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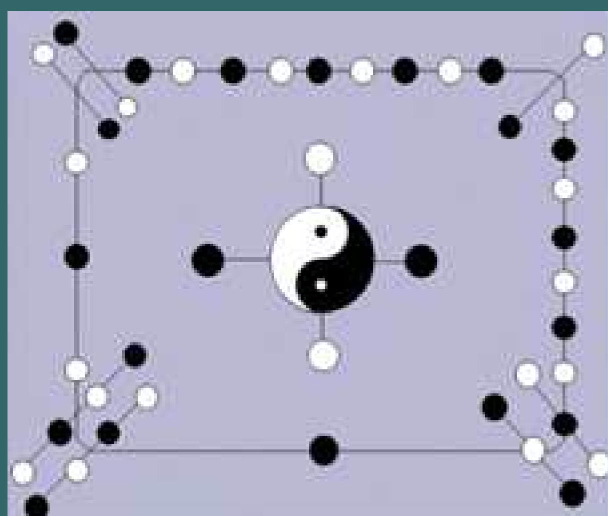

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Eternal truths will be neither true nor eternal unless they have fresh meaning for every new social situation.

By Franklin Roosevelt, an American president.

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On Ruled Surfaces in Minkowski 3-Space

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Abstract: In this paper, we studied the timelike and the spacelike ruled surfaces in Minkowski 3-space by using the angle between unit normal vector of the ruled surface and the principal normal vector of the base curve. We obtained some characterizations on the ruled surfaces by using its rulings, the curvatures of the base curve, the shape operator and Gauss curvature.

Key Words: Minkowski space, ruled surface, striction curve, Gauss curvature

AMS(2010): 53A04, 53A17, 53A35

§1. Introduction

It is safe to report that the many important studies in the theory of ruled surfaces in Euclidean and also in Minkowski and Galilean spaces. A surface M is ruled if through every point of M there is a straight line that lies on M. The most familiar examples are the plane and the curved surface of a cylinder or cone. Other examples are a conical surface with elliptical directrix, the right conoid, the helicoid, and the tangent developable of a smooth curve in space. A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. For example, a cone is formed by keeping one point of a line fixed whilst moving another point along a circle. A developable surface is a surface that can be (locally) unrolled onto a flat plane without tearing or stretching it. If a developable surface lies in three-dimensional Euclidean space, and is complete, then it is necessarily ruled, but the converse is not always true. For instance, the cylinder and cone are developable, but the general hyperboloid of one sheet is not. More generally, any developable surface in three-dimensions is part of a complete ruled surface, and so itself must be locally ruled. There are surfaces embedded in four dimensions which are however not ruled. (for more details see [1])(Hilbert & Cohn-Vossen 1952, pp. 341–342).

In the light of the existing literature, in [8,9,10] authors introduced timelike and spacelike ruled surfaces and they investigated invariants of timelike and spacelike ruled surfaces by Frenet-

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Serret frame vector fields in Minkowski space.

In this study, we investigated timelike ruled surfaces with spacelike rulings, timelike ruled surfaces with timelike rulings and spacelike ruled surfaces with spacelike rulings. Since unit normals of a ruled surface lies in normal planes of the curves on that surface then we investigated all of invariants of base curve of a ruled surface with respect to the angle between unit normal of surface and principal normal.

Now we review some basic concepts on classical differential geometry of space curves and ruled surfaces in Minkowski space. Let $\alpha : I \longrightarrow IR^3$ be a curve with $\alpha'(s) \neq 0$, where $\alpha'(s) = d\,\alpha(s)/ds$. The arc-length s of a curve $\alpha(s)$ is determined such that $\|\alpha'(s)\| = 1$. Let us denote $T(s) = \alpha'(s)$ and we call $T(s)$ a tangent vector of α at $\alpha(s)$. Its well known that there are three types curves in Minkowski space such that if $\langle \alpha', \alpha' \rangle > 0$, α is called spacelike curve, if $\langle \alpha', \alpha' \rangle < 0$, α is called timelike curve and if $\langle \alpha', \alpha' \rangle = 0$, α is called null curve. The curvature of α is defined by $\kappa(s) = \sqrt{\|\alpha''(s)\|}$. If $\kappa(s) \neq 0$, unit principal normal vector $N(s)$ of the curve at $\alpha(s)$ is given by $\alpha''(s) = \kappa(s) N(s)$. The unit vector $B(s) = T(s) \Lambda N(s)$ is called unit binormal vector of α at $\alpha(s)$. If α is a timelike curve, Frenet-Serret formulae is

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = -\tau N, \tag{1}$$

where $\tau(s)$ is the torsion of α at $\alpha(s)$ ([2]). If α is a spacelike curve with a spacelike or timelike principal normal N, the Frenet formulae is

$$T' = \kappa N, \quad N' = -\epsilon\kappa T + \tau B, \quad B' = \tau N, \tag{2}$$

where $\langle T, T \rangle = 1$, $\langle N, N \rangle = \epsilon = \pm 1$, $\langle B, B \rangle = -\epsilon$, $\langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0$ ([4]).

A straight line X in IR^3, such that it is strictly connected to Frenet frame of the curve $\alpha(s)$, is represented uniquely with respect to this frame, in the form

$$X(s) = f(s)N(s) + g(s)B(s), \tag{3}$$

where $f(s)$ and $g(s)$ are the smooth functions. As X moves along $\alpha(s)$, it generates a ruled surface given by the regular parametrization

$$\varphi(s, v) = \alpha(s) + vX(s), \tag{4}$$

where the components f and g are differentiable functions with respect to the arc-lenght parameter of the curve $\alpha(s)$. This surface will be denoted by M. The curve $\alpha(s)$ is called a base curve and the various positions of the generating line X are called the rulings of the surface M.

If consecutive rulings of a ruled surface in IR^3 intersect, the surface is to be developable. All the other ruled surfaces are called skew surfaces. If there is a common perpendicular to two constructive rulings in the skew surface, the foot of the common perpendicular on the main ruling is called a striction point. The set of the striction points on the ruled surface defines the striction curve [3].

The striction curve of M can be written in terms of the base curve $\alpha(s)$ as

$$\overline{\alpha}(s) = \alpha(s) - \frac{\langle T, \nabla_T X\rangle}{\|\nabla_T X\|^2} X(s). \tag{5}$$

If $\|\nabla_T X\| = 0$, the ruled surface doesn't have any striction curves. This case characterizes the ruled surface as cylindrical. Thus, the base curve can be taken as a striction curve.

Let P_x be distribution parameter of M, then

$$P_X = \frac{det(T, X, \nabla_T X)}{\|\nabla_T X\|^2}, \tag{6}$$

where ∇ is Levi-Civita connection on E_v^3 [1]. If the base curve is periodic, M is a closed ruled surface. Let M be a closed ruled surface and W be Darboux vector, then the Steiner rotation and Steiner translation vectors are

$$D = \oint_{(\alpha)} W, \quad V = \oint_{(\alpha)} d\alpha \ , \tag{7}$$

respectively. Furthermore, the pitch of M and the angle of the pitch are

$$L_X = \langle V, X\rangle, \quad \lambda_X = \langle D, X\rangle \ , \tag{8}$$

respectively [3, 5, 6].

§2. Timelike Ruled Surfaces with Spacelike Rulings

Let $\alpha : I \to E_1^3$ be a regular timelike curve with the arc-lenght parameter s and $\{T, N, B\}$ be Frenet vectors. In generally, during one parametric spatial motion, each line X in moving space generates a timelike ruled surface. Since ξ is normal to T, $\xi \in Sp\{N, B\}$ and ξ can be choosen as $\xi = T \Lambda X$ along the spacelike line X depending on the orientation of M. Thus, ξ and X can be written as

$$\xi = -\sin\psi N + \cos\psi B, \quad X = \cos\psi N + \sin\psi B, \tag{9}$$

where $\psi = \psi(s)$ is the angle between ξ and N along α [6]. From (2) and (9), we write

$$\nabla_T X = \kappa \cos\psi T + (\psi' + \tau)\ \xi. \tag{10}$$

We obtain the distribution parameter of the timelike ruled surface M as

$$P_X = \frac{\psi' + \tau}{(\psi' + \tau)^2 - \kappa^2 \cos^2\psi} \tag{11}$$

by using (6) and (10). It is well known that the timelike ruled surface is developable if and only if P_X is zero from [1], so we can state the following theorem.

Theorem 2.1 *A timelike ruled surface with the spacelike rulings is developable if and only if*

$$\psi = -\int \tau ds + c$$

is satisfied, where c is a constant.

In the case $\psi = (2k-1)\pi/2$ and $\psi = k\pi$, $k \in Z$, we get $P_X = P_B$ and $P_X = P_N$, respectively. Thus, the distribution parameters are

$$P_B = \frac{1}{\tau}, \quad P_N = \frac{\tau}{\tau^2 - \kappa^2}$$

and we obtain

$$\frac{P_B}{P_N} = 1 - \left(\frac{\kappa}{\tau}\right)^2.$$

Thus, we get a corollary following.

Corollary 2.2 *The base curve of the timelike ruled surface with the spacelike rulings is a timelike helice if and only if* $\frac{P_B}{P_N}$ *is a constant.*

On the other hand, from (5) the striction curve of M is

$$\overline{\alpha}(s) = \alpha(s) + \frac{\kappa \cos\psi}{(\psi' + \tau)^2 - \kappa^2 \cos^2\psi} X(s).$$

In the case that M is a cylindrical timelike ruled surface with the spacelike rulings, we get the theorem following.

Theorem 2.3 *i) If M is a cylindrical timelike ruled surface with the spacelike rulings, $\kappa \cos\psi = 0$. In the case $\kappa = 0$, the timelike ruled surface is a plane. In the case $\psi = k\pi$, $k \in Z$, unit normal vector of M and binormal vector of the base curve are on the same direction and both the striction curve and the base curve are geodesics of M.*

ii) A cylindrical timelike ruled surface with the spacelike rulings is developable if and only if

$$\kappa \cos\left(\int \tau ds + c\right) = 0$$

is satisfied. In this case, the base curve is a timelike planar curve.

On the other hand, the equation (4) indicates that $\varphi_v : I \times \{v\} \to M$ is a curve of M for each $v \in IR$. Let A be the tangent vector field of the curve φ_v then A is

$$A = (1 + v\kappa \cos\psi) T + v \{\tau + \psi'\} \xi. \tag{12}$$

Since the vector field A is normal to ξ, $\tau + \psi' = 0$ is satisfied. Thus, we get the theorem following.

Theorem 2.4 *The tangent planes of a timelike ruled surface with the spacelike rulings are the* *same along the spacelike generating lines if and only if*

$$\tau + \psi' = 0$$

is satisfied.

Theorem 2.5 *i) Let ψ be $\psi' \neq -\tau$ and M be a closed timelike ruled surface with the spacelike rulings as given in the form (4). The distance between spacelike generating lines of M is minimum along the striction curve.*

ii) Let $\overline{\alpha}(s)$ be a striction curve of a timelike ruled surface with the spacelike rulings, then

$$\frac{\kappa \cos\psi}{(\psi' + \tau)^{2} - \kappa^2 \cos^2\psi}$$

is a constant.

