around axis OC which itself rotates*
 Werend angular velocity Ω . If we let ' = $\angle AOD$
 x at angle \angle from the *z* axis when the body does

sesses it will have an angle \angle + ' t

$$
2f'...' = 2f...\sqrt{1 - \frac{w'^{2}...^{2}}{c^{2}}}
$$
\n(72)

$$
c\sin(\langle +\rangle)\sin\{ = w' ...'}\tag{73}
$$

$$
\frac{d\ldots'}{dt} = c\sin(\kappa + \kappa')\cos\left\{\n\tag{74}
$$

$$
... = ct\sin(\varsigma + 1) \tag{75}
$$

$$
w' = 2f'v \tag{76}
$$

$$
z = ct \cos(\zeta + \zeta) \tag{11}
$$

$$
w = 2f v \tag{78}
$$

Solving the same way we did for the no precession case we find,

$$
\cos \left\{ = \sqrt{\frac{1 - w^2 t^2 \cos^2(\varsigma + \varsigma')}{1 + w^2 t^2 \sin^2(\varsigma + \varsigma')}} \right. \tag{79}
$$
\n
$$
\therefore \frac{1}{\varsigma} = c \int_0^t \sin(\varsigma + \varsigma') \cos \left\{ dt \right. \tag{80}
$$
\n
$$
\text{ observer } O''
$$
\n, the far away observer O'' does not see ripples, or even if

\nbut the straight line average for the radius \dots This average

And

$$
t' = c \int_{0}^{t} \sin(\varsigma + \iota) \cos\left\{ \, dt \right\} \tag{80}
$$

8.3 Non-rotating far away observer *O*

Assuming we have precession, the far away observer O^r does not see ripples, or even if he observes them he cares about the straight line average for the radius ...". This average path is given by a signal that has inclination \langle " but no precession, as if it originates from a signal with inclination \langle and $' = 0$ in the world of observer O. The light signal starting from the center with radial direction that will follow a wavy and curved path (as if the space has ripples) with velocity *c* according to observer O' , will appear to observer O'' as curved but non wavy having velocity \hat{c}_c and inclination $\langle \hat{c} \rangle$. er O'' does not see ripples,
erage for the radius ...". The
no precession, as if it originally
observer O. The light sign
a wavy and curved path (
erver O' , will appear to obtion <".
 O'' are:
 $\frac{2 \cdot m^2}{c^2}$ $2\left(1 - \frac{\sqrt{1 - w^2 t^2 \cos^2(\epsilon + t)}}{1 + w^2 t^2 \sin^2(\epsilon + t)}\right)$ (79)
 $2\left(-\frac{1}{2} \sin(\epsilon + t')\cos\left(\frac{dt}{2}\right)\right)$
 $2\left(-\frac{1}{2} \sin(\epsilon + t')\cos\left(\frac{dt}{2}\right)\right)$
 $2\left(-\frac{1}{2} \sin(\epsilon + t')\cos\left(\frac{t}{2}\right)\right)$
 $2\left(-\frac{1}{2} \sin(\epsilon + t')\right)$
 $2\left(-\frac{1}{2} \cos(\epsilon + t')\right)$
 2 (81) And

And
 $\lim_{x \to \infty} \int_0^x \sin^2(x + x) dx$
 $\lim_{x \to \infty} \int_0^x \sin^2(x + x) dx$ (80)

8.3 Non-rotating far away observer *O'*

Assuming we have precession, the far away observer *O'* does not see ripples, or even if

Assuming we have *w* $w^2t^2 \sin^2(c + 1)$
 $m(c + 1) \cos\left\{dt\right\}$
 Q''
 Q''
 Q''
 Q''
 Q''' server O''
far away observer O'' does not see ripples, or e
e straight line average for the radius ...". This ave
clination \langle " but no precession, as if it originate
) in the world of observer O . The light signal st $= c \int_0^{\pi} \sin(\varsigma + \iota') \cos\left\{ dt \right.$ (80)
 EVEC C^{*t*}

far away observer O^* does not see ripples, or even if

straight line average for the radius This average

straight line average for the radius ..." This averag 8.3 Non-rotating far away observer O^*
Assuming we have precession, the far away observer O^* does not see ripples, or even if
the observes them he cares about the straight line average for the radius ...". This avera Assuming we have precession, the far away observer O^* does not see ripples, or even if

the observes them he cares about the straight line average for the radius ...". This average

path is given by a signal that has i *x* way observer O'' does not see ripples, or even if

aight line average for the radius ...". This average

tition ϵ " but no precession, as if it originates from a

the world of observer O . The light signal startin **Example 12**
 Example 12

The equations describing the problem of observer $Oⁿ$ are: Lorentz contraction of the perimeter,

c c w

$$
\hat{c} \sin \leftarrow \sin \left\{ \mathbf{r} = w \dots \mathbf{r} \right\} \tag{82}
$$

The rate of change of \ldots " must equal the radial velocity,

$$
\frac{d\ldots''}{dt} = \hat{c}_c \sin \langle \cos \{ \, \, \rangle \} \tag{83}
$$

$$
c\cos\zeta = \hat{c}\cos\zeta''\tag{84}
$$

$$
\overline{\ldots} = ct \sin \leftarrow \tag{85}
$$

Since the average of $\prime = 0$.

Solving (81) and denoting for economy \bar{a} as simply \ldots we find

$$
\frac{l...''}{dt} = \hat{e}_c \sin \check{e}'' \cos \check{t}''
$$
\n(83)
\nion is equal for observers *O* and *O''*,
\n
$$
c \cos \check{e} = \hat{e}_c \cos \check{t}''
$$
\n(84)
\n
$$
\therefore c \cos \check{e} = \hat{e}_c \cos \check{t}''
$$
\n(85)
\n
$$
\therefore c \sin \check{t} = ct \sin \check{t}
$$
\n(86)
\n
$$
\therefore \cos \check{t} = \sin \check{t}
$$
\n(87)

Now solving (86) for \dots we find that

$$
... = \frac{m''c}{\sqrt{c^2 - w^2 \cdot w^2}}
$$
 (87)

= $\frac{c}{c} \cos \leftarrow$

er time is $\frac{c}{m}$ given by

ct sin \leftarrow

s simply ... we find
 $\frac{c}{\sqrt{c^2 + w^2 \cdot w^2}}$
 $\frac{w}{c}$
 $\frac{w}{2} - w^2 \cdot w^2$
 $\Rightarrow \frac{c}{w}$. This is also ob *c* with $\sqrt{1-\frac{c^2}{c^2}}$ (81)

sin {" = w..." (82)

dial velocity,
 c sin <"cos {" (83)

qual for observers *O* and *O*",

= $\frac{c}{c}$ cos <" (84)

ver time is $\frac{c}{c}$ given by

as simply ... we find
 $\frac{c}{\sqrt{c^2 + w$ From (87) we see that $m'' < \frac{c}{c}$ and as $m \to \infty$, $m'' \to \frac{c}{c}$. Thi *w w* must equal w_m'' , $\frac{1}{2}$, $\frac{1}{m}$, $\frac{1$ \cdots " \rightarrow $\frac{c}{ }$. This is also obvious by setting **Example 12 Contract Conduct Conduct Conduct Conduct** Conductive *w* \cdots " \rightarrow $\frac{c}{2}$ regardless of the value of \leftarrow . From (86) using (85) and taking the derivative with respect to *t* we find

$$
\frac{d...''}{dt} = \frac{c \sin \varsigma}{(1 + w^2 t^2 \sin^2 \varsigma)^{\frac{3}{2}}}
$$
(88)
\n
$$
\tan \left\{ \frac{d...'}{dt} \right\}
$$
(89)
\n
$$
\tan \left\{ \frac{d...'}{dt} \right\}
$$
(89)
\n
$$
\tan \left\{ \frac{d}{dt} \right\}
$$
(89)
\n
$$
\tan \varsigma \sin \left\{ \frac{d}{dt} \right\}
$$
(90)
\n
$$
\tan \varsigma \sin \left\{ \frac{d}{dt} \right\}
$$
(91)
\n
$$
\sin \frac{d}{dt} \left(\frac{d}{dt} \right)
$$
(92)

From (82) and (83) we have,

$$
= \frac{c \sin \left(\frac{1}{1 + w^2 t^2 \sin^2 \left(\frac{s}{1 + w^2 t^
$$

And using (86) and (88) we find

$$
an\{\,^{\prime\prime} = wt(1 + w^2 t^2 \sin^2 \cdot \,)\tag{90}
$$

Dividing (82) by (84) we find

$$
\tan \leftarrow'' \sin \left\{ \frac{w - w}{c \cos \left(1 + \frac{w}{c^2}\right)} \right\} \tag{91}
$$

And using (86) and (90) we obtain

$$
\frac{d...''}{dt} = \frac{c \sin \leftarrow}{(1 + w^2 t^2 \sin^2 \leftarrow)^{\frac{3}{2}}}
$$
(88)
32) and (83) we have,

$$
\tan \left(\frac{d}{dt} - \frac{w...}{dt} \right)
$$
(89)

$$
\frac{d}{dt} = \frac{w...}{dt}
$$
(89)

$$
\frac{d}{dt} = w t (1 + w^2 t^2 \sin^2 \leftarrow)
$$
(90)

$$
\frac{d}{dt} = w t (1 + w^2 t^2 \sin^2 \leftarrow)
$$
(91)

$$
\tan \leftarrow \frac{d}{dt} = \frac{w...}{c \cos \leftarrow}
$$
(91)

$$
\tan \leftarrow \frac{d}{dt} = \frac{w \sin \leftarrow}{\cos \leftarrow \sqrt{1 + w^2 t^2 \sin^2 \leftarrow}} \sqrt{1 + \cot^2 \left(\frac{d}{dt} \right) \right/ \left(1 + w^2 t^2 \sin^2 \leftarrow \right)^2}
$$
(92)
from (84) and using $\cos \leftarrow \frac{1}{\sqrt{1 + \tan^2 \left(\frac{d}{dt} \right) \left(1 + w^2 t^2 \sin^2 \leftarrow \right)^2}}$

$$
\frac{c}{dt} = \frac{c \cos \leftarrow}{\cos \leftarrow \frac{d}{dt} \left(1 + \frac{\tan^2 \leftarrow (1 + w^2 t^2 \sin^2 \leftarrow)^2}{\left(1 + w^2 t^2 \sin^2 \leftarrow \right)^2} \right)}{\left(1 + w^2 t^2 \sin^2 \leftarrow \right)^2}
$$
(93)

Finally from (84) and using $\cos \leftarrow \frac{1}{\sqrt{1 + \tan^2 \leftarrow}}$ we find \langle " $v' = \frac{1}{\sqrt{1 - \frac{1}{\sqrt$ $+\tan^2$ < " we find

$$
\frac{d_{\cdots}''}{dt} = \frac{c \sin \zeta}{(1 + w^2 t^2 \sin^2 \zeta)^{\frac{3}{2}}}
$$
(88)
(83) we have,

$$
\tan \left(\frac{\pi}{4} - \frac{w \cdots}{4} \right)
$$
(89)
and (88) we find

$$
\tan \left(\frac{\pi}{2} - wt(1 + w^2 t^2 \sin^2 \zeta) \right)
$$
(90)
and (90) we obtain

$$
\frac{w t \sin \zeta}{\cos \zeta \sqrt{1 + w^2 t^2 \sin^2 \zeta}} \sqrt{1 + \cot^2 \left(\frac{\pi}{2} \tan \zeta \right) \frac{\sqrt{1 + w^2 t^2 (1 + w^2 t^2 \sin^2 \zeta)^2}}{(1 + w^2 t^2 \sin^2 \zeta)^{\frac{3}{2}}}}
$$
(92)
1) and using $\cos \zeta = \frac{1}{\sqrt{1 + \tan^2 \zeta^{\frac{3}{2}}}} \text{ we find}$

$$
\hat{\zeta} = \frac{c \cos \zeta}{\cos \zeta^{\frac{3}{2}}} = c \cos \zeta \sqrt{1 + \frac{\tan^2 \zeta (1 + w^2 t^2 (1 + w^2 t^2 \sin^2 \zeta)^2)}{(1 + w^2 t^2 \sin^2 \zeta)^3}}
$$
(93)
10008 $\hat{\zeta}$ to take values greater and smaller than *c*.
ording to observer *O''* are limited in the radial direction to *c/w* which they
rotically, as we remarked in (87) and therefore lie within the cylindrical
c/w without being limited in the z direction since $z'' = z$ due to (84).

This equation allows \hat{c}_c to take values greater and smaller than *c*.

 $\tan \left(\frac{r}{\pi} \frac{W_{\text{max}}}{dt} \right)$ (89)

And using (86) and (88) we find
 $\tan \left(\frac{r}{\pi} \frac{W_{\text{max}}}{\cos \theta} \right)$ (90)

Dividing (82) by (84) we find
 $\tan \left(\frac{r}{\pi} \frac{W_{\text{max}}}{\cos \theta} \right)$ (91)

And using (86) and (90) we obtain
 \t approach asymptotically, as we remarked in (87) and therefore lie within the cylindrical surface of radius c/w without being limited in the z direction since $z'' = z$ due to (84). And using (86) and (88) we find
 $\tan \binom{x}{x} = w\binom{t}{1} + w^2\binom{t}{3} \sin^2 \left(\frac{x}{1}\right)$

(90)

Dividing (82) by (84) we find
 $\tan \binom{x}{n} = \frac{w^2}{\cos \left(\frac{x}{\sqrt{1} + w^2r^2 \sin^2 \left(\frac{x}{\sqrt{1} + w^2r^2 \sin^2 \left(\frac{x}{\sqrt{1} + w^2r^2 \sin^2 \left(\frac{x}{\sqrt{1} + w^2r$

9.1 The two-Dimensional Case for observer *O*

If we allow the angular velocity *w* to vary as a function of the radius it is like having a disc that consists of rings with small width that slide one after the other. Let for example

$$
v = w_0 f(r) \tag{94}
$$

 $(1 + w^2 t^2 \sin^2 \varsigma)^{\frac{3}{2}}$
 $\frac{1}{1 + \tan^2 \varsigma^*}$ we find
 $+ \frac{\tan^2 \varsigma (1 + w^2 t^2 (1 + w^2 t^2 \sin^2 \varsigma)^2)}{(1 + w^2 t^2 \sin^2 \varsigma)^3}$ (93)

greater and smaller than c.

e limited in the radial direction to c/w which they

ed in (87) and tant $\epsilon = \frac{1}{\cos \epsilon \sqrt{1 + w^2 t^2 \sin^2 \epsilon}} \sqrt{1 + \cot t^2 + \cot t^2}$ (a) $\frac{1}{\sqrt{1 + \tan^2 \epsilon^2}}$

Finally from (84) and using $\cos \epsilon^* = \frac{1}{\sqrt{1 + \tan^2 \epsilon^*}}$ we find
 $\frac{1}{\epsilon} = \frac{\cos \epsilon}{\cos \epsilon^*} = \cos \epsilon \sqrt{1 + \frac{\tan^2 \epsilon (1 + w^2 t^2 (1 + w^2 t^2 \sin^2 \epsilon)^2)}{(1 + w$ In this setup there is a multitude of *O* type observers, who stand at the origin O, one for each particular value of the radius *r* , who rotate with the angular velocity of that particular ring. Observer *O* is standing on the center on top of *O*but is not rotating with the disc. The clocks of the two types of observers will run at the same rate. Again equations (11) to (18) continue to hold. Substituting (94) into (18) we find, z as a function of the radius it i

1 that slide one after the other.
 $= w_0 f(r)$

r

2 observers, who stand at the or

rotate with the angular velocit

the center on top of O but is n

servers will run at the same rat

bs Formulation the radial direction to c
are limited in the radial direction to c
read in (87) and therefore lie within th
limited in the z direction since $z'' = z$
of the radius of the radius it is limited that slide one afte re values greater and smaller than *c*.

even O'' are limited in the radial direction to c/u

e remarked in (87) and therefore lie within the c

t being limited in the z direction since $z'' = z$ due

age

1 Case for obser *deretary or a deretary diater* (*diater diater diater diater control x* (*ver or d mergenized in (87) and therefore lie within the c' b being limited in the z direction since* $z'' = z$ due gives the *w*

$$
\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 f(r)^2}} = c \cos\left\{\tag{95}
$$

and

$$
\cos \left\{ = \frac{c}{\sqrt{c^2 + w_0^2 r^2 f(r)^2}} = \frac{1}{\sqrt{1 + w_0^2 t^2 f(ct)^2}} \tag{96}
$$

If we let $f(r) = e^{-\frac{1}{r}}$ we find

$$
\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-2r}}} \tag{97}
$$

or

$$
r' = c \int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-2r}}} + const \tag{98}
$$

To see how r' behaves take the derivative of $\cos \left\{ \frac{d}{dr} \cos \left(\frac{1}{r} - \frac{d}{dt} \cos \left(\frac{1}{r} - \frac{1}{r} \frac{dr}{dt} \right) \right) \right\}$
At $t = 0$ the derivative is negative then as t increases the derivative increases and at

$$
\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-2r}}}
$$
(97)

or

$$
r' = c \int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-2r}}} + const
$$
\n(98)

To see how r' behaves take the derivative of $\cos\{$

$$
\frac{d}{dr}\cos\left\{\right. = \frac{1}{c}\frac{d}{dt}\cos\left\{\right. = -\frac{w_0^2te^{-2}(\cos\left(1-\frac{1}{2}ct\right))}{\cos\left(1+w_0^2t^2e^{-2} - \cos\left(\frac{1}{2}ct\right)\right)^{\frac{3}{2}}}\tag{99}
$$

 $w_0^2 t^2 f(ct)^2$
 mst (9
 $\frac{e^t (1 - 2ct)}{e^t e^{-25ct}}$) $\frac{2}{3}$

e derivative increases and at

big *t* it tends to zero. A plot of $rac{c}{(4t)^4 + w_0^2 r^2 f(r)^2} = \frac{1}{\sqrt{1 + w_0^2 t^2 f(ct)^2}}$ (96)
 $\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-2r}}}$ (97)
 $c \int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-2r}}} + const$ (98)

trivative of cos {
 $\frac{1}{c} \frac{d}{dt} \cos \left(\frac{1}{c} - \frac{w_0^2 t e^{-2t} (1 - 2ct)}{c(1 + w_0^2 t^2 e^{-2t}$ { $=\frac{c}{\sqrt{c^2 + w_0^2 r^2 f(r)^2}} = \frac{1}{\sqrt{1 + w_0^2 t^2 f(ct)^2}}$
d
 $\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-2r}}}$
 $r' = c \int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-2r}}} + const$
e the derivative of $\cos \left\{ \frac{1}{c} \frac{d}{dt} \cos \left(\frac{1}{c} - \frac{w_0^2 t e^{-2r}}{c(1 + w_0^2 t^2 e^{-2r}) \right)^{\frac{3}{2}}} \$ $rac{1}{\sqrt{1 + w_0^2 t^2 f (ct)^2}}$ (96)
 $rac{1}{e^{-2r}}$ (97)
 $rac{1}{e^{-2r}}$ (97)
 $rac{1}{e^{-2r}} + const$ (98)
 $\left(\frac{v_0^2 te^{-2\alpha t}}{(1 - \frac{1}{2}ct)}\right)$ (99)
 $\frac{(1 + w_0^2 t^2 e^{-2\alpha t})^{\frac{3}{2}}}{(1 + w_0^2 t^2 e^{-2\alpha t})^{\frac{3}{2}}}$

ses the derivative increases and $\cos \left\{ \frac{c}{\sqrt{c^2 + w_0^2 r^2 f(r)^2}} \right\} = \frac{1}{\sqrt{1 + w_0^2 t^2 f(ct)^2}}$

and
 $\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-2r}}}$
 $r' = c \int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-2r}}} + const$

ake the derivative of $\cos \left\{ \frac{d}{dr} \cos \left(\frac{d}{dt} \cos \left(\frac{d}{dt} - \frac{w_0^2 t e^{-2t} \cos \left(\frac$ $\cos \left\{ \frac{c}{\sqrt{c^2 + w_0^2 r^2 f(r)^2}} \right\} = \frac{1}{\sqrt{1 + w_0^2 t^2}}$

find
 $\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-2r}}}$
 $r' = c \int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-2r}}} + const$

take the derivative of $\cos \left\{ \frac{d}{dr} \cos \left(\frac{d}{dr} \right) \right\} = -\frac{w_0^2 t e^{-2r}}{c(1 + w_0^2 t^2$ $=\frac{1}{\sqrt{1 + w_0^2 t^2 f(ct)^2}}$ (96)
 $\frac{1}{t^2 e^{-2}}$ (97)
 $\frac{1}{t^2 e^{-2}}$ + const (98)
 $\left\{\frac{w_0^2 te^{-2\lambda\sigma}(1-\lambda ct)}{c(1 + w_0^2 t^2 e^{-2\lambda\sigma})^{\frac{3}{2}}}$ (99)

ases the derivative increases and at

ally for big t it tends to zero. A plot $rac{c}{\sqrt{c^2 + w_0^2 r^2 f(r)^2}} = \frac{1}{\sqrt{1 + w_0^2 t^2 f(ct)^2}}$ (96)
 $\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-2/r}}}$ (97)
 $r' = c \int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-2/r}}} + const$ (98)

e derivative of cos {
 $\left\{ = \frac{1}{c} \frac{d}{dt} \cos \left(= -\frac{w_0^2 t e^{-2l\sigma} (1 - 2ct)}{c(1 + w_0^2$ $rac{c}{(c^2 + w_0^2 r^2 f(r))^2} = \frac{1}{\sqrt{1 + w_0^2 t^2 f(ct)^2}}$ (96)
 $\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-21r}}}$ (97)
 $= c \int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-21r}}} + const$ (98)

derivative of $\cos \left(\frac{1}{c} \frac{1}{dt} \cos \left(\frac{1}{c} - \frac{w_0^2 t^2 e^{-21} (1 - 1) ct}{c(1 + w_0^2 t^2 e^{-$ At $t = 0$ the derivative is negative then as t increases the derivative increases and at $t = \frac{1}{c}$ it becomes zero and then positive and finally for big *t* it tends to zero. A plot of $\cos \{$ appears in Figure 7

Figure 7 The cosine of deflection angle versus *r*. The deflection angle starts at zero ($\cos 0^\circ=1$). Then it increases reaching almost 90[°] degrees (for big enough w_0) at $r = \frac{1}{2}$. Then it falls again to $\}$ $=\frac{1}{2}$. Then it falls again to zero ($cos90^\circ=1$) asymptotically.

The deflection angle initially at 0º increases approaching 90º degrees (closer to 90º for higher w_0) and then drops again to zero. This behavior is similar to the behavior we have examined for the rotation without slippage. Namely, as the angle of deflection increases the signals start rotating in tighter circles until they reach $r = \frac{1}{2}$. Then the signals rotate in $}$ $=\frac{1}{2}$. Then the signals rotate in less and less tight circles until they are directed asymptotically radially outward ($\{ = 0 \}$). The choice of deflection angle versus *r*. The deflection angle starts at zero (cos0^o=1).
 Figure 7 The cosine of deflection angle versus *r*. The deflection angle starts at zero (cos0^o=1).

Then it increases reachi

that for a disc consisting of slipping rings each rings slips with respect to the previous by the same proportion in angular velocity. Consider for example a width *r* of *n* layers. The

first has velocity w_0 , the second $w_0 \mathsf{r}^n$, the third $w_0(\mathsf{r}^n)^2$ *r* w_0 ^{Γ}ⁿ, the third w_0 (Γ ⁿ)² the nth w_0 Sⁿ where S = Γ ⁿ $v_0(r^{\frac{r}{n}})^2$ the nth w_0 Sⁿ where S =
portion S forming a geometric
nave angular velocity w_0r^r . For
ny substitute $r = e^{-3}$ where $\} >$ *r* $w_0(\Gamma^n)^2$ the nth w_0S^n where $S = \Gamma^n$. first has velocity w_0 , the second $w_0 r^{\frac{r}{n}}$, the third $w_0 (r^{\frac{r}{n}})^2$ the nth $w_0 s^n$ where $s = r^{\frac{r}{n}}$.
Each layer slips with respect to the previous by proportion *s* forming a geometric series.
Therefore, first has velocity w_0 , the second $w_0 r^{\frac{r}{n}}$, the third $w_0 (r^{\frac{r}{n}})^2$ the nth $w_0 s^n$ where $s = r^{\frac{r}{n}}$.
Each layer slips with respect to the previous by proportion s forming a geometric series.
Therefore, le Therefore, letting $n \to \infty$ the ring at radius r will have angular velocity $w_0 r^r$. For the first has velocity w_0 , the second $w_0 r^{\frac{r}{n}}$, the third $w_0 (r^{\frac{r}{n}})^2$ the nth $w_0 s^n$ where $s = r^{\frac{r}{n}}$.
Each layer slips with respect to the previous by proportion s forming a geometric series.
Therefore, le $r = e^{-1}$ where $\} > 0$ and first has velocity w_0 , the second w_0 F
Each layer slips with respect to the pr
Therefore, letting $n \to \infty$ the ring at r
case that $0 < r < 1$, where we are inte
then $w_0 r^r = w_0 e^{-\lambda r}$ or $f(r) = e^{-\lambda r}$. as velocity w_0 , the second $w_0 r^{\frac{r}{n}}$, the third *w*
layer slips with respect to the previous by profore, letting $n \to \infty$ the ring at radius *r* will
hat $0 < r < 1$, where we are interested, we may
 $w_0 r^r = w_0 e^{-\frac{1$ then $w_0 r^r = w_0 e^{-\frac{1}{2}r}$ or $f(r) = e^{-\frac{1}{2}r}$. locity w_0 , the second $w_0 r^{\frac{r}{n}}$, the third $w_0 (r^{\frac{r}{n}})^2$ the nth $w_0 s'$
slips with respect to the previous by proportion s forming a
letting $n \to \infty$ the ring at radius r will have angular velocity
 $\langle r \rangle \langle 1$ have angular velocity w_0
ay substitute $\Gamma = e^{-1}$ when

 $(\gamma \sin \leftarrow \cos \leftarrow e)$
($(\gamma \sin \leftarrow \cos \leftarrow e)$
ssuming the amplitude of
complicating the proble and $w_0 r^{\frac{r}{n}}$, the third $w_0 (r^{\frac{r}{n}})^2$ the nth $w_0 s^n$ where $s =$
to the previous by proportion s forming a geometric
ring at radius r will have angular velocity $w_0 r^r$. For
are interested, we may substitute r *x*<sup> $\int_a^b w_0 r^{\frac{L}{n}}$, the third $w_0 (r^{\frac{L}{n}})^2$ the nth $w_0 S''$ where $S = r^{\frac{L}{n}}$.

the previous by proportion *S* forming a geometric series.

g at radius *r* will have angular velocity $w_0 r'$. For the
 i r. So</sup> First has velocity w_0 , the second $w_0 \Gamma^a$, the third $w_0(\Gamma^a)^2$ the nth w_0S^a where $S = \Gamma^b$.

Each layer slips with respect to the previous by proportion S forming a geometric series.