*iii) Let M be a timelike ruled surface as given in the form (**??**), then $\varphi(s, v_o)$ is a striction point if and only if $\nabla_T X$ is normal to the tangent plane at that point on M, where*

$$v_o = \frac{\kappa \cos\psi}{(\psi' + \tau)^{2} - \kappa^2 \cos^2\psi}.$$

Proof i) Let $X_{\alpha(s_1)}$ and $X_{\alpha(s_2)}$ be spacelike generating lines which pass from the points $\alpha(s_1)$ and $\alpha(s_2)$ of the base curve, respectively ($s_1, s_2 \in IR$ and $s_1 < s_2$). Distance between these spacelike generating lines along the orthogonal orbits is

$$J(v) = \int_{s_1}^{s_2} \|A\| \, ds.$$

So we obtain

$$J(v) = \int_{s_1}^{s_2} \left(2v\kappa \cos\psi - 1 + \left((\psi' + \tau)^{2} - \kappa^2 \cos^2\psi\right) v^2\right)^{\frac{1}{2}} ds.$$

If $J(v)$ is minimum for v_0, $J'(v_o) = 0$ and we get

$$v_o = \frac{\kappa \cos\psi}{(\psi' + \tau)^{2} - \kappa^2 \cos^2\psi}.$$

Thus, the orthogonal orbit is the striction curve of M for $v = v_o$.

ii) Since the tangent vector field of the striction curve is normal to X, $\left\langle X, \frac{d\overline{\alpha}}{ds}\right\rangle = 0$. Thus, we get

$$\frac{d}{ds}\left(\frac{\kappa \cos\psi}{(\psi' + \tau)^{2} - \kappa^2 \cos^2\psi}\right) = 0$$

and so

$$\frac{\kappa \cos\psi}{(\psi' + \tau)^{2} - \kappa^2 \cos^2\psi} = \text{constant}.$$

iii) We suppose that $\varphi(s, v_o)$ is a striction point on the timelike base curve $\alpha(s)$ of M, then we must show that $\langle \nabla_T X, X \rangle = 0$ and $\langle \nabla_T X, A \rangle = 0$. Since the vector field X is an unit vector, $T[\langle X, X \rangle] = 0$ and we get $\langle \nabla_T X, X \rangle = 0$.

On the other hand, from (10) and (12), we obtain

$$\langle \nabla_T X, A \rangle = \left\{ (\psi' + \tau)^2 - \kappa^2 \cos^2 \psi \right\} v_o - \kappa \cos \psi.$$

Now, by applying

$$v_o = \frac{\kappa \cos \psi}{(\psi' + \tau)^2 - \kappa^2 \cos^2 \psi},$$

we get $\langle \nabla_T X, A \rangle = 0$. This means that $\nabla_T X$ is normal to the tangent plane at the striction point $\varphi(t, v_o)$ on M.

Conversely, since $\nabla_T X$ is normal to the tangent plane at the point $\varphi(s, v_o)$ on M, $\langle \nabla_T X, A \rangle = 0$ and from (10) and (12), we obtain

$$\left\{ (\psi' + \tau)^2 - \kappa^2 \cos^2 \psi \right\} v_o - \kappa \cos \psi = 0.$$

Thus, we get $v_o = \dfrac{\kappa \cos \psi}{(\psi' + \tau)^2 - \kappa^2 \cos^2 \psi}$ and so $\varphi(s, v_o)$ is a striction point on M. □

Theorem 2.6 *Absolute value of the Gauss curvature of M is maximum at the striction points on the spacelike generating line X and it is*

$$|K|_{\max} = \frac{\left\{ (\psi' + \tau)^2 - \kappa^2 \cos^2 \psi \right\}^2}{(\psi' + \tau)^2}.$$

Proof Let M be a timelike ruled surface as given in the form (4) and Φ be base of the tangent space which is spanned by the unit vectors A_o and X where A_o is the tangent vector of the curve $\varphi(s, v = \text{constant.})$ with the arc-length parameter s^*. Hence, we write $A_o = \dfrac{d\varphi}{ds^*} = \dfrac{d\varphi}{ds}\dfrac{ds}{ds^*}$ where $\dfrac{d\varphi}{ds} = A$, $A_o = \dfrac{1}{\|A\|} A$ and $\dfrac{ds}{ds^*} = \dfrac{1}{\|A\|}$. Thus, we obtain the following equations after the routine calculations.

$$\begin{aligned}
\nabla_{A_o} T &= \frac{\kappa}{\|A\|} \left\{ \cos \psi X - \sin \psi \xi \right\}, \\
\nabla_{A_o} \xi &= \frac{1}{\|A\|} \left\{ -\kappa \sin \psi T - (\psi' + \tau) X \right\}, \\
\nabla_{A_o} A &= \frac{1}{\|A\|} \left\{ \begin{array}{c} \left\{ (1 + v\kappa \cos \psi)' - v(\psi' + \tau) \kappa \sin \psi \right\} T \\ + \left\{ (1 + v\kappa \cos \psi) \kappa \cos \psi + v (\psi' + \tau)^2 \right\} X \\ + \left\{ (-v\kappa \sin \psi - 1) \kappa \sin \psi + (v(\psi' + \tau))' \right\} \xi \end{array} \right\}.
\end{aligned}$$

On the other hand, we denote $\xi_{\varphi(s,v)}$ as the unit normal vector at the points $\varphi(s, v = \text{constant})$, then from (1), (9) and (12) we get

$$\xi_{\varphi(s,v)} = \frac{1}{\|A\|} \left\{ v\left(\psi' + \tau\right) T + \left(1 + v\kappa \cos\psi\right) \xi \right\}. \tag{13}$$

By differentiating both side of (13) with respect to the parameter s, we get

$$\frac{d\xi_{\varphi(s,v)}}{ds} = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left(v\left(\psi'+\tau\right)' - \kappa \sin\psi \left(1 + v\kappa\cos\psi\right)\right) \frac{1}{\|A\|} \\ +v\left(\psi'+\tau\right)\left(\frac{1}{\|A\|}\right)' \end{array} \right\} T \\ + \left\{ \left(1+v\kappa\cos\psi\right)\left(\frac{1}{\|A\|}\right)' + \left(\begin{array}{c} \left(1+v\kappa\cos\psi\right)' \\ -v\kappa\sin\psi\left(\psi'+\tau\right) \end{array} \right) \frac{1}{\|A\|} \right\} \xi \\ -\left(\psi'+\tau\right)\frac{1}{\|A\|} X \end{array} \right\}. \tag{14}$$

Let S be the shape operator of M at the points $\varphi(s,v)$, then we can obtain that the matrix S_Φ is following with respect to base Φ,

$$S_\Phi = \begin{bmatrix} -\langle S(A_o), A_o\rangle & \langle S(A_o), X\rangle \\ -\langle S(X), A_o\rangle & \langle S(X), X\rangle \end{bmatrix}.$$

Since $\langle S(X), X\rangle = 0$ and $\langle S(A_o), X\rangle = \langle S(X), A_o\rangle$, the Gauss curvature is

$$K(s,v) = \det S_\Phi = \langle S(A_o), X\rangle^2.$$

Suppose that s^* is arc-length parameter of A_o, then we get

$$S(A_o) = \nabla_{A_o}\xi_{\varphi(s,v)} = \frac{d\xi_{\varphi(s,v)}}{ds^*} = \frac{d\xi_{\varphi(s,v)}}{ds}\frac{ds}{ds^*} = \frac{1}{\|A\|}\frac{d\xi_{\varphi(s,v)}}{ds}.$$

From (12) and (14), we obtain

$$\begin{aligned} S(A_o) \;=\; & \frac{1}{\|A\|}\left\{ \begin{array}{c} \left(v\left(\psi'+\tau\right)' - \kappa\sin\psi\left(1+v\kappa\cos\psi\right)\right)\frac{1}{\|A\|} \\ +v\left(\psi'+\tau\right)\left(\frac{1}{\|A\|}\right)' \end{array} \right\} T \\ & + \frac{1}{\|A\|}\left\{ \left(1+v\kappa\cos\psi\right)\left(\frac{1}{\|A\|}\right)' + \left(\begin{array}{c} \left(1+v\kappa\cos\psi\right)' \\ -v\kappa\sin\psi\left(\psi'+\tau\right) \end{array} \right)\frac{1}{\|A\|} \right\} \xi \\ & - \left(\psi'+\tau\right)\frac{1}{\|A\|^2} X. \end{aligned}$$

Hence, the Gauss curvature is

$$K(s,v) = \frac{\left(\psi'+\tau\right)^2}{\|A\|^4}. \tag{15}$$

We differentiate both side of (17) with respect to v for finding the maximum value of the

Gauss curvature along X on M. Thus, we obtain

$$\frac{\partial K(s,v)}{\partial v} = \frac{-4\left(\psi'+\tau\right)^2\left\{\left(v\left(\psi'+\tau\right)^2-\kappa^2\cos^2\psi\right)-\kappa\cos\psi\right\}}{\left\{v^2\left\{\left(\psi'+\tau\right)^2-\kappa^2\cos^2\psi\right\}-2v\kappa\cos\psi-1\right\}^3}=0$$

and $v=\dfrac{\kappa\cos\psi}{\left(\psi'+\tau\right)^2-\kappa^2\cos^2\psi}$. It is easy to see that $\varphi(s,v)$ is the striction point and we can say that the absolute value of the Gauss curvature of M is maximum at the striction points on X. Finally, by using (17), we get

$$|K|_{\max}=\frac{\left\{\left(\psi'+\tau\right)^2-\kappa^2\cos^2\psi\right\}^2}{\left(\psi'+\tau\right)^2}. \tag{16}$$

This completes the proof. □

By using (11) and (16), we can write the relation between the Gauss curvature and the distribution parameter as

$$|K|_{\max}=\frac{1}{\left(P_X\right)^2}. \tag{17}$$

Thus, we prove the following corollary too.

Corollary 2.7 *The distribution parameter of the timelike ruled surface with the spacelike rulings depends on the spacelike generating lines.*

Moreover, the Darboux frame of the surface along the timelike base curve is

$$\begin{bmatrix}\nabla_T T\\ \nabla_T X\\ \nabla_T \xi\end{bmatrix}=\begin{bmatrix}0 & \kappa\cos\psi & -\kappa\sin\psi\\ \kappa\cos\psi & 0 & \left(\psi'+\tau\right)\\ -\kappa\sin\psi & -\left(\psi'+\tau\right) & 0\end{bmatrix}\begin{bmatrix}T\\ X\\ \xi\end{bmatrix}$$

and the Darboux vector is

$$W=-\varepsilon_2\left(\psi'+\tau\right)T-\varepsilon_1\kappa\sin\psi X-\varepsilon_1\kappa\cos\psi\xi,$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_3$ are the signs of standart vectors e_1,e_2,e_3, respectively. Thus, we obtain the geodesic curvature, the geodesic torsion and the normal curvature of the timelike ruled surface with the spacelike rulings along its spacelike generating lines as

$$\kappa_g=\varepsilon_1\kappa\sin\psi,\quad \tau_g=-\varepsilon_2\left(\psi'+\tau\right),\quad \kappa_\xi=-\varepsilon_1\kappa\cos\psi\ ,$$

respectively. Note also that if the timelike ruled surface with the spacelike rulings is a constant curvature surface with a nonzero geodesic curvature, P_X is a constant and from (8) and (15), we obtain $\dfrac{\tau_g^2}{\tau_g^2-\kappa_\xi^2}$=constant. Hence, we get the theorem following.