Therefore, letting $n \to \$ First has velocity w_0 , the second $w_0r^{\frac{r}{n}}$, the third $w_0(r)$

Each layer slips with respect to the previous by proportion

Therefore, letting $n \to \infty$ the ring at radius r will have

zase that $0 < r < 1$, where we st has velocity w_0 , the second $w_0 r^{\frac{r}{n}}$, the third $w_0 (r^{\frac{r}{n}})^2$ the nth $w_0 S^n$ where $S = r^{\frac{r}{n}}$.

ch layer slips with respect to the previous by proportion s forming a geometric series.

erefore, letting second $w_0 \Gamma^n$, the third $w_0(\Gamma^n)^2$ the nth w_0 sⁿ where $S = \Gamma^n$.

pect to the previous by proportion 5 forming a geometric series.

the ring at radius r will have angular velocity $w_0 \Gamma^r$. For the

ewe are interes by proportion s forming a geometric series.

will have angular velocity $w_0 r^r$. For the

we may substitute $r = e^{-2}$ where $\} > 0$ and

bserver O'

t there is slippage both in the radial and the z
 $w_0 e^{-\sigma(\frac{1}{2} \sin \alpha +$ *w*₀, the second $w_0 r^{\frac{1}{n}}$, the third $w_0 (r^{\frac{1}{n}})^2$ the nth $w_0 s^n$ where $s = r^{\frac{1}{n}}$
with respect to the previous by proportion s forming a geometric series $n \rightarrow \infty$ the ring at radius *r* will have angular be second $w_0 \Gamma^n$, the third $w_0 (\Gamma^n)^*$ the nth $w_0 S^n$

spect to the previous by proportion s forming a
 ∞ the ring at radius r will have angular velocity

ere we are interested, we may substitute $\Gamma = e^{-1} v$

f(r) = with respect to the previous by proportion s forming a geometric seri-
 $n \rightarrow \infty$ the ring at radius r will have angular velocity w_0r^r . For the
 d , where we are interested, we may substitute $r = e^{-\gamma}$ where $\gamma > 0$ a cond $w_0 r^{\frac{r}{n}}$, the third $w_0 (r^{\frac{r}{n}})^2$ the nth $w_0 s^n$ where *S*
t to the previous by proportion *S* forming a geometric
e ring at radius *r* will have angular velocity $w_0 r^r$. F
ve are interested, we may subs the second $w_0r^{-\frac{r}{n}}$, the third $w_0(r^{-\frac{r}{n}})^2$ the nth w_0S^n where $S = r^{-\frac{r}{n}}$.

respect to the previous by proportion s forming a geometric series.
 $\rightarrow \infty$ the ring at radius r will have angular velocity w_0

9.2 The three-Dimensional Case for observer *O*

For the three dimensional case we assume that there is slippage both in the radial and the z direction. To achieve this we assume that

$$
w = w_0 e^{-\frac{1}{2}w_0 e^{-ct(\frac{1}{2} \sin \left(1 + \cos \left(1 \right))}\right)}
$$
(100)

at radius r will have angular velocitier
ested, we may substitute $\Gamma = e^{-\gamma}$
assume that there is slippage both if
the that
 $\gamma_0 e^{-\gamma_{\text{max}}} = w_0 e^{-ct(\gamma_{\text{sin}\gamma} + \gamma_{\text{cos}\gamma})}$
and precession (assuming the amplitic)
placed by $\$ where, ≥ 0 , $\sim \geq 0$. We disregard precession (assuming the amplitude of precession is function of *t*. Relations (44) to (49) continue to hold, with $w = w_0 e^{-\frac{1}{2} - \frac{1}{2}} = w_0 e^{-ct(\frac{1}{2} \sin \left(1 + \cos \left(\frac{x}{2}\right)\right)}$ denotes 1 Case for ob-

onal case we assume that

this we assume that
 $w = w_0 e^{-\lambda - c z} = w$

We disregard precessio
 \langle must be replaced by \langle

ons (44) to (49) continue t
 $\langle \rangle$ sin $\langle +\rangle$ cos \langle = csin $\langle \sqrt{\frac{1}{1}} \rangle$ The three-Dimensional Case for

e three dimensional case we assum

ion. To achieve this we assume that
 $w = w_0 e^{-\lambda - \tau}$, $\} \ge 0$, $\sim \ge 0$. We disregard pre-

mall) otherwise < must be replace

on of t. Relations (44) to ver O'

re is slippage both in the radial and the z
 $\frac{t(3 \sin \leftarrow + \cos \leftarrow)}{(100)}$

ssuming the amplitude of precession is

complicating the problem since ' is a

bld, with
 $\frac{2t c(3 \sin \leftarrow + \cos \leftarrow)}{2c(3 \sin \leftarrow + \cos \leftarrow)} \frac{v_0^2 t^2 \cos$ ver O'

re is slippage both in the radial and the z
 $\frac{u(3\sin\epsilon + \cos\epsilon)}{2\sin\epsilon}$ (100)

ssuming the amplitude of precession is

complicating the problem since ' is a

ld, with
 $\frac{2u(3\sin\epsilon + -\cos\epsilon)}{2ct(3\sin\epsilon + -\cos\epsilon)} \frac{v_0^2 t$ *c*
is slippage both in the radial ar
in (+ - cos c)
uming the amplitude of precess
omplicating the problem since
, with
 $\frac{1}{2} \sin x + -\cos x \cos^2 y \frac{v^2}{v^2} + \cos^2 z \cos^2 x$
 $\cos^2 x \sin^2 x = \cos^2 y \frac{v^2}{v^2} + \cos^2 y \frac{v^2}{v^2} + \cos^2 z \cos^2$ Dimensional Case for observer O'
 *n*sional case we assume that there is slippage both in the radial and the z
 w = $w_0e^{-1-\tau z} = w_0e^{-\alpha(1)\sin(x+\tau)\cos(x)}$ (100)
 \cdot 0 . We disregard precession (assuming the amplitude of prec ver O'

are is slippage both in the radial and the z
 $\frac{ct(3 \sin x + -\cos x)}{(100)}$

assuming the amplitude of precession is

' complicating the problem since ' is a

old, with
 $\frac{-2tc(3 \sin x + -\cos x)}{w_0^2 t^2 \cos^2 x}$ (101) ver *O'*

ere is slippage both in the radial and the z
 $\frac{ct(3 \sin \leftarrow \cos \leftarrow)}{(100)}$

assuming the amplitude of precession is

' complicating the problem since ' is a

old, with
 $\frac{-2tc(3 \sin \leftarrow \cos \leftarrow)}{2ct(3 \sin \leftarrow \cos \leftarrow)} \frac{v_0^2 t^2$ we assume that
 $w = w_0 e^{-3z-z} = w_0 e^{-ct(3 \sin \epsilon + \cos \epsilon)}$ (100)

e disregard precession (assuming the amplitude of precession is

must be replaced by $\epsilon +'$ complicating the problem since ' is a

44) to (49) continue to hold, with $e^{-z} = w_0 e^{-ct(\frac{1}{3} \sin \left(\frac{1}{2} \cos \theta)\right)}$ (100

recession (assuming the amplitude of precession is

ced by $\left(\frac{1}{2} + \frac{1}{2} \cos \theta\right)$ (assuming the problem since ' is

ntinue to hold, with

sin $\left(\frac{1-e^{-2tc(\frac{1}{3} \sin \left(\frac{1}{2} \$ $w = w_0 e^{-\gamma}$ = $w_0 e^{-\gamma}$ (100)

e disregard precession (assuming the amplitude of precession is

must be replaced by $\langle +'$ complicating the problem since ' is a

44) to (49) continue to hold, with
 $x + cos(x)$

in $\langle cos \{ = c sin$ recession (assuming the amplitude of precession is

ced by $\langle +'$ complicating the problem since ' is a

ntinue to hold, with
 $\sin \langle \sqrt{\frac{1-e^{-2tc(\sin \langle +\cos \langle \cdot \rangle)} w_0^2 t^2 \cos^2 \langle \langle \cdot \rangle \cos \langle \cdot \rangle \cos \langle \cdot \rangle \sin \langle \cdot \rangle \cos \langle \cdot \rangle \cos \langle \cdot \rangle \cos \langle \$ server O'

there is slippage both in the radial and the z
 $e^{-\alpha(3\sin\epsilon + \cos\epsilon)}$ (100)

in (assuming the amplitude of precession is

+' complicating the problem since ' is a

bold, with
 $-e^{-2\alpha(3\sin\epsilon + -\cos\epsilon)}w_0^2t^2\cos^2\epsilon$ (1 server O'

there is slippage both in the radial and the z
 $e^{-ct(\frac{1}{3}\sin(\epsilon + \cos \epsilon))}$ (100)

(assuming the amplitude of precession is

+' complicating the problem since ' is a

hold, with
 $\frac{e^{-2tc(\frac{1}{3}\sin(\epsilon + \cos \epsilon))}w_0^2t^2 \cos^$ e that
 $e^{-\lambda_{m}-z} = w_{0}e^{-ct(\frac{1}{2}\sin\left(\frac{1}{2}+cos\right))}$

d precession (assuming the amplitude

placed by $\left(\frac{1}{2}+e^{-2tC(\frac{1}{2}\sin\left(\frac{1}{2}+cos\right))}w_{0}^{2}t^{2}\cos^{2}\left(\frac{1}{2}+e^{-2tC(\frac{1}{2}\sin\left(\frac{1}{2}+cos\right))}w_{0}^{2}t^{2}\sin^{2}\left(\frac{1}{2}+e^{-2tC(\$ *z* (*x*) $r = e^{-3r}$.
 z $f(x) = e^{-3r}$.
 z consider the example of the matrix of $f(x) = e^{-3r}$.
 z consistent and the seed system that
 $w = w_0 e^{-3 - x z} = w_0 e^{-\alpha(3) \sin(x + \cos x)}$ (100

We disregard precession (assuming the amplit **and** Case for observer O'

al case we assume that there is slippage both in the radial and the

s we assume that
 $w = w_0 e^{-\frac{1}{2} - x z} = w_0 e^{-\alpha(\frac{1}{2} \sin \alpha + \cos \alpha)}$ (100

We disregard precession (assuming the amplitude of pre be assume that
 $= w_0 e^{-\frac{1}{2} - z} = w_0 e^{-ct(\frac{1}{2} \sin \left(\frac{z}{2} \right))}$ (

gard precession (assuming the amplitude of precession

replaced by $\left\langle \frac{1}{2} + \frac{1}{2} \right\rangle$ (

(49) continue to hold, with

s { = c sin $\left\langle \sqrt{\frac{1 - e^{-2tc$ = w_0e^{-3} = v_0e^{-2} = $w_0e^{-ct(3\sin\epsilon + \cos\epsilon)}$ (

ggard precession (assuming the amplitude of precessic

e replaced by $\lt +'$ complicating the problem since '

(49) continue to hold, with

s { = $c \sin \epsilon \sqrt{\frac{1 - e^{-2tc(3\sin\epsilon + \cos$ For observer *O'*

for observer *O'*

me that there is slippage both in the radial and the z

at
 $\int_{-\infty}^{\infty} e^{-\alpha t \sin(x + \cos x)}$ (100)

crossion (assuming the amplitude of precession is

d by $\langle + \rangle$ complicating the problem Even a sume that there is suppage bout in the radial and the z

sissume that there is suppage bout in the radial and the z

sissume that there is suppage bout in the radial and the z

sissume $v = w_0 e^{-1 - z} = w_0 e^{-\alpha t / \sin t + \cos t$ e img at nearas 7 win late argum velocity n_0 . The the example argument interested, we may substitute $\Gamma = e^{-1}$ where $\frac{1}{2} > 0$ and πe^{-2t} .

Onal Case for observer *O'*

sage we assume that there is slippage both where we are interested, we may substitute $\Gamma = e^{-\lambda}$ where $\lambda > 0$ and
 $\int f(r) = e^{-\lambda r}$.

The mensional Case for observer O'

tional case we assume that there is slippage both in the radial and the z
 λ this we assum we are imerested, we may stostature $1 - e$ where $y > 0$ and

can $= e^{-y r}$.

onal Case for observer O'

case we assume that there is slippage both in the radial and the z

we assume that
 $w = w_0 e^{-2x-xz} = w_0 e^{-\sigma(3) \sin(x + \cos(x))}$ **9.2** The three-Dimensional Case for observer *O'*

For the three dimensional case we assume that there is slippage both in the

direction. To achieve this we assume that
 $w = w_0 e^{-3-x-z} = w_0 e^{-ct/3\sin(z + \cos(z))}$

where, $\frac{1}{2$ Case for observer *O'*

assume that there is slippage both in the radial and the z

are that
 $w_0e^{-\lambda-\gamma z} = w_0e^{-\alpha(\lambda \sin \epsilon + \cos \epsilon)}$ (100)

and precession (assuming the amplitude of precession is

eplaced by < +' complicating nal case we assume that there is slippage both in the radial and the z

his we assume that
 $w = w_0 e^{-3 - \gamma z} = w_0 e^{-\alpha(y \sin \alpha + \gamma \cos \alpha)}$ (100)

We disregard precession (assuming the amplitude of precession is
 \langle must be replac conal case we assume that there is slippage both in the radial and the

this we assume that
 $w = w_0 e^{-\frac{1}{2} - x} = w_0 e^{-x(t) \sin(\epsilon + \cos(\epsilon))}$ (100)
 \therefore We disregard precession (assuming the amplitude of precession is
 \langle mus e that there is slippage both in the radia
 $z = w_0 e^{-ct(\sin x + \cos x)}$

ession (assuming the amplitude of pred

by $\langle + \rangle$ complicating the problem sin

nue to hold, with
 $\langle \sqrt{\frac{1 - e^{-2tc(\sin x + \cos x)} w_0^2 t^2 \cos^2 \zeta}{1 + e^{-2ct(\sin x + \cos x)} w_0^2$ msional Case for observer U

al case we assume that there is slippage both in the r

is we assume that
 $w = w_0 e^{-1 - -z} = w_0 e^{-\alpha(1 \sin \alpha + \cos \alpha)}$

We disregard precession (assuming the amplitude of

must be replaced by \leftarrow "co endangle amplitude of precession (assuming the amplitude of precession)

be replaced by $\langle +\rangle$ complicating the problem since \langle

(, (49) continue to hold, with

()
 $\cos \{ = c \sin \left(\sqrt{\frac{1 - e^{-2tc(\sin \left(+ \cos \left(\frac{c}{\sin \left(1 - \cos$ istregard precession (assuming the amplitude of precession
 t be replaced by $\langle +\rangle$ complicating the problem since

to (49) continue to hold, with
 $\cos \left(\frac{1}{2} \cos \right) \sin \left$ *while* the problem since ' is a

dd, with
 $\frac{-2ic(\sin x + \cos x)}{2ic(\sin x + \cos x)} \frac{w_0^2 t^2 \cos^2 x}{w_0^2 t^2 \sin^2 x}$ (101
 $\frac{e^{-2ic(\sin x + \cos x)}w_0^2 t^2 \sin^2 x}{e^{-2ic(\sin x + \cos x)}w_0^2 t^2 \sin^2 x}$ (102
 $\frac{e^{-2ic(\sin x + \cos x)}w_0^2 t^2 \cos^2 x}{e^{-2ic(\sin x + \cos x)}w_0^2 t^$ $e^{-2x-cz} = w_0 e^{-ct(3 \sin \epsilon + \cos \epsilon)}$

d precession (assuming the amplitude of precession

placed by $\epsilon + \epsilon'$ complicating the problem since

continue to hold, with
 $= c \sin \epsilon \sqrt{\frac{1 - e^{-2tc(3 \sin \epsilon + \cos \epsilon)} w_0^2 t^2 \cos^2 \epsilon}{1 + e^{-2ct(3 \sin \epsilon + \cos \epsilon)} w$

elations (44) to (49) continue to hold, with
\n
$$
w_0 e^{-ct(3 \sin \epsilon + -\cos \epsilon)}
$$
\n
$$
\frac{d...'}{dt} = c \sin \epsilon \cos \{ = c \sin \epsilon \sqrt{\frac{1 - e^{-2tc(3 \sin \epsilon + -\cos \epsilon)} w_0^2 t^2 \cos^2 \epsilon}{1 + e^{-2ct(3 \sin \epsilon + \cos \epsilon)} w_0^2 t^2 \sin^2 \epsilon}} \qquad (101)
$$
\n
$$
cos \{ = \sqrt{\frac{c^2 - e^{-2(3 - t - \epsilon)}}{c^2 + e^{-2(3 - t - \epsilon)}} w_0^2 z^2} = \sqrt{\frac{1 - e^{-2tc(3 \sin \epsilon + \cos \epsilon)}}{1 + e^{-2ct(3 \sin \epsilon + \cos \epsilon)}} w_0^2 t^2 \sin^2 \epsilon}} \qquad (102)
$$
\n
$$
cos \epsilon \text{ and } ... = ct \sin \epsilon \text{ . Therefore,}
$$
\n
$$
...' = c \sin \epsilon I(\epsilon, t, \epsilon), \quad (103)
$$
\n
$$
I(\epsilon, t, \epsilon), -\epsilon = \int_0^t \sqrt{\frac{1 - e^{-2tc(3 \sin \epsilon + \cos \epsilon)}}{1 + e^{-2ct(3 \sin \epsilon + \cos \epsilon)}} w_0^2 t^2 \sin^2 \epsilon} dt \qquad (104)
$$
\n
$$
a \text{ have,}
$$
\n
$$
I(\frac{f}{2}, t, \epsilon) = -\int_0^t \sqrt{\frac{1}{1 + w_0^2 t^2 e^{-2ct}}} dt \qquad (105)
$$
\n
$$
2) \text{ is the same as (59), where } w = w_0 e^{-1 - \epsilon} \text{ . The same is true for (58)}
$$
\n
$$
te^{-tc(3 \sin \epsilon + \cos \epsilon)} \le \frac{1}{w_0 \cos \epsilon} \qquad (106)
$$
\n
$$
for t \text{ close to 0 and } t \text{ very big the above condition is satisfied, while the}
$$

where

$$
\frac{d_{\text{m}}'}{dt} = c \sin \leftarrow \cos \left\{ \frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1 - e^{-2c(t) \sin \leftarrow t - \cos \leftarrow t} \sqrt{\frac{1
$$

$$
t' = c \sin \left(I(\langle t, t, \rangle, \gamma) \right) \tag{103}
$$

where

$$
\sqrt{\frac{c^2 - e^{-2(3e^{i} + z)}}{c^2 + e^{-2(3e^{i} + z)}} \frac{w_0^2 z^2}{w_0^2 ...^2}} = \sqrt{\frac{1 - e^{-2tc(\frac{3 \text{sin} (x + z \cos \epsilon)}{3 \text{sin} (x + z \cos \epsilon)} \frac{w_0^2 t^2 \cos^2 \epsilon}{w_0^2 t^2 \sin^2 \epsilon}}}{1 + e^{-2ct(\frac{3 \text{sin} (x + z \cos \epsilon)}{3 \text{sin} (x + z \cos \epsilon)} \frac{w_0^2 t^2 \sin^2 \epsilon}{w_0^2 t^2 \sin^2 \epsilon}}}
$$
(102)

$$
I(\epsilon, t, \epsilon, \epsilon) = \int_0^t \sqrt{\frac{1 - e^{-2tc(\frac{3 \text{sin} (x + z \cos \epsilon)}{3 \text{sin} (x + z \cos \epsilon)} \frac{w_0^2 t^2 \cos^2 \epsilon}{w_0^2 t^2 \sin^2 \epsilon}}}{1 + e^{-2ct(\frac{3 \text{sin} (x + z \cos \epsilon)}{3 \text{sin} (x + z \cos \epsilon)} \frac{w_0^2 t^2 \sin^2 \epsilon}{w_0^2 t^2 \sin^2 \epsilon}}}
$$
(104)

$$
I(\frac{f}{2}, t, \epsilon, \epsilon) = \int_0^t \sqrt{\frac{1}{1 + w_0^2 t^2 e^{-2ct}}} dt
$$
(105)
the same as (59), where $w = w_0 e^{-\frac{3}{2} - z \epsilon}$. The same is true for (58)
red condition for (102) to be real, is

$$
te^{-tc(\frac{3}{2} \sin \epsilon + z \cos \epsilon)} \le \frac{1}{w_0 \cos \epsilon}
$$
(106)
to 12. Use to 0 and t very big the above condition is satisfied, while the
you can be maximum. Therefore, for each ϵ , it is either satisfied for all t

For $\epsilon = 90^{\circ}$ we have,

$$
I(\frac{f}{2},t,\},,-) = \int_{0}^{t} \sqrt{\frac{1}{1 + w_0^2 t^2 e^{-2ct}}}\,dt\tag{105}
$$

Observe that (102) is the same as (59), where $w = w_0 e^{-\frac{1}{2}x}$. The same is true for (58) (60), (61). The required condition for (102) to be real, is

$$
te^{-tc(\frac{1}{2}\sin\left(\frac{1}{2} - \cos\left(\frac{1}{2}\right)\right))} \le \frac{1}{w_0 \cos\left(\frac{1}{2}\right)}\tag{106}
$$

It is obvious that for *t* close to 0 and *t* very big the above condition is satisfied, while the left hand side has only one maximum. Therefore, for each $\left\langle \right\rangle$, it is either satisfied for all *t* or there are positive t_1 and t_2 with $t_1 \le t_2$ so that it is $\frac{e^{-2(1)^{2}+1+1)}w_0^2 z^2}{e^{-2(1)^{2}+1+1}w_0^2 z^2} = \sqrt{\frac{1-e^{-2\pi t} \sin^2 t - \cos^2 w_0^2 t^2 \cos^2 t}{1+e^{-2\pi (1)\sin t - \cos^2 w_0^2 t^2 \sin^2 t}}}}$ (102)
 $= ct \sin \leftarrow$ Therefore,
 $\therefore t = c \sin \leftarrow I(\leftarrow, t, 1, \rightarrow)$ (103)
 $\left(\frac{1}{1+e^{-2\pi t} \sin^{\frac{2}{3}+1} \cos^2 \frac{$ $\sqrt{c^2 + e^{-2c(x-2z)}}w_0^{z-2z}$ $\sqrt{1+e^{-2c(x)sin(x-2z)}}$

because $z = ct \cos \leftarrow$ and $\ldots = ct \sin \leftarrow$. Therefore,
 $\ldots' = c \sin \leftarrow I(\leftarrow, t, 1, -\right)$

where
 $I(\leftarrow, t, 1, -\right) = \int_0^t \sqrt{\frac{1-e^{-2(c(\sin(x+cos(x))}w_0^2t^2)}{1+e^{-2c(\sin(x+cos(x))}w_0^2t^2}}$

For $\leftarrow =$

Given a \leftarrow , if it is satisfied for all *t*, then the angle of deflection { does not become 90^o degrees (except perhaps at a single point). Therefore, the signal is allowed to increase its radius for all *t* and asymptotically become radial.

Given a \leftarrow , if it is satisfied for all *t*, then the angle of deflection { does not become 90 degrees (except perhaps at a single point). Therefore, the signal is allowed to increase irradius for all *t* and asymptotic If it is not satisfied for an interval ($t \in (t_1, t_2)$), it means that at t_1 the signal has reached Given a ϵ , if it is satisfied for all *t*, then the angle of deflection { does not become 90^o degrees (except perhaps at a single point). Therefore, the signal is allowed to increase its radius for all *t* and asympt after t_1 and until it reaches t_2 the signal is not observable in the interval (t_1, t_2) . After t_2 , for all *t*, then the angle of deflection { does not become 90°
a single point). Therefore, the signal is allowed to increase its
tically become radial.
terval $(t \in (t_1, t_2))$, it means that at t_1 the signal has reached the signal is again allowed to increase its radius, increase $\cos\{-\text{and become}\}$ asymptotically radial. Given a \leftarrow , if it is satisfied for all *t*, then the angle of ddegrees (except perhaps at a single point). Therefore, the radius for all *t* and asymptotically become radial.
If it is not satisfied for an interval ($t \$ Given a \leftarrow , if it is satisfied for all *t*, then the angle of deflection { does not become 90°
degrees (except perhaps at a single point). Therefore, the signal is allowed to increase its
radius for all *t* and asympto is satisfied for all *t*, then the angle of deflection { does not become 90°
perhaps at a single point). Therefore, the signal is allowed to increase its
nd asymptotically become radial.
Field for an interval $(t \in (t_1, t_2$ 1 < , if it is satisfied for all *t*, then the angle

(except perhaps at a single point). Therefore all *t* and asymptotically become radial.

1 of satisfied for an interval $(t \in (t_1, t_2))$, it m

0 (angle of deflection 90 Example 1 at a singled of or all *t*, then the angle of deflection
 comparison α , if it is satisfied for all *t*, then the angle of deflection

as for all *t* and asymptotically become radial.

is not satisfied for a then the angle of deflection { does not beard to income radial.
 $\in (t_1, t_2)$), it means that at t_1 the signal has grees) and cannot increase its radius anym final is not observable in the interval (t_1, t_2) , we its *n*, then the angle of deflection { does not become 90°

point). Therefore, the signal is allowed to increase its

become radial.
 $(t \in (t_1, t_2))$, it means that at t_1 the signal has reached

degrees) and cannot increas $\epsilon(t_1, t_2)$, it means that at t_1 the signal has
grees) and cannot increase its radius anym
gnal is not observable in the interval (t_1, t_2)
se its radius, increase cos { and become
 ϵ , (106) is satisfied for all *z* point). Therefore, the signal is allowed to increase it
become radial.
 $(t \in (t_1, t_2))$, it means that at t_1 the signal has reached

degrees) and cannot increase its radius anymore. So

signal is not observable in **Example 10. Consideration** of a matter of $V_1 + V_2$, V_1 are most and u_1 in the signal ans concerned $\left(\frac{1}{2}, 0 \right)$, and the interval $\left(\frac{1}{2}, 0 \right)$. After t_2 , $\left(\frac{1}{2} \right)$ (angle of deflection 90^{*} d

What is the condition so that given $\left\langle \right\rangle$, (106) is satisfied for all t? It is that the maximum of 0 cos $\sqrt{ }$ $1 - \lambda$ ad what is the maximum? Takin $w_0 \cos \leftarrow$. And what is the maximum? Taking the derivative $\frac{d}{dt}te^{-tc(\frac{1}{2}sin\theta + \cos\theta)} = e^{-tc(\frac{1}{2}sin\theta + \cos\theta)}(1-tc(\frac{1}{2}sin\theta + \cos\theta))$, and maximum occurs Fr t_1 and until it reaches t_2 the signal is not observable in the interval (t_1, t_2) . After the signal is again allowed to increase its radius, increase cos { and become mptotically radial.

Moreover this the condi ther t_1 and until it reaches t_2 the signal is not observable

e signal is again allowed to increase its radius, increase

symptotically radial.

That is the condition so that given $\left\langle \frac{106}{w_0 \cos \varsigma} \right\rangle$. And wh cos { = 0 (angle of deflection 90° degrees) and cannot increase its radius anymore. So

after t_i and until it reaches t_i , the signal is not observable in the interioral (t_i, t_i) . After t_2 ,

the signal is again allo 1.

In so that given $\left\langle \frac{(106)}{w_0 \cos \varsigma} \right\rangle$. So satisfied for all t ? It is that the maximum of

than or equal to $\frac{1}{w_0 \cos \varsigma}$. And what is the maximum? Taking the
 $\frac{(1 + \cos \varsigma)}{w_0 \cos \varsigma} = e^{-\kappa (3 \sin \varsigma + \cos \varsigma)} (1 - t$ al.

an so that given \leftarrow , (106) is satisfied for all *t*? It is that the maximum of

s than or equal to $\frac{1}{w_0 \cos \epsilon}$. And what is the maximum? Taking the
 $\frac{w_0 \cos \epsilon}{w_0 \cos \epsilon}$. And what is the maximum? Taking the
 Example 10 increase is radius, increase cost and become

so that given $\left\langle \cdot, \cdot \right\rangle$ are signal is not observable in the metrical v_1, v_2 . Anter v_2 ,

so well to increase its radius, increase cost and become

so t allowed to increase its radius, increase cos { and become

ial.

iai.

iai.

is so that given $\left\langle \cdot \right\rangle$ (106) is satisfied for all t ? It is that the maximum of

ses than or equal to $\frac{1}{w_0 \cos \epsilon}$. And what is the m to $\frac{1}{w_0} \cos \leftarrow$
 $\frac{1}{\sin(\leftarrow + \cos \leftarrow)} (1 - tc(\leftarrow) \sin \leftarrow + \leftarrow \cos \leftarrow) \right)$, and maximum occurs

stituting in (106), if
 $\frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow} \leq \frac{ce}{w_0}$ (107)
 $\frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow} \leq \frac{ce}{w_0}$ (107)

and there is no t_1, t_2 than or equal to $\frac{1}{w_0 \cos \epsilon}$. And what is the maximum? Taking the
 $\pi(\epsilon + \cos \epsilon) = e^{-i\epsilon/(3i\alpha + \cos \epsilon)} (1 - t c(\epsilon)) \sin \epsilon + \cos \epsilon)$, and maximum occurs
 $\frac{\cos \epsilon}{3 \sin \epsilon} = e^{-i\epsilon/(3i\alpha + \cos \epsilon)} (1 - t c(\epsilon)) \sin \epsilon + \cos \epsilon)$, and maximum occurs
 $\frac{\cos \epsilon}{3$ fied for all t? It is that the maximum of

what is the maximum? Taking the
 $\sin \leftarrow + \cos \leftarrow$)), and maximum occurs

if

if
 $\frac{ce}{w_0}$ (107)
 $\frac{ce}{w_0}$ (108)
 $\frac{\cos \leftarrow}{w_0}$ and using the Lambert function
 $\frac{\cos \leftarrow}{w_0 \cos \$ r equal to $\frac{1}{w_0 \cos \epsilon}$. And what is the maximum? Taking the
 $= e^{-x(\frac{1}{2} \sin \epsilon + \frac{1}{\cos \epsilon})} (1-tc(\frac{1}{2} \sin \epsilon + \frac{1}{\cos \epsilon}))$, and maximum occurs