Theorem 2.8 *A timelike ruled surface with the spacelike rulings is a constant curvature surface* *with a nonzero geodesic curvature if and only if* $\dfrac{\tau_g^2}{\tau_g^2-\kappa_\xi^2}$ *is a constant. In the case that the base curve is one of the timelike geodesics of the timelike ruled surface, the timelike ruled surface is developable.*

On the other hand, the Steiner rotation vector is

$$D=-\varepsilon_2\left(\oint_{(\alpha)}(\psi'+\tau)\,ds\right)T-\varepsilon_1\left(\oint_{(\alpha)}\kappa\sin\psi ds\right)X-\varepsilon_1\left(\oint_{(\alpha)}\kappa\cos\psi ds\right)\xi.$$

Furthermore, the angle of pitch of M is

$$\lambda_X=-\varepsilon_1\oint_{(\alpha)}\kappa\sin\psi ds.$$

From (4), (5) and (17), we obtain that $L_N=\lambda_N=0$, $L_B=0$ and $\lambda_B=-\varepsilon_1\oint_{(\alpha)}\kappa ds$ for the special cases, $X=N$ and $X=B$.

§3. Timelike Ruled Surfaces with Spacelike Rulings

Let $\alpha:I\to E^3$ be a regular spacelike curve with the arc-lenght parameter s. Since T and X are spacelike vectors, ξ is a timelike vector and the functions f and g satisfy

$$f^{\,2}-g^2=-\epsilon$$

along α ([6]). From (2) and (3), we write

$$\nabla_T X=-\epsilon f\kappa T+\{\tau+\epsilon(f'g-fg')\}\,\xi. \tag{18}$$

The distribution parameter of the timelike ruled surface with timelike rulings is obtained by a direct computation as

$$P_X=\frac{\tau+\epsilon(f'g-fg')}{f^2\kappa^2+\{\tau+\epsilon(f'g-fg')\}^{\,2}}. \tag{19}$$

Thus, we have the following result.

Theorem 3.1 *A timelike ruled surface with timelike rulings is developable if and only if*

$$\tau+\epsilon(f'g-fg')=0$$

is satisfied.

In the cases $f=0$, $g=1$ and $f=1$, $g=0$, we get $P_X=P_B$ and $P_X=P_N$, respectively.

Thus, the distribution parameters are

$$P_B = \frac{1}{\tau}, \quad P_N = \frac{\tau}{\kappa^2 + \tau}$$

and we obtain

$$\frac{P_B}{P_N} = 1 + \left(\frac{\kappa}{\tau}\right)^2.$$

Thus, we get the following corollary.

Corollary 3.2 *The base curve of the timelike ruled surface with timelike rulings is a spacelike helice if and only if* $\frac{P_B}{P_N}$ *is a constant.*

On the other hand, from (5), the striction curve of M is

$$\overline{\alpha}(s) = \alpha(s) + \frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2} X(s).$$

In the case that M is a cylindrical timelike ruled surface with timelike rulings, we find the following result.

Theorem 3.3 *i) If M is a cylindrical timelike ruled surface with timelike rulings, $\kappa f = 0$. In the case $\kappa = 0$, the timelike ruled surface is a plane. In the case $\kappa \neq 0$, unit normal vector of M and binormal vector of the base curve are on the same direction and both the striction curve and the base curve are geodesics of M.*

ii) A cylindrical timelike ruled surface with timelike rulings is developable if and only if the base curve is a planar spacelike curve.

The tangent vector field of the curve $\varphi_v : I \times \{v\} \to M$ is

$$A = (1 - v\epsilon\kappa f)\,T + v\,\{\tau + \epsilon(f'g - fg')\}\,\xi \tag{20}$$

on M for each $v \in IR$. Since the vector field A is normal to ξ, $\tau + \epsilon(f'g - fg') = 0$ is satisfied along the curve φ_v. Thus, we have the following theorem.

Theorem 3.4 *The tangent planes of a timelike ruled surface with timelike rulings are the same along a timelike generating lines if and only if*

$$\tau + \epsilon(f'g - fg') = 0$$

is satisfied.

Theorem 3.5 *i) Let $\tau + \epsilon(f'g - fg') \neq 0$ and M be a closed timelike ruled surface with timelike rulings as given in the form (4). The distance between timelike generating lines of M is minimum along the striction curve.*

ii) Let $\overline{\alpha}(s)$ be a striction curve of M, then $\frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2}$ *is a constant.*

iii) Let M be a timelike ruled surface with timelike rulings as given in the form (4), then $\varphi(t, v_o)$ is a striction point if and only if $\nabla_T X$ is normal to the tangent plane at that point on M, where

$$v_o = \frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2}.$$

Proof *i)* Let $X_{\alpha(s_1)}$ and $X_{\alpha(s_2)}$ be timelike generating lines which pass from the points $\alpha(s_1)$ and $\alpha(s_2)$ of the base curve, respectively($s_1, s_2 \in IR$, *and* $s_1 < s_2$). Distance between these timelike generating lines along the orthogonal orbits is

$$J(v) = \int_{s_1}^{s_2} \|A\|\, ds.$$

Then, we obtain

$$J(v) = \int_{s_1}^{s_2} \left(\left\{ f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2 \right\} v^2 - 2v\epsilon f\kappa + 1 \right)^{\frac{1}{2}} ds.$$

If $J(v)$ is minimum for v_0, $J'(v_o) = 0$ and we get

$$v_o = \frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2}.$$

Thus, the orthogonal orbit is the striction curve of M for $v = v_o$.

ii) Since the tangent vector field of the striction curve is normal to X, $\left\langle X, \dfrac{d\overline{\alpha}}{ds} \right\rangle = 0$. Thus, we get

$$\frac{d}{ds}\left(\frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2} \right) = 0$$

and so

$$\frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2} = \text{constant}.$$

iii) We suppose that $\varphi(t, v_o)$ is a striction point on the spacelike base curve $\alpha(t)$ of M, then we must show that $\langle \nabla_T X, X \rangle = 0$ and $\langle \nabla_T X, A \rangle = 0$. Since the vector field X is an unit vector, $T[\langle X, X \rangle] = 0$ and we get $\langle \nabla_T X, X \rangle = 0$.

On the other hand, from (18) and (20), we obtain

$$\langle \nabla_T X, A \rangle = -\epsilon f\kappa\,(1 - \epsilon v_o \kappa f) + v_o\,\{\tau + \epsilon(f'g - fg')\}.$$

By using

$$v_o = \frac{-\kappa\cosh\psi}{(\psi' + \tau)^2 + \kappa^2\cosh^2\psi}$$

we get $\langle \nabla_T X, A \rangle = 0$. This means that $\nabla_T X$ is normal to the tangent plane at the striction point $\varphi(s, v_o)$ on M.

Conversely, since $\nabla_T X$ is normal to the tangent plane at the point $\varphi(s, v_o)$ on M, $\langle \nabla_T X, A \rangle =$

0 and from (18) and (20), we obtain

$$\left\{f^2\kappa^2+\{\tau+\epsilon(f'g-fg')\}^2\right\}v_o-\epsilon\kappa f=0.$$

Thus, we get

$$v_o=\frac{\epsilon\kappa f}{f^2\kappa^2+\{\tau+\epsilon(f'g-fg')\}^2}$$

and so $\varphi(s,v_o)$ is a striction point on M. □

Theorem 3.6 *Absolute value of the Gauss curvature of M is maximum at the striction points on the timelike generating line X and it is*

$$|K|_{\max}=\frac{\left\{f^2\kappa^2+\{\tau+\epsilon(f'g-fg')\}^2\right\}^2}{\{\tau+\epsilon(f'g-fg')\}^2}.$$

Proof Let M be a timelike ruled surface with timelike rulings as given in the form (4) and Φ be base of the tangent space which is spanned by the unit vectors A_o and X where A_o is the tangent vector of the curve $\varphi(s,v=\text{constant.})$ with the arc-length parameter s^*. Hence, we write $A_o=\dfrac{d\varphi}{ds^*}=\dfrac{d\varphi}{ds}\dfrac{ds}{ds^*}$ where $\dfrac{d\varphi}{ds}=A$, $A_o=\dfrac{1}{\|A\|}A$ and $\dfrac{ds}{ds^*}=\dfrac{1}{\|A\|}$. Thus, we obtain the following equations after the routine calculations.

$$\begin{aligned}
\nabla_{A_o}T &= \frac{\epsilon\kappa}{\|A\|}\{-fX+g\xi\}\\
\nabla_{A_o}\xi &= \frac{1}{\|A\|}\{-\epsilon\kappa gT+\{\tau+\epsilon(f'g-fg')\}X\}\\
\nabla_{A_o}A &= \frac{1}{\|A\|}\left\{\begin{array}{c}
\left\{(1-\epsilon v\kappa f)'-\epsilon v\kappa g\left(\tau+\epsilon(f'g-fg')\right)\right\}T\\
+\left\{-\epsilon\kappa f\left(1-\epsilon v\kappa f\right)+v\left\{\tau+\epsilon(f'g-fg')\right\}^2\right\}X\\
+\left\{\epsilon\kappa g\left(1-\epsilon v\kappa f\right)+\left\{v\left\{\tau+\epsilon(f'g-fg')\right\}\right\}'\right\}\xi
\end{array}\right\}
\end{aligned}$$

On the other hand, we denote $\xi_{\varphi(s,v)}$ as the unit normal vector at the points φ $(s,v=$ constant), then using (2), (3) and (20) we get

$$\xi_{\varphi(s,v)}=\frac{1}{\|A\|}\left\{-v\left\{\tau+\epsilon(f'g-fg')\right\}T+\left(1-\epsilon v\kappa f\right)\xi\right\}. \tag{21}$$

By differentiating both side of (21) with respect to the parameter s, we get

$$\frac{d\xi_{\varphi(s,v)}}{ds}=\left\{\begin{array}{c}
\left\{\begin{array}{c}
-\left\{v\left\{\tau+\epsilon(f'g-fg')\right\}'+\epsilon\kappa g\left(1-\epsilon v\kappa f\right)\right\}\frac{1}{\|A\|}\\
-v\left\{\tau+\epsilon(f'g-fg')\right\}\left(\frac{1}{\|A\|}\right)'
\end{array}\right\}T\\
+\left\{\left(1-\epsilon v\kappa f\right)\left(\frac{1}{\|A\|}\right)'+\left(\begin{array}{c}
\left(1-\epsilon v\kappa f\right)'\\
-\epsilon v\kappa g\left\{\tau+\epsilon(f'g-fg')\right\}
\end{array}\right)\frac{1}{\|A\|}\right\}\xi\\
+\left\{\tau+\epsilon(f'g-fg')\right\}\frac{1}{\|A\|}X
\end{array}\right\}. \tag{22}$$