So substituting in (106), if
 $\frac{\cos \epsilon}{\frac{1}{2} \sin \epsilon + \frac{1}{\cos \epsilon}} \leq \frac{ce}{w_0}$ (107) so that given ϵ , (106) is satisfied for all t? It is that the maximum of

han or equal to $\frac{1}{w_0 \cos \epsilon}$. And what is the maximum? Taking the
 $\frac{1}{\epsilon}$ $\cos \epsilon$ $= e^{-\pi (1) \sin(\epsilon + \cos \epsilon)} (1 - t c) \sin \epsilon + -\cos \epsilon$), and maximum occur

at $t = \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2}}}$. So substituting in (106), if $+ \sim \cos \left(\right)$. So substituting in (106), if - . So substituting in (10
 $\frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow}$

ed for all *t* and there is no

en
 $\frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow}$

we solve (106)
 $\frac{\sin \leftarrow + \cos \leftarrow}{\cos \leftarrow}$

in $\leftarrow + \cos \leftarrow$) $\geq -\frac{c(\frac{1}{2} \sin \leftarrow + \frac{1}{2} \sin \leftarrow)}{w_0 \cos \leftarrow}$

$$
\frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow} \leq \frac{ce}{w_0} \tag{107}
$$

condition (106) is satisfied for all *t* and there is no t_1 , t_2 $t₂$ In the opposite case, when

$$
\frac{\cos \leftarrow}{\sin \leftarrow + \leftarrow \cos \leftarrow} > \frac{ce}{w_0} \tag{108}
$$

In order to find t_1 and t_2 we solve (106)

derivative
$$
\frac{-te^{-i\omega t} \sin \omega t - \cos \omega t}{dt} = e^{-i\omega t} \cos \omega t + \cos \omega t
$$
 (1 – $tc(\frac{1}{2} \sin \omega + \omega \cos \omega)$), and maximum occurs
\nat $t = \frac{1}{c(\frac{1}{2} \sin \omega + \omega \cos \omega)}$. So substituting in (106), if
\n $\frac{\cos \omega t}{\frac{1}{2} \sin \omega + \omega \cos \omega \omega} \leq \frac{ce}{w_0}$ (107)
\ncondition (106) is satisfied for all *t* and there is no t_1 , t_2
\nIn the opposite case, when
\n $\frac{\cos \omega t}{\frac{1}{2} \sin \omega + \omega \cos \omega} \geq \frac{ce}{w_0}$ (108)
\nIn order to find t_1 and t_2 we solve (106)
\n $-c(\frac{1}{2} \sin \omega + \omega \cos \omega)te^{-t\omega t} \sin \omega t + \omega \cos \omega t \geq -\frac{c(\frac{1}{2} \sin \omega + \omega \cos \omega)}{w_0 \cos \omega}$ and using the Lambert function
\n(*W*(.)) we obtain $-c(\frac{1}{2} \sin \omega t + \omega \cos \omega)t \geq W(-\frac{c(\frac{1}{2} \sin \omega t + \omega \cos \omega)}{w_0 \cos \omega})$ and finally

 $(W(.))$ we obtain $-c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow) t \geq W(-\frac{C(1 - \sin \leftarrow + \cos \leftarrow)}{2})$ $\int_0^{\infty} \cos \left(\frac{\sin \left(\cos \left(\frac{\pi}{2} \right) \right)}{\sin \left(\frac{\pi}{2} \right)} \right) \sin \left(\frac{\pi}{2} \right) d\theta d\theta$ $c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow)$ and finally $w_0 \cos \leftarrow$ $+ \sim \cos \left(\frac{1}{2} \right)$ $\begin{aligned}\n\text{Equation 1: } \n\frac{1}{w_0 \cos \epsilon} &\text{Area in } \frac{1}{w_0 \cos \epsilon} \\
&= e^{-ac(\frac{1}{2} \sin \epsilon + -\cos \epsilon)} (1 - tc(\frac{1}{2} \sin \epsilon + \frac{1}{2} \cos \epsilon)) , \text{ and maximum occurs} \\
\text{So substituting in (106), if} \\
&\frac{\cos \epsilon}{\frac{1}{2} \sin \epsilon + \frac{1}{2} \cos \epsilon} \leq \frac{ce}{w_0} \\
&\text{for all } t \text{ and there is no } t_1, t_2 \\
&\frac{\cos \epsilon}{\frac{1}{2} \sin \epsilon + \frac{1$ $c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow)$ (100) or equal to $\frac{w_0 \cos x}{w_0 \cos x}$. And what is the maximum: Faxing the
 $e^{x/2} = e^{-ac(1)\sin x + \cos x} (1-tc(\frac{1}{2}\sin x + \cos x))$, and maximum occurs
 $\frac{\cos x}{\frac{1}{2}\sin x + \cos x} \le \frac{ce}{w_0}$ (107)

d for all t and there is no t₁, t₂

n
 $\frac{\cos$ $+ \sim \cos \left(\frac{1}{2} \right)$ $w_0 \cos \left(\frac{1}{2} \right)$ (109) vative $\frac{d}{dt}te^{-sc(3\sin c + \cos c)} = e^{-ic(3\sin c + \cos c)}(1-tc(3\sin c + \cos c))$
 $= \frac{1}{c(\frac{1}{2}\sin \left(\frac{1}{2}\cos \left(\frac{1}{2}\sin \left(\frac{1}{2}\cos \left(\frac{1}{2}\sin \left(\frac{1}{2}\cos \left(\frac{1}{2}\sin \left(\frac{1}{2}\cos \left(\frac{1}{2}\cos \left(\frac{1}{2}\cos \left(\frac{1}{2}\cos \left(\frac{1}{2}\cos \left(\frac{1}{2}\cos \left(\frac{1}{2}\cos \left(\frac{1}{2}\cos \left(\frac{1$

 0 cos $\sqrt{ }$ This, by the theory on Lambert functions, gives two solutions, when

$$
\frac{c(\frac{1}{2}\sin\leftarrow + \cos\leftarrow)}{w_0 \cos\leftarrow} > -\frac{1}{e}
$$
, which by the way is the same as condition (108) as expected.

Wative $\frac{1}{dt}te^{-t/(2\pi t + \pi t)} = e^{-t/(2\pi t + \pi t)}(1-t(t) \sin \pi t + \cos \pi t)$
 $= \frac{1}{c(\pi t + \cos \pi)}$. So substituting in (106), if
 $\frac{\cos \pi}{\sin \pi t + \cos \pi} \leq \frac{ce}{w_0}$

dition (106) is satisfied for all t and there is no t_1 , t_2

the oppos $\frac{1}{\sin \leftarrow + \cos \leftarrow}$. So substituting in (106), if
 $\frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow} \leq \frac{ce}{w_0}$

on (106) is satisfied for all *t* and there is no *t*₁, *t*₂

pposite case, when
 $\frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow} \leq \frac{ce}{w_0}$
 $\frac{\cos \leftarrow}{\sin$ derivative $\frac{d}{dt}te^{-\pi x/3 \cot x - \cot x} = e^{-\pi x/3 \cot x - \cot x} (1 - t c(\frac{1}{2}) \sin x + \cos x \cos x)$, and maximum occurs

at $t = \frac{\cos x}{c(\frac{1}{2} \sin x + \cos x)}$. So substituting in (106), if
 $\frac{\cos x}{\frac{1}{2} \sin x + \cos x} \leq \frac{ce}{w_0}$ (107)

condition (106) is sa $-c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow) te^{-tc(\frac{1}{2} \sin \leftarrow + \cos \leftarrow)} \geq -\frac{c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow)}{w_0 \cos \leftarrow}$ and to $W(0,0)$ we obtain $-c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow) t \geq W(-\frac{c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow)}{w_0 \cos \leftarrow} t \leq -\frac{1}{c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow)} W(-\frac{c(\frac{1}{2} \sin \leftarrow + \cos \left$ 0 cos $\sqrt{ }$ $\frac{\cos x}{\sin x + \cos x} \le \frac{ce}{w_0}$ (107)

or all *t* and there is no *t*₁, *t*₂

or all *t* and there is no *t*₁, *t*₂
 $\frac{\cos x}{\sin x + \cos x} > \frac{ce}{w_0}$ (108)

solve (106)
 $+\cos x$ > $\ge -\frac{c(3 \sin x + \cos x)}{w_0 \cos x}$ and using the L $\frac{\cos \zeta}{\sin \zeta + -\cos \zeta} \leq \frac{ce}{w_0}$ (1)

Sisting for all t and there is no t₁, t₂

when
 $\frac{\cos \zeta}{\sin \zeta + -\cos \zeta} > \frac{ce}{w_0}$ (1)

Sin $\zeta + -\cos \zeta > \frac{ce}{w_0}$ (1)

d t₂ we solve (106)
 $\frac{ce^{-tc(1)\sin(\zeta + \cos \zeta)}}{w_0 \cos \zeta}$ a $c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow)$ $x + -\cos \epsilon$)
 $\cos \epsilon$
 $\cos \epsilon$ $\frac{\cos \epsilon}{w_0}$ (107)

5) is satisfied for all *t* and there is no *t*₁, *t*₂
 \csc \csc , when
 $\frac{\cos \epsilon}{\sin \epsilon + \cos \epsilon} > \frac{ce}{w_0}$ (108)
 $d t_1$ and *t*₂ we solve (106)
 $\cos \epsilon$) $te^{-\epsilon(1)\sin(\epsilon + \$ $\frac{\cos \epsilon}{\frac{1}{2} \sin \epsilon + \cos \epsilon} \leq \frac{ce}{w_0}$ (107)

atisfied for all *t* and there is no *t*₁, *t*₂

when
 $\frac{\cos \epsilon}{\sin \epsilon + \cos \epsilon} > \frac{ce}{w_0}$ (108)

and *t*₂ we solve (106)
 $\int \frac{1}{\sin \epsilon + \cos \epsilon} \geq \frac{c(1) \sin \epsilon + - \cos \epsilon}{w_0 \cos \epsilon}$ an 106), if
 $\frac{cos x}{cos x} \le \frac{ce}{w_0}$ (107)

is no t_1 , t_2
 $\frac{ce}{cos x} > \frac{ce}{w_0}$ (108)
 $\frac{c+2 cos x}{cos x}$ and using the Lambert function
 $\frac{c(\frac{1}{2} sin x + -cos x)}{w_0 cos x}$ and finally
 $\frac{c(-\frac{c(\frac{1}{2} sin x + -cos x)}{w_0 cos x})}{w_0 cos x}$ (109) + - cos <)
 $\frac{1}{3}$ sin c + - cos < $\frac{1}{3}$ sin c + - cos < $\frac{1}{w_0}$ (107)

is satisfied for all t and there is no t₁, t₂

axe, when
 $\frac{\cos \varsigma}{\frac{1}{3} \sin \varsigma + - \cos \varsigma} > \frac{ce}{w_0}$ (108)

t₁ and t₂ we solve (106 $+\infty$ cos < $\frac{1}{2}$ w_0 cos < $\frac{1}{2}$ and W(.)) we obtain $-c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow) t \ge W(-\frac{c(\frac{1}{2} \sin \leftarrow) t}{c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow) t})$
 $t \le -\frac{1}{c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow) t}$

This, by the theory on Lambert functions, gives two
 $-\frac{c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow) t}{w_0 \cos \leftarrow} - \frac{1}{e},$ wh 0 cos $\sqrt{ }$ } sin ϵ + ~ cos ϵ ⁻ w_0

is satisfied for all t and there is no t_1 , t_2

case, when
 $\frac{\cos \epsilon}{\sin \epsilon + \cos \epsilon} > \frac{ce}{w_0}$
 t_1 and t_2 we solve (106)
 $\cos \epsilon$) $te^{-\pi c(1 \sin \epsilon + \cos \epsilon)} \ge -\frac{c(1 \sin \epsilon + \cos \epsilon)}{w_0 \cos \epsilon}$ a on (106) is satisfied for all t and there is no t_1 , t_2

pposite case, when
 $\frac{\cos \varsigma}{\frac{1}{3} \sin \varsigma + - \cos \varsigma} > \frac{ce}{w_0}$

t to find t_1 and t_2 we solve (106)
 $n \varsigma + - \cos \varsigma$) $te^{-\pi(\frac{1}{2} \sin \varsigma + \frac{\varsigma}{\cos \varsigma})} \ge -\frac{c(\frac$ $c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow)$ where $W(\cdot)$ is condition (106) is satisfied for all *t* and there is no t_1 , t_2

In the opposite case, when
 $\frac{\cos x}{\sin x + \cos x} > \frac{ce}{w_0}$ (108)

In order to find t_1 and t_2 we solve (106)
 $-c(\} \sin x + \cos x)ie^{-x(x)\sin x + \cos x} \ge -\frac{c(\} \sin x + \$ ion (106) is satisfied for all *t* and there is no t_1 , t_2

opposite case, when
 $\frac{\cos x}{\sin x + \cos x} > \frac{ce}{w_0}$ (108)
 $\frac{\sin x}{\sin x + \cos x}$ we solve (106)
 $\sin x - \cos x$ $Ie^{-x(1\sin x + \cos x)} \ge -\frac{c(1 \sin x + \cos x)}{w_0 \cos x}$ and using the La $\frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow \cos \leftarrow} \frac{ce}{w_0}$ (10
 t and there is no *t*₁, *t*₂
 $\frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow \cos \leftarrow} \frac{ce}{w_0}$ (10
 $\frac{\sin \leftarrow + \cos \leftarrow \cos \leftarrow}{w_0 \cos \leftarrow}$ and using the Lambert functions $w_0 \cos \leftarrow$ $w_0 \cos \leftarrow$ and using the La **a** (106) is satisfied for all t and there is no t_1 , t_2
posite case, when
 $\frac{\cos \leftarrow}{\sin x + \cos \leftarrow} \frac{c e}{w_0}$
to find t_1 and t_2 we solve (106)
 $\leftarrow + -\cos \leftarrow \sec \leftarrow \frac{c}{w_0}$
to find t_1 and t_2 we solve (106)
 $\leftarrow +$ mathion (106) is satisfied for all *t* and there is $\cos \leftarrow \cos \leftarrow \sin \theta$

mathion (106) is satisfied for all *t* and there is no t_1 , t_2

the opposite case, when
 $\frac{\cos \leftarrow \cos \leftarrow \sin \theta t_1 t_2}{\sin \leftarrow + \cos \leftarrow \cos \leftarrow \sin \theta t_2}$ (108)
 $\frac{W}{(1 + \infty) \cos \left(\frac{1}{2}\right)} W_{-1}(-\frac{C(y \sin \left(\frac{1}{2}\right) \cos \left(\frac{1}{2}\right))}{w_0 \cos \left(\frac{1}{2}\right)})$ where $W_0(.)$ is the solution near the

origin and W_{-1} (.) is the solution further away from -1 on the negative branch of the Lambert function.

Looking now at $cos\{\theta$ we take the derivative with respect to t to find its interior minimum (that corresponds to a maximum of $\{\)$ along the path of the signal. After some straight forward manipulation we find,

$$
\frac{d}{dt}\cos\left\{\frac{\sinh(\theta)}{2}\right\} = \begin{cases}\n-\frac{w_0^2te^{-2ct(\sin(\epsilon + \cos\epsilon))}(1-ct(\sin\epsilon + \cos\epsilon))e_0^2t^2\sin^2\epsilon}{\sqrt{1+e^{-2ct(\sin(\epsilon + \cos\epsilon))}w_0^2t^2\sin^2\epsilon}} + \frac{\sin^2\epsilon\sqrt{1-e^{-2ct(\sin(\epsilon + \cos\epsilon))}w_0^2t^2\cos^2\epsilon}}{\sqrt{1+e^{-2ct(\sin(\epsilon + \cos\epsilon))}w_0^2t^2\sin^2\epsilon}}\n\end{cases}
$$
\n(110)
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\n(110)
\n(111)
\n(111)
\n(121)
\n(131)
\n(142)
\n(15)
\n(16)
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\n(110)
\n(111)
\n(112)
\n(113)
\n(114)
\n(115)
\n(116)
\n

The second factor, in big parentheses, is positive. Therefore, looking at the first factor we see that it starts negative for $t = 0$ and then changes sign at

$$
t_{\text{max}} = \frac{1}{c(\} \sin \leftarrow + \cos \leftarrow) = \frac{\sqrt{\dots^2 + z^2}}{c(\} \dots + \cos \leftarrow)
$$
 (111)

$$
z_{\text{f max}} = ct_{\text{f max}} \cos \leftarrow \frac{\cos \leftarrow \sqrt{3 \sin \left(\cos \left(\frac{\pi}{2}\right)\right)}}{\sin \left(\cos \left(\frac{\pi}{2}\right)\right)} \tag{113}
$$

1^{24+-cos(x)} $w_0^2 t^2 \cos^2 \left\{\sqrt{1+e^{-2ct(3\sin\epsilon + \cos\epsilon)}} w_0^2 t^2 \sin^2 \left\{\sqrt{1+e^{-2ct(3\$ Equations (112) and (113) can be combined using the definition of cos $\left\langle \right\rangle$ and sin $\left\langle \right\rangle$ in a single equation giving the locus of points where $\{$ attains its maximum along the path of

$$
\}_{\cdots \{ \max} + \sim z_{\{ \max}} = 1 \tag{114}
$$

$$
\frac{d}{dt} \cos \xi = \begin{cases}\n\cos^2 \xi - \frac{1}{\sqrt{1 - e^{-2\alpha(\xi)\sin(\xi + \cos \xi)}} \frac{\sin^2 \xi}{\sqrt{1 + e^{-2\alpha(\xi)\sin(\xi + \cos \xi)}} \frac{\cos^2 \xi}{\sqrt{1 + e^{-2\alpha(\xi)\sin(\xi + \cos \xi)}} \frac{\cos^2 \xi}{\sqrt{1 + e^{-2\alpha(\xi)\sin(\xi + \cos \xi)}} \frac{\cos^2 \xi}{\sqrt{1 + e^{-2\alpha(\xi)\sin(\xi + \cos \xi)}} \frac{\sqrt{1 + e^{-2\alpha(\xi)\sin(\xi + \cos \xi)}} \frac{\sqrt{1 + e^{-2\alpha(\xi)\sin^2 \xi}}}{\sqrt{1 + e^{-2\alpha(\xi)\sin^2 \xi}}} \\
\frac{d}{dx} \tan \xi = \frac{1}{c(\xi)\sin \xi + \cos \xi} = \frac{\sin \xi}{c(\xi)\sin \xi + \cos \xi} = \frac{\sin \xi}{c(\xi)\sin \xi + \cos \xi} \tag{112}
$$
\nThis corresponds to\n
$$
\frac{d}{dx} = ct_{\text{max}} \sin \xi = \frac{\sin \xi}{\sin \xi + \cos \xi} \tag{112}
$$
\nEquations (112) and (113) can be combined using the definition of cos ξ and sin ξ in a single equation giving the locus of points where ξ attains its maximum along the path of a signal,
\n
$$
\frac{1}{2} \cos \xi_{\text{max}} = ct_{\text{max}} \cos \xi = \frac{1}{\sin \xi + \cos \xi} \tag{113}
$$
\nwhich is a straight line. The value of cos ξ at the minimum is in (\ldots, z) space,
\n
$$
\cos \xi_{\text{max}} = \cos \xi \big|_{t = \tan \xi} = \sqrt{\frac{(\xi \sin \xi + \cos \xi)^2 c^2 - \mu_0^2 e^{-2} \cos^2 \xi}{(\xi \sin \xi + \cos \xi)^2 \cos^2 \xi}} \frac{\cos \xi}{\cos \xi} \tag{114}
$$
\nwhich is the same as\n
$$
\cos \xi_{\text{max}} = \frac{\sqrt{c^2 - \mu_0^2 e^{-2} \cos^2 \xi}}{\sqrt{c^2 + \mu_0^2 e^{-2} \cos^2 \xi}} \frac{\cos \xi}{\cos \xi} \tag{116}
$$
\nThis by the way is the same

which is the same as

$$
\cos\left\{\right._{\text{max}} = \sqrt{\frac{c^2 - w_0^2 e^{-2} z_{\text{max}}^2}{c^2 + w_0^2 e^{-2} ..._{\text{max}}^2}}\tag{116}
$$

The condition in (115) is a condition on \langle ,

$$
z_{\text{max}} = \frac{\cos \leftarrow}{\sin \leftarrow + \cos \leftarrow \cos \leftarrow} \leq \frac{ce}{w_0}
$$
 (117)

This by the way is the same condition as (107) that is required for the non existence of the solutions t_1 , t_2 .

The parametric plot of z_{max} versus m_{max} is a straight line given by (114). The plot appears in Figure 8 where two cases are shown. In Figure 8 (a) the condition (117) is satisfied for all z and all \langle . In order for this to be true, we require the maximum over \langle of the left hand side of (117) to be less or equal to the right hand side. This maximum occurs at $\epsilon = 0$ and at this point condition (117) becomes

$$
\frac{1}{\sim} \le \frac{ce}{w_0} \tag{118}
$$

In this case the minimum of $cos\{\cdot\}$ in the direction of the path of the signal occurs on the rhombus by revolution ABCD for all $\left\langle \right\rangle$.

In Figure 8 (b), (118) does not hold. This means that for some $\left(117\right)$ is valid and for the rest it does not hold. In particular, talking about the first quadrant, because the same hold for the rest by symmetry, for $\epsilon \leq G\hat{O}B$ it does not hold, but it holds for $\epsilon > G\hat{O}B$. So for $\zeta \leq G\hat{O}B$, cos { does not attain a minimum on the rhombus, because it becomes zero before reaching it, as it encounters the curve AB (marked as t_1), which is the solution of t_1 *t* . This solution as well as t_2 (the curve between AB marked as t_2) exist, when condition (117), which the same as (107), holds. Between t_1 and t_2 the signal is not observable by O' , because $cos\{\theta$ becomes imaginary, unless we allow O' to observe signals travelling with speed greater than c . After t_2 the radius starts to increase again and the angle of deflection is of the signal occurs on the
 $\epsilon \leftarrow (117)$ is valid and for the

frant, because the same hold

holds for $\epsilon > G\hat{O}B$. So for

because it becomes zero
 t_1 , which is the solution of t_1
 t_2) exist, when conditio th of the signal occurs on the

me < (117) is valid and for the

adrant, because the same hold

it holds for $\langle \rangle < G\hat{O}B$. So for

s, because it becomes zero
 $s t_1$), which is the solution of t_1

as t_2) exist,

{ returns asymptotically to zero. Observe that at A and B $\frac{\cos x}{\cos x}$ 0 $\cos \leftarrow$ *ce*_{Le the} *w* ζ ce I_{rad} $+ \sim \cos \leftarrow w_0$. In the

opposite case, when $\langle \rangle GOB$, the local minimum of cos is attained on the remaining of the rhombus BCD and EFA. After that the signal returns slowly to the radial direction. Note that in Figure 8 we plot z_{max} vs \ldots _{max}. But observer *O'* sees \ldots ' instead of \ldots , which is contracted with respect to \dots . Therefore, the shape that observer O' will see will not be a rhombus but will be deformed since it will be contracted in the ... direction.

 $\cos \xi = 0$, when they arrive at the curve marked t_1 . So the deflection angle ξ has reached 90° degrees and cannot increase any more. Between t_1 and t_2 the signal is not observable by observer *O*[']. After that, the deflection starts decreasing and asymptotically becomes zero, thus the signal returns to the radial direction. If on the other hand, $\langle \cdot \rangle GOB$, then the signal attains its maximum deflection on the line of the rest of the rhombus BCD and EFA and after that it decreases asymptotically towards zero returning to the radial direction. The diagrams show z_{max} vs \ldots _{{ max}, while observer O' sees \ldots' . So we must imagine a contraction in the \ldots direction to reflect what observer O' sees, and then the rhombus will be deformed accordingly.

In Figure 9 we plot $cos \{_{max}$ as a function of \leftarrow when (101) does hold. In (a) for a small value of \sim and in (b) for a big value of \sim .

Figure 9 Plot of cos $\{\mathbf{r}_{\text{max}}\}$ as a function of the inclination of the signal, \langle , as it varies **Figure 9** Plot of $\cos \{\frac{1}{\text{max}}\}$ as a function of the inclination of the signal, $\langle \cdot \rangle$, as it varies
from 0 to $f/2$ when condition (118) is valid. (a) corresponds to small value of \sim while
(b) to a bigger one. (b) to a bigger one. In (b) the deflection of the signal in the z direction is smallest and it increases (the cosine decreases) gradually as we approach the radial direction. **Figure 9** Plot of $\cos\left(\frac{1}{\max}\right)$ as a function of the inclination of the signal, \leftarrow , as it varies
from 0 to $f/2$ when condition (118) is valid. (a) corresponds to small value of \sim while
(b) to a bigger one. In (ation of the signal, $\left\langle \right\rangle$, as it varies
rresponds to small value of \sim while in the z direction is smallest and it
roach the radial direction.
roblem is described by equations (
roblem is described by equations (
 nclination of the signal, $\left\langle \right\rangle$, as it varies

(a) corresponds to small value of \sim while

signal in the z direction is smallest and it

e approach the radial direction.
 z O''

the problem is described by eq External in the signal, $\left\langle \right\rangle$, as it varies

corresponds to small value of \sim while

grad in the z direction is smallest and it

approach the radial direction.
 O''

ne problem is described by equations (81)

relo **Figure 9** Plot of $\cos\left(\frac{1}{\tan x}\right)$ as a function of the inclination of the signal, $\langle \cdot \rangle$, as it varies

from 0 to $f/2$ when condition (118) is valid. (a) corresponds to small value of \sim while

increases (the cosi s {_{max} as a function of the inclination of the signal, ϵ , as it varies

n condition (118) is valid. (a) corresponds to small value of - while

In (b) the deflection of the signal in the z direction is smallest and it of $\cos\left(\frac{1}{\tan x}\right)$ as a function of the inclination of the signal, $\langle \cdot \rangle$, as it varies
when condition (118) is valid. (a) corresponds to small value of \sim while
me. In (b) the deflection of the signal in the z dire Plot of $\cos\left(\frac{1}{n\omega_0}\right)$ as a function of the inclination of the signal, ϵ , as it varies $f / 2$ when condition (118) is vailed, (a) corresponds to small value of \sim while digere one. In (b) the deflection of the s **EXECUTE:** The included of the inclination of the signal, $\langle \cdot \rangle$, as it varies ondition (118) is valid. (a) corresponds to small value of \sim while the signal in the *z* direction is smallest and it be checked by gradu Plot of $\cos\left(\frac{1}{\text{max}}\right)$ as a function of the inclination of the signal, ϵ , as it varies $f/2$ when condition (118) is valid. (a) corresponds to small value of \sim while the cosine deterates gradually as we approac cos {_{max} as a function of the inclination of the signal, \leftarrow , as it varies
en condition (118) is valid. (a) corresponds to small value of \sim while
t. In (b) the deflection of the signal in the z direction is smalles cos $\{\mathbf{m}_{\text{max}}$ as a function of the inclination of the signal, $\langle \cdot \rangle$, as it varies
hen condition (118) is valid. (a) corresponds to small value of \sim while
e. In (b) the deflection of the signal in the z directio function of the inclination of the signal, \langle (118) is valid. (a) corresponds to small v

efflection of the signal in the z direction is

gradually as we approach the radial dire

e for Observer O"

server. For him the as a function of the inclination of the signal, $\langle \cdot \rangle$, as it varies

ition (118) is valid. (a) corresponds to small value of \sim while

the deflection of the signal in the z direction is smallest and it

asses) gradua tion of the inclination of
8) is valid. (a) corresponding the signal in the r
colling as we approach the resolution of the angular velocity w
cording t $s\{_\text{max}$ as a function of the inclination of the signal, $\langle \cdot \rangle$, as it varies

condition (118) is valid. (a) corresponds to small value of \sim while

(b) the deflection of the signal in the z direction is smallest an x as a function of the inclination of the signal, \leftarrow , as it varies
oldition (118) is valid. (a) corresponds to small value of \sim while
the deflection of the signal in the z direction is smallest and it
reases) gradua nction of the inclination of the signal, ϵ , as it varies

18) is valid. (a) corresponds to small value of \sim while

ection of the signal in the z direction is smallest and it

radually as we approach the radial direc

9.3 Rotation with Slippage for Observer *O*

Observer O'' is the far away observer. For him the problem is described by equations (81) to (87) with the only difference that the angular velocity *w* is now not constant but varies sine decreases) gradually as we approach the radial direction.

with Slippage for Observer O''

the far away observer. For him the problem is described by equations (81)

only difference that the angular velocity w is now Observer O"

er. For him the problem is described by equat

the angular velocity w is now not constant bording to $w = w_0 e^{-1}e^{-x} = w_0 e^{-ct(\frac{1}{2} \sin x + \cos x)}$

call from (85) that we denote $\frac{1}{w}$ which is the
 $\frac{w}{2}e^{-2(\frac{$ 1 the problem is described by α

in velocity w is now not const
 $w = w_0 e^{-\lambda - z} = w_0 e^{-ct(\frac{1}{2} \sin \alpha + \cos \alpha)}$

(85) that we denote $\overline{...}$ which
 $\overline{z_0} = \frac{ct \sin \alpha}{\sqrt{1 + w_0^2 t^2 \sin^2 \alpha e^{-2ct(\frac{1}{2} \sin \alpha + \alpha)}}} = \frac{1}{\sqrt{1 + w_0^2 t^2$ the signal in the z direction is smallest and it

since signal in the z direction is smallest and it

since approach the radial direction.