From (20) and (22), we obtain

$$
\begin{aligned}
S(A_o) &= \frac{1}{\|A\|}\left\{\begin{array}{c} -\left\{v\left\{\tau+\epsilon(f'g-fg')\right\}'+\epsilon\kappa g\left(1-\epsilon v\kappa f\right)\right\}\frac{1}{\|A\|} \\ -v\left\{\tau+\epsilon(f'g-fg')\right\}\left(\frac{1}{\|A\|}\right)' \end{array}\right\}T \\
&\quad+\frac{1}{\|A\|}\left\{(1-\epsilon v\kappa f)\left(\frac{1}{\|A\|}\right)'+\left(\begin{array}{c} (1-\epsilon v\kappa f)' \\ -\epsilon v\kappa g\left\{\tau+\epsilon(f'g-fg')\right\} \end{array}\right)\frac{1}{\|A\|}\right\}\xi \\
&\quad+\left\{\tau+\epsilon(f'g-fg')\right\}\frac{1}{\|A\|^2}X.
\end{aligned}
$$

So the Gauss curvature is

$$
K(s,v)=\frac{\left\{\tau+\epsilon(f'g-fg')\right\}^2}{\|A\|^4}. \tag{23}
$$

We differentiate both side of (23) with respect to v for finding the maximum value of the Gauss curvature along X on M. Thus, we obtain

$$
\frac{\partial K(s,v)}{\partial v}=\frac{4\left\{\tau+\epsilon(f'g-fg')\right\}^2\left\{v\left\{f^2\kappa^2+\left\{\tau+\epsilon(f'g-fg')\right\}^2\right\}-\epsilon\kappa f\right\}}{\left\{v^2\left\{\tau+\epsilon(f'g-fg')\right\}^2+\left(1-\epsilon v\kappa f\right)^2\right\}^3}=0
$$

and $v=\dfrac{\epsilon\kappa f}{f^2\kappa^2+\left\{\tau+\epsilon(f'g-fg')\right\}^2}$. It is easy to see that $\varphi(s,v)$ is the striction point and we can say that the absolute value of the Gauss curvature of M is maximum at the striction points on X. Finally, by using (23), we get

$$
|K|_{\max}=\frac{\left\{f^2\kappa^2+\left\{\tau+\epsilon(f'g-fg')\right\}^2\right\}^2}{\left\{\tau+\epsilon(f'g-fg')\right\}^2}. \tag{24}
$$

This completes the proof. □

By using (19) and (24), we can write the relation between the Gauss curvature and the distribution parameter as similar to the equation (17). Thus, we prove the following corollary, too.

Corollary 3.7 *The distribution parameter of the timelike ruled surface with timelike rulings depends on the timelike generating lines.*

Moreover, the Darboux frame of the surface along the spacelike base curve is

$$
\begin{bmatrix} \nabla_T T \\ \nabla_T X \\ \nabla_T \xi \end{bmatrix}=\begin{bmatrix} 0 & -\epsilon f\kappa & \epsilon g\kappa \\ -\epsilon f\kappa & 0 & \left\{\tau+\epsilon(f'g-fg')\right\} \\ -\epsilon g\kappa & \left\{\tau+\epsilon(f'g-fg')\right\} & 0 \end{bmatrix}\begin{bmatrix} T \\ X \\ \xi \end{bmatrix}
$$

and the Darboux vector is

$$
W=-\varepsilon_2\left\{\tau+\epsilon(f'g-fg')\right\}T+\varepsilon_1\epsilon g\kappa X+\varepsilon_1\epsilon f\kappa\xi.
$$

Thus, we obtain the geodesic curvature, the geodesic torsion and the normal curvature of the timelike ruled surface with timelike rulings along its timelike generating lines as

$$\kappa_g = \varepsilon_1 \epsilon g \kappa \quad \tau_g = -\varepsilon_2 \left\{ \tau + \epsilon (f' g - f g') \right\} \quad \kappa_\xi = \varepsilon_1 \epsilon f \kappa,$$

respectively. If the timelike ruled surface with timelike rulings is a constant curvature surface with a nonzero geodesic curvature, P_X is a constant and from (8) and (15), we obtain $\frac{\tau_g^2}{\kappa_g^2 + \tau_g^2} =$ constant. Hence, we get the following theorem.

Theorem 3.8 *A timelike ruled surface with timelike rulings is a constant curvature surface with a nonzero geodesic curvature if and only if* $\frac{\tau_g^2}{\kappa_g^2 + \tau_g^2}$ *is a constant. In the case that the spacelike base curve is one of the geodesics of the timelike ruled surface, the timelike ruled surface is developable.*

The Steiner rotation vector is

$$D = -\varepsilon_2 \left(\oint_{(\alpha)} \left\{ \tau + \epsilon (f' g - f g') \right\} ds \right) T + \varepsilon_1 \epsilon \left(\oint_{(\alpha)} g \kappa ds \right) X + \varepsilon_1 \epsilon \left(\oint_{(\alpha)} f \kappa ds \right) \xi.$$

The angle of pitch of M is

$$\lambda_X = -\varepsilon_1 \epsilon \oint_{(\alpha)} g \kappa ds.$$

From (4), (5) and (17), we obtain that $L_N = \lambda_N = 0$, $L_B = 0$ and $\lambda_B = -\varepsilon_1 \epsilon \oint_{(\alpha)} \kappa ds$ for the special cases, $X = N$ and $X = B$.

§4. Spacelike Ruled Surfaces with Spacelike Rulings

Let $\alpha : I \to E^3$ be a regular spacelike curve with the arc-length parameter s. Since T and X are spacelike vectors, ξ is a timelike vector and the functions f and g satisfy

$$f^2 - g^2 = \epsilon$$

along α ([6]). From (2) and (3), we write

$$\nabla_T X = -\epsilon f \kappa T + \left\{ \tau - \epsilon (f' g - f g') \right\} \xi. \tag{25}$$

The distribution parameter of the spacelike ruled surface with spacelike rulings is

$$P_X = \frac{-\left\{ \tau - \epsilon (f' g - f g') \right\}}{f^2 \kappa^2 - \left\{ \tau - \epsilon (f' g - f g') \right\}^2}. \tag{26}$$

Thus we have the following result.

Theorem 4.1 *A spacelike ruled surface with spacelike rulings is developable if and only if*

$$\tau - \epsilon(f'g - fg') = 0$$

is satisfied.

In the cases $f = 0$, $g = 1$ and $f = 1$, $g = 0$, we get $P_X = P_B$ and $P_X = P_N$, respectively. Thus, the distribution parameters are

$$P_N = \frac{-\tau}{\kappa^2 - \tau}, \quad P_B = \frac{1}{\tau}$$

and we obtain

$$\frac{P_B}{P_N} = \left(\frac{\kappa}{\tau}\right)^2 - 1.$$

Thus, we get the following conclusion.

Corollary 4.2 *The base curve of the spacelike ruled surface is a spacelike helice if and only if* $\frac{P_B}{P_N}$ *is a constant.*

On the other hand, from (5), the striction curve of M is

$$\overline{\alpha}(s) = \alpha(s) + \frac{\epsilon\kappa f}{f^2\kappa^2 - \{\tau - \epsilon(f'g - fg')\}^2} X(s).$$

In the case that M is a cylindirical spacelike ruled surface, we know the result following.

Theorem 4.3 *i) If M is a cylindrical spacelike ruled surface, $\kappa f = 0$. In the case $\kappa = 0$, the spacelike ruled surface is a plane. In the case $f = 0$, unit normal vector of M and binormal vector of the base curve are on the same direction and both the striction curve and the spacelike base curve are geodesics of M.*

ii) A cylindrical spacelike ruled surface is developable if and only if the base curve is a planar spacelike curve.

The tangent vector field of the curve $\varphi_v : I \times \{v\} \to M$ is

$$A = (1 - v\epsilon\kappa f)\, T + v\,\{\tau - \epsilon(f'g - fg')\}\,\xi \tag{27}$$

on M for each $v \in IR$. Since the vector field A is normal to ξ, $\tau - \epsilon(f'g - fg') = 0$ is satisfied. Thus, the following theorem is true.

Theorem 4.4 *The tangent planes of a spacelike ruled surface are the same along a spacelike generating lines if and only if*

$$\tau - \epsilon(f'g - fg') = 0$$

is satisfied.

Theorem 4.5 *i) Let $\kappa f \neq 0$ and M be a closed spacelike ruled surface as given in the form (4).*

The distance between spacelike generating lines of M is minimum along the spacelike striction curve.

ii) Let $\overline{\alpha}(s)$ be a striction curve of M, then $\dfrac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}$ *is a constant.*

iii) Let M be a spacelike ruled surface as given in the form (4), then $\varphi(s,v_o)$ is a striction point if and only if $\nabla_T X$ is normal to the tangent plane at that point on M, where

$$v_o=\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}.$$

Proof *i)* Let $X_{\alpha(s_1)}$ and $X_{\alpha(s_2)}$ be spacelike generating lines which pass from the points $\alpha(s_1)$ and $\alpha(s_2)$ of the base curve, respectively$(s_1,s_2\in IR,\ and\ s_1<s_2)$. The distance between these spacelike generating lines along the orthogonal orbits is

$$J(v)=\int_{s_1}^{s_2}\|A\|\,ds.$$

Then, we obtain

$$J(v)=\int_{s_1}^{s_2}\left(\left\{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2\right\}v^2-2v\epsilon f\kappa+1\right)^{\frac{1}{2}}ds.$$

If $J(v)$ is minimum for v_0, $J'(v_o)=0$ and we get

$$v_o=\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}.$$

Thus, the orthogonal orbit is the spacelike striction curve of M for $v=v_o$.

ii) Since the tangent vector field of the spacelike striction curve is normal to X, $\left\langle X,\frac{d\overline{\alpha}}{ds}\right\rangle=0$. Thus, we get

$$\frac{d}{ds}\left(\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}\right)=0$$

and so

$$\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}=\text{constant}.$$

iii) We suppose that $\varphi(s,v_o)$ is a spacelike striction point on the spacelike base curve $\alpha(s)$ of M, then we must show that $\langle\nabla_T X,X\rangle=0$ and $\langle\nabla_T X,A\rangle=0$. Since the vector field X is an unit vector, $T[\langle X,X\rangle]=0$ and we get $\langle\nabla_T X,X\rangle=0$. On the other hand, from (25) and (27), we obtain

$$\langle\nabla_T X,A\rangle=-\epsilon f\kappa\left(1-\epsilon v_o\kappa f\right)-v_o\left\{\tau-\epsilon(f'g-fg')\right\}.$$

By using

$$v_o=\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2},$$

we get $\langle\nabla_T X,A\rangle=0$. This means that $\nabla_T X$ is normal to the tangent plane at the spacelike striction point $\varphi(s, v_o)$ on M.

Conversely, since $\nabla_T X$ is normal to the tangent plane at the point $\varphi(s, v_o)$ on M, $\langle \nabla_T X, A \rangle = 0$ and from (25) and (27), we obtain

$$-\epsilon f \kappa (1 - \epsilon v_o \kappa f) - v_o \{\tau - \epsilon(f'g - fg')\} = 0.$$

So we get $v_o = \dfrac{\epsilon \kappa f}{f^2\kappa^2 - \{\tau - \epsilon(f'g - fg')\}^2}$ and so $\varphi(s, v_o)$ is a striction point on M. □

Theorem 4.6 *Absolute value of the Gauss curvature of M is maximum at the striction points on the spacelike generating line X and it is*

$$|K|_{\max} = \frac{\left\{f^2\kappa^2 - \{\tau - \epsilon(f'g - fg')\}^2\right\}^2}{\{\tau - \epsilon(f'g - fg')\}^2}$$

Proof Let M be a spacelike ruled surface as given in the form (1) and Φ be base of the tangent space which is spanned by the unit vectors A_o and X where A_o is the tangent vector of the curve $\varphi(s, v = const.)$ with the arc-length parameter s^*. Hence, we write $A_o = \dfrac{d\varphi}{ds^*} = \dfrac{d\varphi}{ds}\dfrac{ds}{ds^*}$ where $\dfrac{d\varphi}{ds} = A$, $A_o = \dfrac{1}{\|A\|}A$ and $\dfrac{ds}{ds^*} = \dfrac{1}{\|A\|}$. Thus, we obtain the following equations after the routine calculations,

$$\begin{aligned}
\nabla_{A_o} T &= \frac{\epsilon\kappa}{\|A\|}\{fX - g\xi\}, \\
\nabla_{A_o} \xi &= \frac{1}{\|A\|}\{-\epsilon\kappa g T + \{\tau - \epsilon(f'g - fg')\} X\}, \\
\nabla_{A_o} A &= \frac{1}{\|A\|}\left\{\begin{array}{c}
\left\{(1 - \epsilon v\kappa f)' - \epsilon v \kappa g\,(\tau - \epsilon(f'g - fg'))\right\} T \\
+\left\{-\epsilon\kappa f\,(1 - \epsilon v\kappa f) + v\,\{\tau - \epsilon(f'g - fg')\}^2\right\} X \\
+\left\{-\epsilon\kappa g\,(1 - \epsilon v\kappa f) + \{v\,\{\tau - \epsilon(f'g - fg')\}\}'\right\} \xi
\end{array}\right\}.
\end{aligned}$$

We denote $\xi_{\varphi(s,v)}$ as the unit normal vector at the points φ $(s, v =$constant), then from (2), (3) and (27) we get

$$\xi_{\varphi(s,v)} = \frac{1}{\|A\|}\{v\,\{\tau - \epsilon(f'g - fg')\} T + (1 - \epsilon v\kappa f)\,\xi\}. \tag{28}$$

By differentiating both side of (28) with respect to the parameter s, we get

$$\frac{d\xi_{\varphi(s,v)}}{ds} = \left\{\begin{array}{c}
\left\{\begin{array}{c}
\left\{v\,\{\tau - \epsilon(f'g - fg')\}' - \epsilon\kappa g\,(1 - \epsilon v\kappa f)\right\} \frac{1}{\|A\|} \\
+v\,\{\tau - \epsilon(f'g - fg')\}\left(\frac{1}{\|A\|}\right)'
\end{array}\right\} T \\
+\left\{(1 - \epsilon v\kappa f)\left(\frac{1}{\|A\|}\right)' + \left(\begin{array}{c}
(1 - \epsilon v\kappa f)' \\
-\epsilon v\kappa g\,\{\tau - \epsilon(f'g - fg')\}
\end{array}\right)\frac{1}{\|A\|}\right\}\xi \\
+\{\tau - \epsilon(f'g - fg')\}\frac{1}{\|A\|} X
\end{array}\right\}. \tag{29}$$

From (27) and (29), we obtain

$$
\begin{aligned}
S\left(A_o\right) \; = \; & \frac{1}{\|A\|}\left\{\begin{array}{c} \left\{v\left\{\tau-\epsilon(f'g-fg')\right\}'-\epsilon\kappa g\left(1-\epsilon v\kappa f\right)\right\}\frac{1}{\|A\|} \\ +v\left\{\tau-\epsilon(f'g-fg')\right\}\left(\frac{1}{\|A\|}\right)' \end{array}\right\}T \\
& +\frac{1}{\|A\|}\left\{(1-\epsilon v\kappa f)\left(\frac{1}{\|A\|}\right)'+\left(\begin{array}{c} (1-\epsilon v\kappa f)' \\ -\epsilon v\kappa g\left\{\tau-\epsilon(f'g-fg')\right\} \end{array}\right)\frac{1}{\|A\|}\right\}\xi \\
& +\left\{\tau-\epsilon(f'g-fg')\right\}\frac{1}{\|A\|^2}X.
\end{aligned}
$$

Hence, the Gauss curvature is

$$
K(s,v)=\frac{\left\{\tau-\epsilon(f'g-fg')\right\}^2}{\|A\|^4}. \tag{30}
$$

We differentiate both side of (30) with respect to v for finding the maximum value of the Gauss curvature along X on M. Thus, we obtain

$$
\frac{\partial K(s,v)}{\partial v}=\frac{4\left\{\tau-\epsilon(f'g-fg')\right\}^2\left\{v\left\{f^2\kappa^2-\left\{\tau-\epsilon(f'g-fg')\right\}^2\right\}-\epsilon\kappa f\right\}}{\left\{v^2\left\{f^2\kappa^2-\left\{\tau-\epsilon(f'g-fg')\right\}^2\right\}-2\epsilon v\kappa f+1\right\}^3}=0
$$

and $v=\dfrac{\epsilon\kappa f}{f^2\kappa^2-\left\{\tau-\epsilon(f'g-fg')\right\}^2}$. It is easy to see that $\varphi(s,v)$ is the spacelike striction point and we can say that the absolute value of the Gauss curvature of M is maximum at the striction points on X. Finally, by using (30), we get

$$
|K|_{\max}=\frac{\left\{f^2\kappa^2-\left\{\tau-\epsilon(f'g-fg')\right\}^2\right\}^2}{\left\{\tau-\epsilon(f'g-fg')\right\}^2}. \tag{31}
$$

This completes the proof. □

We can write the relation between the Gauss curvature and the distribution parameter as similar to (17) by using (26) and (31).Thus, we prove the following corollary, too.

Corollary 4.7 *The distribution parameter of the spacelike ruled surface depends on the spacelike generating lines.*

Moreover, the Darboux frame of the surface along the spacelike base curve is

$$
\begin{bmatrix} \nabla_T T \\ \nabla_T X \\ \nabla_T \xi \end{bmatrix}=\begin{bmatrix} 0 & \epsilon f\kappa & -\epsilon g\kappa \\ -\epsilon f\kappa & 0 & \left\{\tau-\epsilon(f'g-fg')\right\} \\ -\epsilon g\kappa & \left\{\tau-\epsilon(f'g-fg')\right\} & 0 \end{bmatrix}\begin{bmatrix} T \\ X \\ \xi \end{bmatrix}
$$

and the Darboux vector is

$$
W=-\varepsilon_2\left\{\tau-\epsilon(f'g-fg')\right\}T-\varepsilon_1\epsilon g\kappa X-\varepsilon_1\epsilon f\kappa\xi.
$$

Thus, we obtain the geodesic curvature, the geodesic torsion and the normal curvature of the ruled surface along its spacelike generating lines as

$$\kappa_g = -\varepsilon_1 \epsilon g \kappa \quad \tau_g = -\varepsilon_2 \left\{ \tau - \epsilon (f'g - fg') \right\} \quad \kappa_\xi = -\varepsilon_1 \epsilon f \kappa \ ,$$

respectively. Note also that if the ruled surface is a constant curvature surface with a nonzero geodesic curvature, P_X is a constant and from (8) and (15), we obtain $\dfrac{\tau_g^2}{\kappa_g^2 + \tau_g^2} = \text{constant}$. Hence, we have the following theorem.

Theorem 4.8 *A spacelike ruled surface is a constant curvature surface with a nonzero geodesic curvature if and only if* $\dfrac{\tau_g^2}{\kappa_g^2+\tau_g^2}$ *is a constant. In the case that the spacelike base curve is one of the geodesics of the spacelike ruled surface, the spacelike ruled surface is developable.*

On the other hand, the Steiner rotation vector is

$$D = -\varepsilon_2 \left(\oint\limits_{(\alpha)} \left\{ \tau - \epsilon (f'g - fg') \right\} ds \right) T - \varepsilon_1 \epsilon \left(\oint\limits_{(\alpha)} g\kappa ds \right) X - \varepsilon_1 \epsilon \left(\oint\limits_{(\alpha)} f\kappa ds \right) \xi.$$

The angle of pitch of M is

$$\lambda_X = -\varepsilon_1 \epsilon \oint\limits_{(\alpha)} g\kappa ds.$$

From (4), (5) and (17), we obtain that $L_N = \lambda_N = 0$, $L_B = 0$ and $\lambda_B = -\varepsilon_1 \epsilon \oint\limits_{(\alpha)} \kappa ds$ for the special cases, $X = N$ and $X = B$.

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Enumeration of k-Fibonacci Paths Using Infinite Weighted Automata

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Abstract: In this paper, we introduce a new family of generalized colored Motzkin paths, where horizontal steps are colored by means of $F_{k,l}$ colors, where $F_{k,l}$ is the l-th k-Fibonacci number. We study the enumeration of this family according to the length. For this, we use infinite weighted automata.

Key Words: Fibonacci sequence, Generalized colored Motzkin path, k-Fibonacci path, infinite weighted automata, generating function.

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§1. Introduction

A lattice path of length n is a sequence of points $P_1, P_2, \ldots, P_n$ with $n \geqslant 1$ such that each point P_i belongs to the plane integer lattice and each two consecutive points P_i and P_{i+1} connect by a line segment. We will consider lattice paths in $\mathbb{Z} \times \mathbb{Z}$ using three step types: a rise step $U = (1,1)$, a fall step $D = (1,-1)$ and a $F_{k,l}-$colored length horizontal step $H_l = (l, 0)$ for every positive integer l, such that H_l is colored by means of $F_{k,l}$ colors, where $F_{k,l}$ is the l-th k-Fibonacci number.

Many kinds of generalizations of the Fibonacci numbers have been presented in the literature [10,11] and the corresponding references. Such as those of *k-Fibonacci numbers* $F_{k,n}$ and the *k-Smarandache-Fibonacci numbers* $S_{k,n}$. For any positive integer number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$, is defined recurrently by

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \text{ for } n \geqslant 1.$$

The generating function of the k-Fibonacci numbers is $f_k(x) = \dfrac{x}{1 - kx - x^2}$, [4,6]. This sequence was studied by Horadam in [9]. Recently, Falcón and Plaza [6] found the k-Fibonacci numbers by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. The interested reader is also referred to [1, 3, 4, 5, 6, 12, 13, 16] for further information about this.

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A *generalized $F_{k,l}$-colored Motzkin path* or simply *k-Fibonacci path* is a sequence of rise, fall and $F_{k,l}-$colored length horizontal steps ($l = 1, 2, \cdots$) running from $(0,0)$ to $(n,0)$ that never pass below the x-axis. We denote by $\mathcal{M}_{F_{k,n}}$ the set of all k-Fibonacci paths of length n and $\mathcal{M}_k = \bigcup_{n=0}^{\infty} \mathcal{M}_{F_{k,n}}$. In Figure 1 we show the set $\mathcal{M}_{F_{2,3}}$.