The problem is described by equations (81)

ular velocity w is now not constant b as we approach the radial direction.

Prver O''

thim the problem is described by equations (81)

gular velocity w is now not constant but varies

to $w = w_0 e^{-1 - -z} = w_0 e^{-\alpha(1) \sin(x + \cos x)}$

(91) on (85) that we denote $\overline{...}$ signal in the z direction i

e approach the radial dire

r O"

the problem is described

r velocity w is now not of
 $w = w_0 e^{-\lambda - 2z} = w_0 e^{-ct(\lambda \sin \theta)}$

85) that we denote $\frac{1}{w}$ wl
 $\frac{ct \sin \theta}{\sqrt{1 + w_0^2 t^2 \sin^2 \theta} e^{-2ct(\theta)}}$
 we approach the radial direction.

er O"

m the problem is described by equations (81)

ar velocity w is now not constant but varies
 $w = w_0 e^{-1 - -z} = w_0 e^{-\alpha t \sin x + \cos x}$

(85) that we denote \overline{w} which is the average
 \frac deltary of the problem is described by equations (81)
 r O''

the problem is described by equations (81)
 r O''

the problem is described by equations (81)
 r velocity w is now not constant but varies
 $w = w_0 e^{-2x-xz}$ Where **Solution** (the cosine decreases) gradually as we approach the radial direction.

9.3 Rotation with Slippage for Observer O''

(b) Observer O'' is the far away observer. For him the problem is described by equations O' is the far away observer. For him the problem is described by equations (81)

(ith the only difference that the angular velocity w is now not constant but varies

itsincate from the origin according to $w = w_0e^{-3x-5t} = w$ r away observer. For him the problem is described by equations (81)

ifference that the angular velocity w is now not constant but varies

the origin according to $w = w_0 e^{-1.5-xz} = w_0 e^{-\alpha(1)\sin(z + \cos(z))}$
 $w = c t \sin \left(\frac{\arcsin(\pi \cos(z))$ lippage for Observer O''

way observer. For him the problem is described by equations (81)

ference that the angular velocity w is now not constant but varies

he origin according to $w = w_0 e^{-3-c-x} = w_0 e^{-\alpha(3 \sin x + \cos x)}$
 $= c t$ bit with suppage for Observer C

is the far away observer. For him the problem is described by

the only difference that the angular velocity w is now not constance

from the origin according to $w = w_0 e^{-1 - x z} = w_0 e^{-\alpha t/ \$ away observer. For him the problem is described by equations (81)
 fference that the angular velocity w is now not constant but varies

the origin according to $w = w_0 e^{-3x-cx} = w_0 e^{-ctt \sin x + \cos x}$
 $= ct \sin \epsilon$ (recall from (85) For an analytic or USE vertical By equations (81)

s the far away observer. For him the problem is described by equations (81)

cond volifference that the angular velocity w is now not constant but varies

using $\lim_{m \to \in$

of \ldots by simply \ldots)

$$
\dots'' = \frac{\dots c}{\sqrt{c^2 + w^2 \dots^2}} = \frac{\dots c}{\sqrt{c^2 + w_0^2 \dots^2 e^{-2(1 - x + z)}}} = \frac{ct \sin \leftarrow}{\sqrt{1 + w_0^2 t^2 \sin^2 \leftarrow e^{-2ct(1 \sin \leftarrow + \cos \leftarrow)}}}
$$
(119)

Taking the time derivative and equating to the radial velocity using equation (83) we obtain,

the far away observer. For him the problem is described by equations (81)
\nonly difference that the angular velocity *w* is now not constant but varies
\n
$$
\sinh y = ct \sin \left(\arctan \left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)\right)
$$

\n $\frac{d}{dx} = ct \sin \left(\arctan \left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)\right)$
\n $\frac{d}{dx} = -ct \sin \left(\arctan \left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)\right)$
\n $\frac{d}{dx} = -\frac{c}{\sqrt{c^2 + w_{0}^2 - c^2(3 - 1 - 1)}} = \frac{ct \sin \left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)}{\sqrt{1 + w_{0}^2 t^2 \sin^2 \left(\frac{3\pi}{4} + \frac{3\pi}{4}\right) \cdot \left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)}} = \frac{ct \sin \left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)}{\sqrt{1 + w_{0}^2 t^2 \sin^2 \left(\frac{3\pi}{4} + \frac{3\pi}{4}\right) \cdot \left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)}} = \frac{c}{c} \sin \left(\arctan \left(\frac{3\pi}{4}\right)\right)$
\n $\cos \left(\frac{1 + c}{\sin \left(\frac{3\pi}{4}\right)}\right) = \frac{3}{\sqrt{1 + w_{0}^2 t^2 \sin^2 \left(\frac{3\pi}{4}\right)}} = \frac{c}{c} \sin \left(\arctan \left(\frac{3\pi}{4}\right)\right)$
\n $\cos \left(\frac{1 + c}{\sin \left(\frac{3\pi}{4}\right)}\right) = \frac{1 + w_{0}^2 t^2 \sin^2 \left(\frac{3\pi}{4}\right)}{1 + c \sin^2 \left(\frac{3\pi}{4}\right)^2}$
\n $\cos \left(\frac{1}{\cos \left(\frac{3\pi}{4}\right)}\right) = \frac{w_{0}r}{r} = \tan \left(\frac{r}{\sqrt{1 + w_{0}^2 t^2 \sin^2 \left(\frac{3\pi}{4}\right)}}\right)$
\n $\cos \left(\frac{r}{\cos \left(\frac{3\pi}{4}\$

By dividing (82) by (120) we find

$$
\tan\left\{\right.^{n} = wt \frac{1 + w^{2} t^{2} \sin^{2} \leftarrow 1 + c s t^{3} w^{2} \sin^{2} \leftarrow 121}
$$
\n(121)

Dividing (82) by (84) and using (86) and (121) we find

$$
\tan \leftarrow'' = \frac{w...''}{c \cos \leftarrow \sin \left\{ r \right\}} = \tan \leftarrow \frac{wt}{\sqrt{1 + w^2 t^2 \sin^2 \left(\sqrt{1 + \frac{(1 + c \sin^2 w^2 \sin^2 \left(\sqrt{1 + w^2 t^2 \sin^2 \left(\sqrt{1 + w^2 t
$$

Using (84) we obtain

Observe *O*ⁿ is the far away observer. For him the problem is described by equations (81)
to (87) with the only difference that the angular velocity *w* is now not constant but varies
with the distance from the origin according to
$$
w = w_0 e^{-3-x-z} = w_0 e^{-ct(3 \sin t - \cos x)}
$$

From (86) and using $... = ct \sin \leftarrow$ (recall from (85) that we denote $...$ which is the average
of $...$ by simply $...$)

$$
...'' = \frac{...c}{\sqrt{c^2 + w^2 ...^2}} = \frac{...c}{\sqrt{c^2 + w_0^2 ...^2 e^{-2(2-x+z)}}} = \frac{ct \sin \leftarrow}{\sqrt{1 + w_0^2 t^2 \sin^2 \leftarrow e^{-2ct(3 \sin \leftarrow + \cos \leftarrow t \right)}}}
$$
(119)
Taking the time derivative and equating to the radial velocity using equation (83) we
obtain,

$$
c \sin \leftarrow \frac{(1 + c(\sin \leftarrow + \cos \leftarrow t \right))^3 w^2 \sin^2 \leftarrow)}{(1 + w^2 t^2 \sin^2 \leftarrow t \right)^2} = \frac{1}{c} \sin \leftarrow \frac{1}{\cos \leftarrow t \right)} \frac{(120)}{(120)}
$$

By dividing (82) by (120) we find

$$
\tan \leftarrow \frac{1 + w^2 t^3 \sin^2 \leftarrow}{1 + c S t^3 w^2 \sin^2 \leftarrow}
$$
(121)
Where $s = \frac{1}{2} \sin \leftarrow + \cos \leftarrow$
Dividing (82) by (84) and using (86) and (121) we find

$$
\tan \leftarrow \frac{1}{c \cos \leftarrow \sin \leftarrow t \right)} = \tan \leftarrow \frac{wt}{\sqrt{1 + w^2 t^2 \sin^2 \leftarrow \sqrt{1 + \frac{(1 + cSt^3 w^2 \sin^2 \leftarrow)^2}{w^2 t^2 (1 + w^2 t^2 \sin^2 \leftarrow)^2}}}\right}
$$
(122)
Using (84) we obtain

$$
\int_{c} = c \frac{\cos \leftarrow}{\cos \leftarrow} = c \cos \leftarrow \sqrt{1 + \tan^2 \leftarrow \frac{1}{1 + w^2 t^2 \sin^2 \leftarrow \sqrt{1 + \frac{(1 + cSt^3 w^2 \sin^2 \leftarrow
$$

The ratio \int_{c}^{b} / c tends to 1 when $t \to 0$ or $t \to \infty$. Also for $t \geq -\frac{c}{t}$, $\int_{c}^{c} \geq 1$. (Reg. cS , $c = C$ $\geq \frac{1}{cS}$, $\frac{\hat{c}}{c} \geq 1$. (Recall that

 1 is the time where Letteins its maxim cS is the time where $\{$ attains its maximum for observer O' as we found in (111)). Since \int_{c}^{c} / *c* starts at 1 and tends at 1 at infinity and for $t \ge \frac{1}{\sqrt{2}}$ it is $\int_{c}^{c} \ge 1$, it must either be flat cS c c $\geq \frac{1}{cS}$ it is $\frac{c}{c} \geq 1$, it must either be flat equal to 1 or have at least one maximum for $t \geq \frac{1}{t}$. cS and cS and cS

The plot of tan $\{$ ["] vs t appears in Figure 10(a) and shows that it has a maximum that corresponds to a max for {''. In Figure 10(b) we plot \int_{c}^{c} / c for small angular velocity w_0 and in Figure 10(c) \int_{c}^{c} / c for big w_0 .

Figure 10 In (a) is the plot of tan { " vs t . In (b) and (c) is the ratio \int_{c}^{c} / c vs t, for small w_0 small (about10 rad/sec) and for w_0 big (about $3*10⁵$ rad/sec) respectively.

Taking $\frac{d...'}{d}$ we see that it is positive: \mathbf{r} we see that it is positive:

$$
\frac{d\ldots''}{d\ldots} = c \frac{c^2 + \frac{1}{2} \ldots^3 w_0^2 e^{-2(\frac{1}{2} \ldots + z)}}{(c^2 + \ldots^2 w_0^2 e^{-2(\frac{1}{2} \ldots + z)})^{\frac{3}{2}}}
$$
\n(124)

 $\lim_{n \to \infty}$ $\frac{c}{c}$, which agrees with the $\lim_{w \to \infty} w = \frac{c}{w_0}$, which agrees with the $rac{d...''}{d...} = c \frac{c^2 + \frac{1}{2} \cdot \frac{3}{2} w_0^2 e^{-2(\frac{1}{2} + \frac{1}{2})}}{(c^2 + \frac{9}{2} \cdot 2^{2(\frac{1}{2} + \frac{1}{2})})^{\frac{3}{2}}}$ (124)

aless $\} = 0$ in which case $\lim_{z \to \infty} \frac{w}{w_0} = \frac{c}{w_0}$, which agrees with the

case. Further, for *t* big, *t* $\frac{d...''}{d...} = c \frac{c^2 + 1 \cdot \cdot \cdot^3 w_0^2 e^{-2(1 - x + z)}}{(c^2 + ...^2 w_0^2 e^{-2(1 - x + z)})^{\frac{3}{2}}}$ (124)
 $= \infty$ unless $\} = 0$ in which case $\lim_{n \to \infty} x^n = \frac{c}{w_0}$, which agrees with the

lippage case. Further, for *t* big, ..." $\to ...$

$$
w_{w} = \int_{0}^{t} w_{0} e^{-ct(\frac{1}{2}\sin\left(\epsilon + \cos\left(\frac{x}{2}\right)\right)} dt = \frac{w_{0}}{c(\frac{1}{2}\sin\left(\epsilon + \cos\left(\frac{x}{2}\right)\right)} \left(1 - e^{-ct(\frac{1}{2}\sin\left(\epsilon + \cos\left(\frac{x}{2}\right)\right)}\right)
$$
(125)

appears in Figure 11, where the inclination \langle is a parameter.

Figure 11 The signal path as it advances in time upward in the z direction, while revolving around the z axis at increasing in time radial distance ...". (a) A single signal path. (b) Many signal paths for the same time interval with different $\langle \cdot \rangle$. The signals paths towards the positive z semiaxis only are drawn. To complete the picture one must imagine the same jet of signals towards the negative z direction.

We see how most of the signal except those close to $\epsilon \approx 90^\circ$ travel tight to each other in the z direction until they break up to return asymptotically to the radial direction. The jet like formation is symmetric with respect to the plane of rotation and another jet emanates towards the negative z direction.