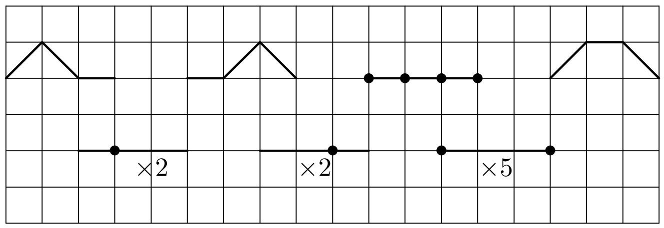


Figure 1 k-Fibonacci Paths of length 3, $|\mathcal{M}_{F_{2,3}}| = 13$

A *grand k-Fibonacci path* is a k-Fibonacci path without the condition that never going below the x-axis. We denote by $\mathcal{M}^*_{F_{k,n}}$ the set of all grand k-Fibonacci paths of length n and $\mathcal{M}^*_k = \bigcup_{n=0}^{\infty} \mathcal{M}^*_{F_{k,n}}$. A *prefix k-Fibonacci path* is a k-Fibonacci path without the condition that ending on the x-axis. We denote by $\mathcal{PM}_{F_{k,n}}$ the set of all prefix k-Fibonacci paths of length n and $\mathcal{PM}_k = \bigcup_{n=0}^{\infty} \mathcal{PM}_{F_{k,n}}$. Analogously, we have the family of *prefix grand k-Fibonacci paths*. We denote by $\mathcal{PM}^*_{F_{k,n}}$ the set of all prefix grand k-Fibonacci paths of length n and $\mathcal{PM}^*_k = \bigcup_{n=0}^{\infty} \mathcal{PM}^*_{F_{k,n}}$.

In this paper, we study the generating function for the k-Fibonacci paths, grand k-Fibonacci paths, prefix k-Fibonacci paths, and prefix grand k-Fibonacci paths, according to the length. We use Counting Automata Methodology (CAM) [2], which is a variation of the methodology developed by Rutten [14] called Coinductive Counting. Counting Automata Methodology uses infinite weighted automata, weighted graphs and continued fractions. The main idea of this methodology is find a counting automaton such that there exist a bijection between all words recognized by an automaton $\mathcal{M}$ and the family of combinatorial objects. From the counting automaton $\mathcal{M}$ is possible find the ordinary generating function (GF) of the family of combinatorial objects [4].

§2. Counting Automata Methodology

The terminology and notation are mainly those of Sakarovitch [13]. An *automaton* $\mathcal{M}$ is a 5-tuple $\mathcal{M} = (\Sigma, Q, q_0, F, E)$, where Σ is a nonempty input alphabet, Q is a nonempty set of states of $\mathcal{M}$, $q_0 \in Q$ is the initial state of $\mathcal{M}$, $\emptyset \neq F \subseteq Q$ is the set of final states of $\mathcal{M}$ and $E \subseteq Q \times \Sigma \times Q$ is the set of transitions of $\mathcal{M}$. The language recognized by an automaton $\mathcal{M}$ is denoted by $L(\mathcal{M})$. If Q, Σ and E are finite sets, we say that $\mathcal{M}$ is a finite automaton [15].

Example 2.1 Consider the finite automaton $\mathcal{M} = (\Sigma, Q, q_0, F, E)$ where $\Sigma = \{a, b\}$, $Q = \{q_0, q_1\}$, $F = \{q_0\}$ and $E = \{(q_0, a, q_1), (q_0, b, q_0), (q_1, a, q_0)\}$. The transition diagram of $\mathcal{M}$ is as shown in Figure 2. It is easy to verify that $L(\mathcal{M}) = (b \cup aa)^*$.

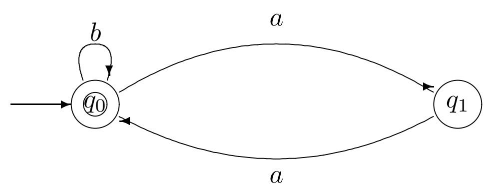


Figure 2 Transition diagram of $\mathcal{M}$, Example 1

Example 2.2 Consider the infinite automaton $\mathcal{M}_{\mathcal{D}} = (\Sigma, Q, q_0, F, E)$, where $\Sigma = \{a, b\}$, $Q = \{q_0, q_1, \cdots\}$, $F = \{q_0\}$ and $E = \{(q_i, a, q_{i+1}), (q_{i+1}, b, q_i) : i \in \mathbb{N}\}$. The transition diagram of $\mathcal{M}_{\mathcal{D}}$ is as shown in Figure 3.

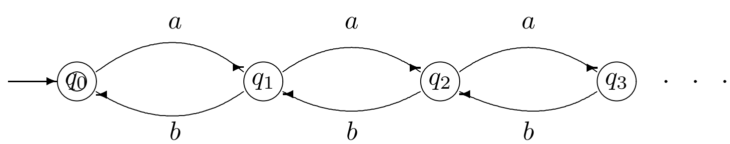


Figure 3 Transition diagram of $\mathcal{M}_{\mathcal{D}}$

The language accepted by $\mathcal{M}_{\mathcal{D}}$ is

$$L(\mathcal{M}_{\mathcal{D}}) = \{w \in \Sigma^* : |w|_a = |w|_b \text{ and for all prefix } v \text{ of } w, |v|_b \leq |v|_a\}.$$

An ordinary generating function $F = \sum_{n=0}^{\infty} f_n z^n$ corresponds to a formal language L if $f_n = |\{w \in L : |w| = n\}|$, i.e., if the n-th coefficient f_n gives the number of words in L with length n.

Given an alphabet Σ and a semiring $\mathbb{K}$. A *formal power series* or *formal series* S is a function $S : \Sigma^* \to \mathbb{K}$. The image of a word w under S is called the *coefficient* of w in S and is denoted by s_w. The series S is written as a formal sum $S = \sum_{w \in \Sigma^*} s_w w$. The set of formal power series over Σ with coefficients in $\mathbb{K}$ is denoted by $\mathbb{K}\langle\langle \Sigma^* \rangle\rangle$.

An automaton over Σ^* with weights in $\mathbb{K}$, or $\mathbb{K}$-*automaton* over Σ^* is a graph labelled with elements of $\mathbb{K}\langle\langle \Sigma^* \rangle\rangle$, associated with two maps from the set of vertices to $\mathbb{K}\langle\langle \Sigma^* \rangle\rangle$. Specifically, a *weighted automaton* $\mathcal{M}$ over Σ^* with weights in $\mathbb{K}$ is a 4-tuple $\mathcal{M} = (Q, I, E, F)$ where Q is a nonempty set of *states* of $\mathcal{M}$, E is an element of $\mathbb{K}\langle\langle \Sigma^* \rangle\rangle^{Q \times Q}$ called *transition matrix.* I is an element of $\mathbb{K}\langle\langle \Sigma^* \rangle\rangle^{Q}$, i.e., I is a function from Q to $\mathbb{K}\langle\langle \Sigma^* \rangle\rangle$. I is the *initial function* of $\mathcal{M}$ and can also be seen as a row vector of dimension Q, called *initial vector* of $\mathcal{M}$ and F is an element of $\mathbb{K}\langle\langle \Sigma^* \rangle\rangle^{Q}$. F is the *final function* of $\mathcal{M}$ and can also be seen as a column vector of dimension Q, called *final vector* of $\mathcal{M}$.

We say that $\mathcal{M}$ is a *counting automaton* if $\mathbb{K} = \mathbb{Z}$ and $\Sigma^* = \{z\}^*$. With each automaton, we can associate a counting automaton. It can be obtained from a given automaton replacing every transition labelled with a symbol a, $a \in \Sigma$, by a transition labelled with z. This transition is called a *counting transition* and the graph is called a *counting automaton* of $\mathcal{M}$. Each transition from p to q yields an equation

$$L(p)(z) = zL(q)(z) + [p \in F] + \cdots .$$

We use L_p to denote $L(p)(z)$. We also use Iverson's notation, $[P] = 1$ if the proposition P is true and $[P] = 0$ if P is false.

2.1 Convergent Automata and Convergent Theorems

We denote by $L^{(n)}(\mathcal{M})$ the number of words of length n recognized by the automaton $\mathcal{M}$, including repetitions.

Definition 2.3 *We say that an automaton $\mathcal{M}$ is convergent if for all integer $n \geqslant 0$, $L^{(n)}(\mathcal{M})$ is finite.*

The proof of following theorems and propositions can be found in [2].

Theorem 2.4(First Convergence Theorem) *Let $\mathcal{M}$ be an automaton such that each vertex (state) of the counting automaton of $\mathcal{M}$ has finite degree. Then $\mathcal{M}$ is convergent.*

Example 2.5 The counting automaton of the automaton $\mathcal{M}_{\mathcal{D}}$ in Example 2 is convergent.

The following definition plays an important role in the development of applications because it allows to simplify counting automata whose transitions are formal series.

Definition 2.6 *Let $\mathcal{M}$ be an automaton, and let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be a formal power series with $f_n \in \mathbb{N}$ for all $n \geqslant 0$ and $f_0 = 0$. In a counting automaton of $\mathcal{M}$ the set of counting transitions from state p to state q, without intermediate final states, see Figure 4 (left), is represented by a graph with a single edge labeled by $f(z)$, see Figure 4(right).*

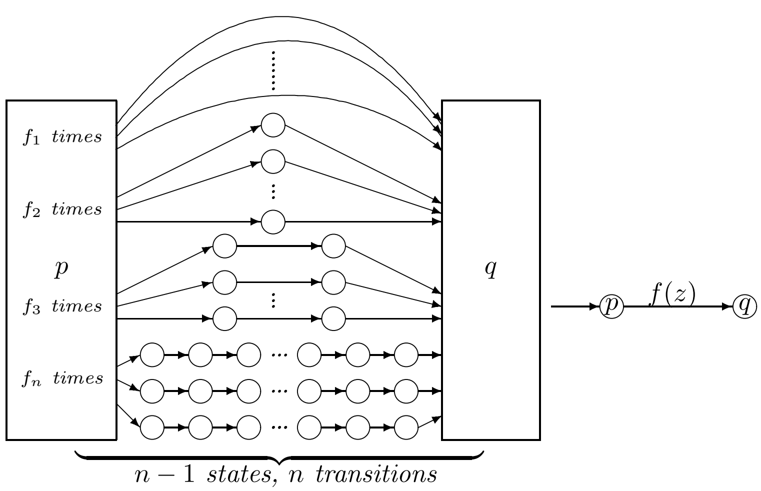


Figure 4 *Transitions from the state p to q and its transition in parallel*

This kind of transition is called a transition in parallel. The states p and q are called visible states and the intermediate states are called hidden states.

Example 2.7 In Figure 5 (left) we display a counting automaton $\mathcal{M}_1$ without transitions in parallel, i.e., every transition is label by z. The transitions from state q_1 to q_2 correspond to the series $\dfrac{1-\sqrt{1-4z}}{2} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + \cdots$. However, this automaton can also be represented using transitions in parallel. Figure 5 (right) displays two examples.

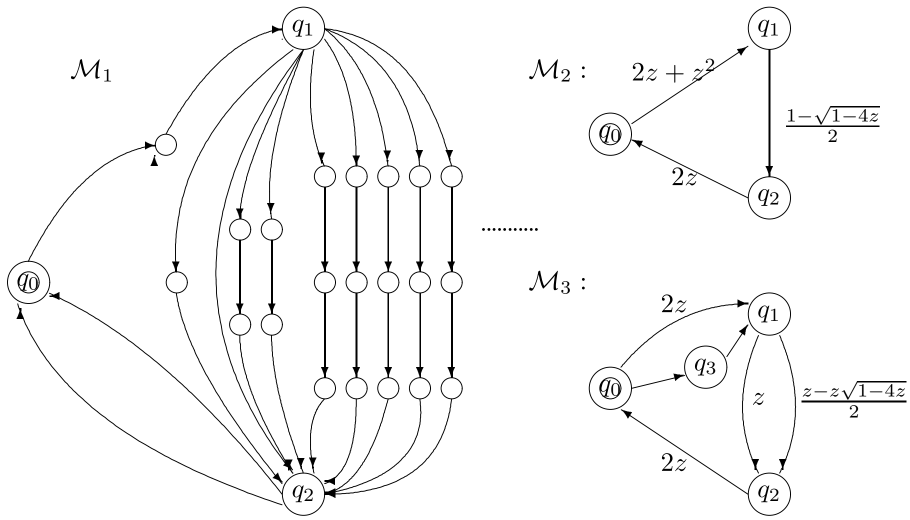


Figure 5 Counting automata with transitions in parallel

Theorem 2.8(Second Convergence Theorem) *Let $\mathcal{M}$ be an automaton, and let $f_1^q(z), f_2^q(z), \cdots$, be transitions in parallel from state $q \in Q$ in a counting automaton of $\mathcal{M}$. Then $\mathcal{M}$ is convergent if the series*

$$F^q(z) = \sum_{k=1}^{\infty} f_k^q(z)$$

is a convergent series for each visible state $q \in Q$ of the counting automaton.

Proposition 2.9 *If $f(z)$ is a polynomial transition in parallel from state p to q in a finite counting automaton $\mathcal{M}$, then this gives rise to an equation in the system of GFs equations of $\mathcal{M}$*

$$L_p = f(z)L_q + [p \in F] + \cdots .$$

Proposition 2.10 *Let $\mathcal{M}$ be a convergent automaton such that a counting automaton of $\mathcal{M}$ has a finite number of visible states $q_0, q_1, \cdots, q_r$, in which the number of transitions in parallel starting from each state is finite. Let $f_1^{q_t}(z), f_2^{q_t}(z), \cdots, f_{s(t)}^{q_t}(z)$ be the transitions in parallel from the state $q_t \in Q$. Then the GF for the language $L(\mathcal{M})$ is $L_{q_0}(z)$. It is obtained by solving* *the system of* $r+1$ *GFs equations*

$$L(q_t)(z) \;=\; f_1^{q_t}(z)L(q_{t_1})(z) \;+\; f_2^{q_t}(z)L(q_{t_2})(z) \;+\; \cdots \;+\; f_{s(t)}^{q_t}(z)L(q_{t_{s(t)}})(z) \;+\; [q_t \;\in\; F],$$

with $0 \le t \le r$*, where* q_{t_k} *is the visible state joined with* q_t *through the transition in parallel* $f_k^{q_t}$*, and* $L(q_{t_k})$ *is the GF for the language accepted by* $\mathcal{M}$ *if* q_{t_k} *is the initial state.*

Example 2.11 *The system of GFs equations associated with* $\mathcal{M}_2$*, see Example* 2.7*, is*

$$\begin{cases} L_0 &= (2z+z^2)L_1 + 1 \\ L_1 &= \dfrac{1-\sqrt{1-4z}}{2}L_2 \\ L_2 &= 2zL_0. \end{cases}$$

Solving the system for L_0*, we find the GF for the language* $\mathcal{M}_2$ *and therefore of* $\mathcal{M}_1$ *and* $\mathcal{M}_3$

$$L_0 = \frac{1}{1-(2z^2+z^3)(1-\sqrt{1-4z})} = 1 + 4z^3 + 6z^4 + 10z^5 + 40z^6 + 114z^7 + \cdots.$$

2.2 An Example of the Counting Automata Methodology (CAM)

A counting automaton associated with an automaton $\mathcal{M}$ can be used to model combinatorial objects if there is a bijection between all words recognized by the automaton $\mathcal{M}$ and the combinatorial objects. Such method, along with the previous theorems and propositions constitute the *Counting Automata Methodology (CAM)*, see [2].

We distinguish three phases in the CAM:

(1) Given a problem of enumerative combinatorics, we have to find a convergent automaton $\mathcal{M}$ (see Theorems 2.4 and 2.8), whose GF is the solution of the problem.

(2) Find a general formula for the GF of $\mathcal{M}'$, where $\mathcal{M}'$ is an automaton obtained from $\mathcal{M}$ truncating a set of states or edges see Propositions 2.9 and 2.10. Sometimes we find a relation of iterative type, such as a continued fraction.

(3) Find the GF $f(z)$ to which converge the GFs associated to each $\mathcal{M}'$, which is guaranteed by the convergences theorems.

Example 2.12 A *Motzkin path* of length n is a lattice path of $\mathbb{Z}\times\mathbb{Z}$ running from $(0,0)$ to $(n,0)$ that never passes below the x-axis and whose permitted steps are the up diagonal step $U=(1,1)$, the down diagonal step $D=(1,-1)$ and the horizontal step $H=(1,0)$. The number of Motzkin paths of length n is the *n-th Motzkin number* m_n, sequence A001006[1]. The number of words of length n recognized by the convergent automaton $\mathcal{M}_{\text{Mot}}$, see Figure 6, is the nth Motzkin number and its GF is

$$M(z) = \sum_{i=0}^{\infty} m_i z^i = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}.$$

[1]Many integer sequences and their properties are found electronically on the On-Line Encyclopedia of Sequences [17].

$\mathcal{M}_{\text{Mot}}$:

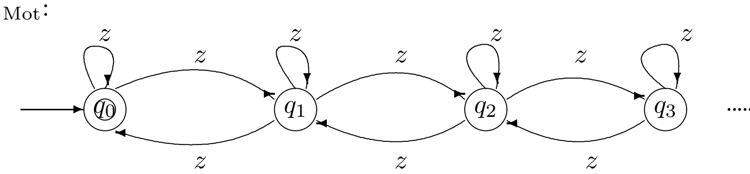


Figure 6 Convergent automaton associated with Motzkin paths

In this case the edge from state q_i to state q_{i+1} represents a rise, the edge from the state q_{i+1} to q_i represents a fall and the loops represent the level steps, see Table 1.

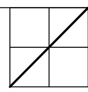
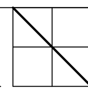
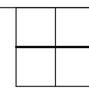
$(q_i, z, q_{i+1}) \in E \Leftrightarrow$		$(q_{i+1}, z, q_i) \in E \Leftrightarrow$		$(q_i, z, q_i) \in E \Leftrightarrow$	

Table 1 Bijection between $\mathcal{M}_{\text{Mot}}$ and Motzkin paths

Moreover, it is clear that a word is recognized by $\mathcal{M}_{\text{Mot}}$ if and only if the number of steps to the right and to the left coincide, which ensures that the path is well formed. Then

$$m_n = |\{w \in L(\mathcal{M}_{\text{Mot}}) : |w| = n\}| = L^{(n)}(\mathcal{M}_{\text{Mot}}).$$

Let $\mathcal{M}_{\text{Mot}s}$, $s \geq 1$ be the automaton obtained from $\mathcal{M}_{\text{Mot}}$, by deleting the states $q_{s+1}, q_{s+2}, \ldots$. Therefore the system of GFs equations of $\mathcal{M}_{\text{Mot}s}$ is

$$\begin{cases} L_0 = zL_0 + zL_1 + 1, \\ L_i = zL_{i-1} + zL_i + zL_{i+1}, \quad 1 \leq i \leq s-1, \\ L_s = zL_{s-1} + zL_s. \end{cases}$$

Substituting repeatedly into each equation L_i, we have

$$L_0 = \left.\frac{H}{1 - \dfrac{F^2}{1 - \dfrac{F^2}{\dfrac{\vdots}{1 - F^2}}}}\right\} s \text{ times},$$

where $F = \dfrac{z}{1-z}$ and $H = \dfrac{1}{1-z}$. Since $\mathcal{M}_{\text{Mot}}$ is convergent, then as $s \to \infty$ we obtain a convergent continued fraction M of the GF of $\mathcal{M}_{\text{Mot}}$. Moreover,

$$M = \frac{H}{1 - F^2\left(\frac{M}{H}\right)}.$$

Hence $z^2M^2 - (1-z)M + 1 = 0$ and

$$M(z) = \frac{1 - z \pm \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

Since $\epsilon \in L(\mathcal{M}_{\mathrm{Mot}})$, $M \to 0$ as $z \to 0$. Hence, we take the negative sign for the radical in $M(z)$.

§3. Generating Function for the k-Fibonacci Paths

In this section we find the generating function for k-Fibonacci paths, grand k-Fibonacci paths, prefix k-Fibonacci paths and prefix grand k-Fibonacci paths, according to the length.

Lemma 3.1([2]) *The GF of the automaton $\mathcal{M}_{\mathrm{Lin}}$, see Figure 7, is*

$$E(z) = \cfrac{1}{1 - h_0(z) - \cfrac{f_0(z)\, g_0(z)}{1 - h_1(z) - \cfrac{f_1(z)\, g_1(z)}{\ddots}}},$$

where $f_i(z), g_i(z)$ and $h_i(z)$ are transitions in parallel for all integer $i \geqslant 0$.

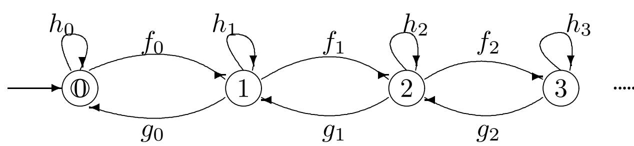


Figure 7 Linear infinite counting automaton $\mathcal{M}_{Lin}$

The last lemma coincides with Theorem 1 in [7] and Theorem 9.1 in [14]. However, this presentation extends their applications, taking into account that $f_i(z), g_i(z)$ and $h_i(z)$ are GFs, which can be GFs of several variables.

Corollary 3.2 *If for all integers $i \geq 0$, $f_i(z) = f(z), g_i(z) = g(z)$ and $h_i(z) = h(z)$ in $\mathcal{M}_{\mathrm{Lin}}$, then the GF is*

$$B(z) = \frac{1 - h(z) - \sqrt{(1 - h(z))^2 - 4f(z)g(z)}}{2f(z)g(z)} \tag{1}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m+2n}{m} \left(f(z)\, g(z)\right)^n (h(z))^m \tag{2}$$

$$= \cfrac{1}{1 - h(z) - \cfrac{f(z)\, g(z)}{1 - h(z) - \cfrac{f(z)\, g(z)}{1 - h(z) - \cfrac{f(z)\, g(z)}{\ddots}}}}, \tag{3}$$

where C_n is the nth Catalan number, sequence A000108.