The maximum of tan $\{$ ["] (or max for $\{$ ["]) is hard to calculate although manipulation of the graph that appears in Figure 10(a) shows clearly that a maximum occurs very close to $m_{\text{max}} = \frac{1}{\sqrt{2 \sin (n \cdot 1 - \cos n)}}$, the minimum of cos { (or 1) (a)

(a)
 **The signal path as it advances in time upwared as a start increasing in time radial distance ..."

the same time interval with different** \leftarrow **. The same drawn. To complete the picture one must in

z direction.
** $t_{\text{max}} = \frac{1}{\sqrt{2} + \frac{1}{2}}$ $c(\frac{1}{2} \sin \leftarrow + \cos \leftarrow)$ **Example 12**
 Example 11 The signal path as it advances in time upward in the z dialy
 CONFIGUATE: The signal path as it advances in time upward in the z dialy

round the z axis at increasing in time radial distance . $+ \sim \cos \left(\frac{1}{2} \right)$, the minimum of $cos\{$ (or max of $\{$) as we found in (111), when we studied the case of observer O' . It is logical that they must be very close since observer O'' sees an average of what O' sees. We will, therefore, proceed to a first like formation is symmetric with respect to the plane of rotation and another jet emanates
towards the negative z direction.
The maximum of tan {" (or max for {") is hard to calculate although manipulation of the
graph th he upward in the z direction, while revolving
tance ...". (a) A single signal path. (b) Many si₃. The signals paths towards the positive z sem
e must imagine the same jet of signals towards
close to $\langle \approx 90^0 \rangle$ travel it advances in time upward in th
in time radial distance ...". (a) *E*
if with different $\langle \cdot \rangle$. The signals
ete the picture one must imagine
nal except those close to $\langle \cdot \rangle \rangle$
as a up to return asymptotically
with re The maximum of $\tan \{-n\}$ or max for $\{-n\}$ is hard to calculate although manipulation of the
graph that appears in Figure 10(a) shows clearly that a maximum occurs very close to
 $t_{\text{t,max}} = \frac{1}{c(3 \sin \leftarrow -\cos \leftarrow)}$, the minim axis only are drawn. To complete the picture one must imagine the same jet of signals towards the regative *z* direction.
We see how most of the signal except those close to $\epsilon \approx 90^9$ travel tight to each other in the z

$$
t_{\text{t'max}} \simeq t_{\text{t max}} = \frac{1}{c(\text{)}} \sin \leftarrow + \cos \leftarrow)
$$
 (126)

This, as we mentioned in (111) to (114) implies that our maximum tan $\{$ ["] is very close to

$$
\}_{\cdots \{ \max} + \sim z_{\{ \max}} = 1 \tag{127}
$$

Where

$$
\dots_{\text{max}} = ct_{\text{max}} \sin \leftarrow \frac{\sin \leftarrow \sin \leftarrow}{\sin \leftarrow + \cos \leftarrow}
$$
\n(128)\n
\n
$$
z_{\text{max}} = ct_{\text{max}} \cos \leftarrow \frac{\cos \leftarrow \cos \leftarrow}{\sin \leftarrow + \cos \leftarrow}
$$
\n(129)

$$
\frac{1}{\sqrt{2}} \int \frac{\sin x}{\sin x} dx = ct_{\text{max}} \sin x = \frac{\sin x}{\sin x + \cos x} \qquad (128)
$$
\n
$$
z_{\text{max}} = ct_{\text{max}} \cos x = \frac{\cos x}{\sin x + \cos x} \qquad (129)
$$
\n
$$
z_{\text{max}} = \frac{\cos x}{\sin x + \cos x} \qquad (129)
$$
\n
$$
z_{\text{max}} = \frac{\cos x}{\sin x} \qquad (130)
$$

Then we calculate what shape observer O'' sees as the rhombus where $\{$ is maximized. For this we use (119) and calculate it at t_{max} to find

sin cos *z ct* max max max 2 2 2 0 0 2 2 2 2 2 2 2 2 sin (130)

 $\begin{aligned} \n\pi_{\text{max}} &= ct_{\text{f,max}} \sin \varsigma = \frac{\sin \varsigma}{\frac{1}{3} \sin \varsigma + \varsigma \cos \varsigma} \n\end{aligned}$ (128)
 $\pi_{\text{max}} = ct_{\text{f,max}} \cos \varsigma = \frac{\cos \varsigma}{\frac{1}{3} \sin \varsigma + \varsigma \cos \varsigma}$ (129)

pe observer *O''* sees as the rhombus where { is maximized.

alculate it at t $= ct_{\text{f max}} \sin \leftarrow \frac{\sin \leftarrow}{3 \sin \leftarrow + \cos \leftarrow}$ (1)
 $= ct_{\text{f max}} \cos \leftarrow \frac{\cos \leftarrow}{3 \sin \leftarrow + \cos \leftarrow}$ (1)

Subserver *O''* sees as the rhombus where { is maximize

de it at $t_{\text{f max}}$ to find
 $\frac{\sin \leftarrow}{6 \cdot \left(\frac{2}{\cos \theta}\right)} = \frac{\sin \leftarrow}{\sqrt{(3^2 + \frac{w_0^$ $C_{\text{max}} = ct_{\text{f max}} \sin \varsigma = \frac{\sin \varsigma}{\frac{\varsigma}{\sin \varsigma + \cos \varsigma}}$ (128)
 $z_{\text{f max}} = ct_{\text{f max}} \cos \varsigma = \frac{\cos \varsigma}{\frac{\varsigma}{\sin \varsigma + \cos \varsigma}}$ (129)

Then we calculate what shape observer *O''* sees as the thombus where { is maximized.

For this we use ($\therefore_{\text{t max}} = ct_{\text{t max}} \sin \zeta = \frac{\sin \zeta}{\frac{1}{3} \sin \zeta + \cos \zeta}$ (128)
 $z_{\zeta_{\text{max}}} = ct_{\zeta_{\text{max}}} \cos \zeta = \frac{\cos \zeta}{\frac{1}{3} \sin \zeta + \cos \zeta}$ (129)

Then we calculate what shape observer *O*" sees as the rhombus where { is maximized.

For thi a cigar like locus but it may be wider close to a rectangle with rounded corners depending on the values of $\}$ and \sim . The width of the locus decreases with increasing w_0 . Observe

$$
m_{\text{t}} = ct_{\text{t}} \sin \theta = \frac{\sin \theta}{\sin \theta} = \frac{\sin \theta}{\sin \theta} = \frac{\sin \theta}{\sin \theta} = \frac{\cos \theta}{\sin \theta} = ct_{\text{t}} \cos \theta = \frac{\cos \theta}{\sin \theta} = \frac{\sin \theta}{\sqrt{1 + \frac{m_{\text{t}}^2 - m_{\text{t}}^2}} = \frac{\sin \theta}{\sqrt{1 + \frac{m_{\text{t}}^2 - m_{\text{t}}^2}}} = \frac{\sin \theta}{\sqrt{1 + \frac{m_{\text{t}}^2 - m_{\text{t}}^2}} = \frac{\sin \theta}{\sqrt{1 + \frac{m_{\text{t}}^2 - m_{\text{t}}^2}} = \frac{\sin \theta}{\sqrt{1 + \frac{m_{\text{t}}^2 - m_{\text{t}}^2}} = \frac{\sin \theta}{\sqrt{1 + \frac{m_{\text{t}}^2 - m_{\text{t}}^2}}} = \frac{\sin \theta}{\sqrt{1 + \frac{m_{\text{t}}^2 - m_{\text{t}}^2}} = \frac{\sin \theta}{\sqrt{1 + \frac{m_{\text{t}}^2 - m_{\text{t}}^2}}} = \frac{\sin \theta}{\sqrt{1 + \frac{m_{\
$$

Figure 12 The locus of points $(\ldots_{\text{max}}^r, z_{\text{max}})$, (where $\{\text{ attains its maximum}\}\$ as parameter $\left\{\text{varies from 0 to } f/2\right\}$. The other quadrants are obtained by symmetry. The width narrows dramatically as w increases. The shape and r **Figure 12** The locus of points (... $\binom{n}{nmax}$, $z_{\text{(max)}}$), (where { attains its maximum) as parameter varies from 0 to $f/2$. The other quadrants are obtained by symmetry. The width narrows dramatically as w increases. dramatically as *w* increases. The shape and roundness also varies with parameters $\}$ and \sim . For $\frac{1}{2}$ and $\frac{1}{2}$ a c_{max} , (where { attains its maximum)

tts are obtained by symmetry. The width

and roundness also varies with parameters
 $\frac{w_0^2}{c^2 e^2}$.

e maximum at
 $\frac{w_0}{c^2 e^{2}}$

e maximum at Figure 12 The locus of points (..._{(max}, z_{{max}), (where { attains its
varies from 0 to *f* / 2. The other quadrants are obtained by symmetry
dramatically as w increases. The shape and roundness also varies wit
 \langle = 9 bocus of points (... $\binom{n}{\text{max}}, z_{\text{max}}$), (where { attain $\binom{n}{2}$. The other quadrants are obtained by symmeler increases. The shape and roundness also varies also varies $\sqrt{3^2 + \frac{w_0^2}{c^2 e^2}}$.

angle at its approx Figure 12 The locus of points (... ℓ_{max} , $z_{\text{t max}}$), (where { attains its maximum
varies from 0 to $f/2$. The other quadrants are obtained by symmetry. The width
dramatically as w increases. The shape and roundness **Example 12** The locus of points $\left(\dots_{\text{max}}^m, z_{\text{max}}\right)$, (where $\{\text{ attains its maximum}\}\$ as paramatics from 0 to $f/2$. The other quadrants are obtained by symmetry. The width narrows lramatically as w increases. The shape and rou **gure 12** The locus of points $\left(\ldots_{\text{f max}}^{\pi}, z_{\text{f max}}\right)$, (where { attains its maximum) as parameter <

ricis from 0 to $f/2$. The other quadrants are obtained by symmetry. The width narrows

annatically as w increases the statemental sum parameters $\frac{1}{s}$ and

es with parameters $\frac{1}{s}$ and

($\frac{1}{s}$ sin $s + \frac{1}{s}$ cos s)
 $\frac{1}{s}$ + $\frac{1}{s}$ cos $\frac{1}{s}$ $\frac{1}{(x_{\text{max}}}, z_{\text{max}})$, (where { attains its maximum) as parameter <
quadrants are obtained by symmetry. The width narrows
shape and roundness also varies with parameters } and ~ . For
 $\sqrt{3^2 + \frac{w_0^2}{c^2 e^2}}$.
oximate c_{max}), (where { attains its maximum) as parameter <
ts are obtained by symmetry. The width narrows
and roundness also varies with parameters } and ~ . For
 $\frac{w_0^2}{2c^2e^2}$
e maximum at
 cS
 $\frac{w}{1 + \cos \zeta} = \frac{w_0 e^{ \left(\frac{n}{\cos x}, z_{\text{max}}\right)$, (where { attains its maximum) as parameted quadrants are obtained by symmetry. The width narrows

shape and roundness also varies with parameters } and ~.
 $\frac{1}{\sqrt{2^2 + \frac{w_0^2}{c^2 e^2}}}$.

froximat Its its maximum) as parameter κ

metry. The width narrows

es with parameters $\}$ and \sim . For
 $\frac{1}{2} \sin \left(\frac{1}{2} \arccos \left(\frac{1}{2} \$ $\int_{\tan x}^{\pi} s(z_{\text{max}})$, (where { attains its maximum) as parameter <
uadrants are obtained by symmetry. The width narrows
hape and roundness also varies with parameters } and ~ . For
 $\sqrt{2^2 + \frac{w_0^2}{c^2 c^2}}$
vimate maxim ains its maximum) as parameter \langle

7 mmetry. The width narrows

aries with parameters $\}$ and \sim . For
 $-ct(\frac{1}{2} \sin \left(\frac{1}{2} + \cos \left(\frac{1}{2}\right)\right))$
 $\sin \left(\frac{1}{2} + \cos \left(\frac{1}{2}\right)\right)$ (131) s (... $\frac{1}{\sqrt{1-\cos^2 x}}$, (where { attains its maximum) as parameter <

encry quadrants are obtained by symmetry. The width narrows

The shape and roundness also varies with parameters } and ~ . For
 $\frac{1}{\sqrt{2^2 + \frac{w_0^2$

$$
\epsilon = 90^\circ
$$
 the distance OA=OB =
$$
\frac{1}{\sqrt{3^2 + \frac{w_0^2}{c^2 e^2}}}.
$$

$$
t_{\text{f}''\text{max}} \simeq t_{\text{f}''\text{max}} = \frac{1}{c(\text{f} \sin \leftarrow + \infty \cos \leftarrow)} = 1/c\text{S}
$$

Assumes the approximate value

$$
\tan\left\{\frac{n}{\max} \simeq \frac{W}{\sqrt{(1-\sin\left(\frac{1}{2}\right)\sin\left(\frac{1}{2}\right) - \cos\left(\frac{1}{2}\right))}} = \frac{W_0 e^{-ct(\frac{1}{2}\sin\left(\frac{1}{2}\right) - \cos\left(\frac{1}{2}\right))}}{\sqrt{(1-\sin\left(\frac{1}{2}\right) - \cos\left(\frac{1}{2}\right))}} \tag{131}
$$

10. Conclusion

Starting from the assumption that two observers rotating with respect to the other around a common axis will agree on epicenter angles as fractions of a circle but not necessarily on the value of , we find the length of the radius of a rotating disc as seen by the non rotating observer. The radius will be contracted but not with the same factor as the perimeter because we allow the value of to change for the non rotating observer with regards to his measurements on the rotating disc. We argued that we have to consider two types of non rotating observers. One within the radius *c/w* and one outside. For the non rotating observer O' within c/w , a light signal starting radially from the origin of the rotating disc (frame) that rotates with the frame at an angle $\epsilon = 90^{\circ}$ from the z axis, will gradually turn sideways forming tighter circles until it asymptotically reaches 90º degrees deflection from the radial as the radius tends to infinity. His space is distorted and is different. If a signal is emitted from the origin at an angle $\langle 90^\circ$ from the z axis it again expands 10. Conclusion

Starting from the assumption that two observers rotating with respect to the other around a

common axis will agree on epicenter angles as fractions of a circle but not necessarily on

the value of , we fi 10. Conclusion
Starting from the assumption that two observers rotating with respect to the other around a
common axis will agree on epicenter angles as fractions of a circle but not the
censearily on
observer. The radius

rotating in ever tighter to one another circles, until it reaches $|z| = \frac{c}{z}$. After that the signal *w* $=$ - $\frac{c}{c}$. After that the signal

is not observable by observer *O* .

exponentially as the radius and ζ increases with slippage parameters $\}$ and \sim respectively). In this case, the space is not divided by a cylinder of radius *c/w,* but still there is contraction of the radial distances. The light rays originating from the origin at an angle \leftarrow from the z axis change their initial direction sideways until they reach a maximum deflection from the radial direction and then asymptotically turn back to the radial. The locus of points, where the maximum deflection occurs is determined. For an observer *O* within or near this locus it (the locus) looks close to a deformed rhombus by revolution

with the radial distance contracted, while for an observer O^r located far away it looks like a cigar depending on the slippage parameters.

For the slippage case and for observer O'' , it is worth noting the jet like formations, in the direction of the positive and negative z axis, of the signals that emanate from the origin of the rotating frame. Similar jet effects, but less pronounced, should appear for observer O' for the slippage case.

11. References

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