Theorem 3.3 *The generating function for the k-Fibonacci paths according to the their length is*

$$\begin{aligned} T_k(z) &= \sum_{i=0}^{\infty} |\mathcal{M}_{F_{k,i}}| z^i && (4)\\ &= \frac{1-(k+1)z-z^2-\sqrt{(1-(k+1)z-z^2)^2-4z^2(1-kz-z^2)^2}}{2z^2(1-kz-z^2)} && (5)\\ &= \cfrac{1}{1-\frac{z}{1-kz-z^2}-\cfrac{z^2}{1-\frac{z}{1-kz-z^2}-\cfrac{z^2}{1-\frac{z}{1-kz-z^2}-\cfrac{z^2}{\ddots}}}} && (6) \end{aligned}$$

and

$$\left[z^t\right] T_k(z) = \sum_{n=0}^{t} \sum_{m=0}^{t-2n} \binom{m+2n}{m} C_n F_{k,t-2n-m+1}^{(m)},$$

where C_n is the n-th Catalan number and $F_{k,j}^{(r)}$ is a convolved k-Fibonacci number.

Convolved k-Fibonacci numbers $F_{k,j}^{(r)}$ are defined by

$$f_k^{(r)}(x) = (1-kx-x^2)^{-r} = \sum_{j=0}^{\infty} F_{k,j+1}^{(r)} x^j, \quad r \in \mathbb{Z}^+.$$

Note that

$$F_{k,m+1}^{(r)} = \sum_{j_1+j_2+\cdots+j_r=m} F_{k,j_1+1} F_{k,j_2+1} \cdots F_{k,j_r+1}.$$

Moreover, using a result of Gould[8, p.699] on Humbert polynomials (with $n=j, m=2, x=k/2, y=-1, p=-r$ and $C=1$), we have

$$F_{k,j+1}^{(r)} = \sum_{l=0}^{\lfloor j/2 \rfloor} \binom{j+r-l-1}{j-l} \binom{j-l}{l} k^{j-2l}.$$

Ramírez [13] studied some properties of convolved k-Fibonacci numbers.

Proof Equations (5) and (6) are clear from Corollary 3.2 taking $f(z)=z=g(z)$ and $h(z)=\dfrac{z}{1-kz-z^2}$. Note that $h(z)$ is the GF of k-Fibonacci numbers. In this case the edge from state q_i to state q_{i+1} represents a rise, the edge from the state q_{i+1} to q_i represents a fall and the loops represent the $F_{k,l}-$colored length horizontal steps $(l=1,2,\cdots)$. Moreover, from

Equation (2), we obtain

$$\begin{aligned}
T_k(z) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n} \left(\frac{z}{1-kz-z^2}\right)^m \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n+m} \left(\frac{1}{1-kz-z^2}\right)^m \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n+m} \sum_{i=0}^{\infty} F_{k,i+1}^{(m)} z^i \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} C_n F_{k,i+1}^{(m)} \binom{m+2n}{m} z^{2n+m+i},
\end{aligned}$$

taking $s = 2n + m + i$

$$T_k(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=2n+m}^{\infty} C_n F_{k,s-2n-m+1}^{(m)} \binom{m+2n}{m} z^s.$$

Hence

$$\left[z^t\right] T_k(z) = \sum_{n=0}^{t}\sum_{m=0}^{t-2m} C_n F_{k,t-2n-m+1}^{(m)} \binom{m+2n}{m}. \qquad \square$$

In Table 2 we show the first terms of the sequence $|\mathcal{M}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 1, 3, 8, 23, 67, 199, 600, 1834, 5674, 17743, ...
2	1, 1, 4, 13, 47, 168, 610, 2226, 8185, 30283, 112736, $\cdots$
3	1, 1, 5, 20, 89, 391, 1735, 7712, 34402, 153898, 690499, $\cdots$
4	1, 1, 6, 29, 155, 820, 4366, 23262, 124153, 663523, 3551158, $\cdots$

Table 2 Sequences $|\mathcal{M}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$

Definition 3.4 *For all integers $i \geq 0$ we define the continued fraction $E_i(z)$ by:*

$$E_i(z) = \cfrac{1}{1 - h_i(z) - \cfrac{f_i(z)\, g_i(z)}{1 - h_{i+1}(z) - \cfrac{f_{i+1}(z)\, g_{i+1}(z)}{\ddots}}},$$

where $f_i(z), g_i(z), h_i(z)$ are transitions in parallel for all integers positive i.

Lemma 3.5([2]) *The GF of the automaton $\mathcal{M}_{\mathrm{BLin}}$, see Figure 8, is*

$$E_b(z) = \frac{1}{1 - h_0(z) - f_0(z)g_0(z)E_1(z) - f_0'(z)g_0'(z)E_1'(z)},$$

where $f_i(z), f_i'(z), g_i(z), g_i'(z), h_i(z)$ and $h_i'(z)$ are transitions in parallel for all $i \in \mathbb{Z}$.

$\mathcal{M}_{BLin}$:

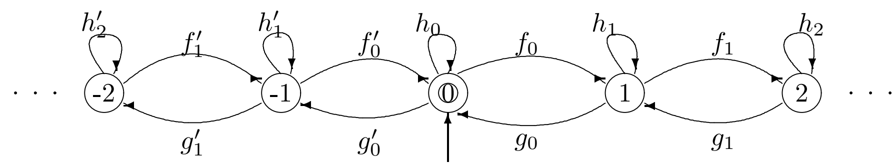


Figure 8 Linear infinite counting automaton $\mathcal{M}_{BLin}$

Corollary 3.6 *If for all integers i, $f_i(z) = f(z) = f_i'(z), g_i(z) = g(z) = g_i'(z)$ and $h_i(z) = h(z) = h_i'(z)$ in $\mathcal{M}_{\mathrm{BLin}}$, then the GF*

$$B_b(z) = \frac{1}{\sqrt{(1-h(z))^2 - 4f(z)g(z)}} \tag{7}$$

$$= \cfrac{1}{1 - h(z) - \cfrac{2f(z)g(z)}{1 - h(z) - \cfrac{f(z)g(z)}{1 - h(z) - \cfrac{f(z)g(z)}{\ddots}}}}, \tag{8}$$

where $f(z), g(z)$ and $h(z)$ are transitions in parallel. Moreover, if $f(z) = g(z)$, then the GF

$$B_b(z) = \frac{1}{1-h(z)} + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^n \frac{n}{n+2k}\binom{n+2k}{k}\binom{l+2n+2k}{l} f(z)^{2n+2k} h(z)^l. \tag{9}$$

Theorem 3.7 *The generating function for the grand k-Fibonacci paths according to the their length is*

$$T_k^*(z) = \sum_{i=0}^{\infty} |\mathcal{M}_{F_{k,i}}^*| z^i = \frac{1 - kz - z^2}{\sqrt{(1-(k+1)z - z^2)^2 - 4z^2(1 - kz - z^2)^2}} \tag{10}$$

$$= \cfrac{1}{1 - \frac{z}{1-kz-z^2} - \cfrac{2z^2}{1 - \frac{z}{1-kz-z^2} - \cfrac{z^2}{1 - \frac{z}{1-kz-z^2} - \cfrac{z^2}{\ddots}}}} \tag{11}$$

and

$$\left[z^t\right] T_k^*(z) = F_{k+1,t}^{(1)} + \sum_{n=1}^{t}\sum_{m=0}^{t}\sum_{l=0}^{t-2n-2m} 2^n \frac{n}{n+2m}\binom{n+2m}{m}\binom{l+2n+2m}{l} F_{k,t-2n-2m-l+1}^{(l)}, \tag{12}$$

with $t \geqslant 1$.

Proof Equations (10) and (11) are clear from Corollary 3.6, taking $f(z) = z = g(z)$ and $h(z) = \frac{z}{1-kz-z^2}$. Moreover, from Equation (9), we obtain

$$\begin{aligned} T_k^*(z) &= \frac{1}{1-\frac{z}{1-kz-z^2}} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} 2^n \frac{n}{n+2m}\binom{n+2m}{m}\binom{l+2n+2m}{l} z^{2n+2m}\left(\frac{z}{1-kz-z^2}\right)^l \\ &= 1 + \sum_{j=0}^{\infty} F_{k+1,j}^{(1)} z^j + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty} 2^n \frac{n}{n+2m}\binom{n+2m}{m}\binom{l+2n+2m}{l} F_{k,u}^{(l)} z^{2n+2m+u+1}, \end{aligned}$$

taking $s = 2n + 2m + l + u$

$$T_k^*(z) = 1 + \sum_{j=0}^{\infty} F_{k+1,j}^{(1)} z^j + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{s=2n+2m+l}^{\infty} 2^n \frac{n}{n+2m}\binom{n+2m}{m}\binom{l+2n+2m}{l} F_{k,s-2n-2m-l}^{(l)} z^s.$$

Therefore, Equation (12) is clear. □

In Table 3 we show the first terms of the sequence $|\mathcal{M}_{F_{k,i}}^*|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 4, 11, 36, 115, 378, 1251, 4182, 14073, 47634, $\cdots$
2	1, 5, 16, 63, 237, 920, 3573, 14005, 55156, 218359, $\cdots$
3	1, 6, 23, 108, 487, 2248, 10371, 48122, 223977, 1046120, $\cdots$
4	1, 7, 32, 177, 949, 5172, 28173, 153963, 842940, 4624581, $\cdots$

Table 3 Sequences $|\mathcal{M}_{F_{k,i}}^*|$ for $k = 1, 2, 3, 4$ and $i \geqslant 1$

In Figure 9 we show the set $\mathcal{M}_{F_{2,3}}^*$.

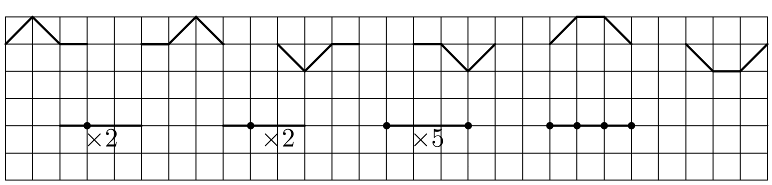


Figure 9 Grand k-Fibonacci Paths of length 3, $|\mathcal{M}_{F_{2,3}}^*| = 16$

Lemma 3.8([2]) *The GF of the automaton* $\textsc{Fin}_{\mathbb{N}}(\mathcal{M}_{Lin})$, *see Figure* 10, *is*

$$G(z) = E(z) + \sum_{j=1}^{\infty}\left(\prod_{i=0}^{j-1}(f_i(z)E_i(z))E_j(z)\right),$$

where $E(z)$ *is the GF in Lemma* 3.1.

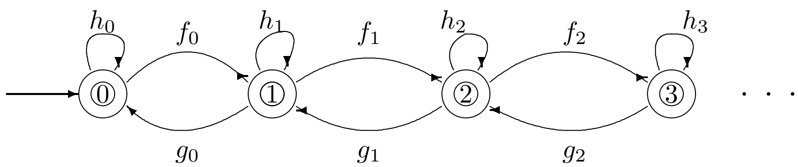


Figure 10 Linear infinite counting automaton $\textsc{Fin}_{\mathbb{N}}(\mathcal{M}_{Lin})$

Corollary 3.9 *If for all integer* $i \geqslant 0$, $f_i(z) = f(z), g_i(z) = g(z)$ *and* $h_i(z) = h(z)$ *in* $\textsc{Fin}_{\mathbb{N}}(\mathcal{M}_{Lin})$, *then the GF is:*

$$G(z) = \frac{1-2f(z)-h(z)-\sqrt{(1-h(z))^2-4f(z)g(z)}}{2f(z)\left(f(z)+g(z)+h(z)-1\right)} \tag{13}$$

$$= \cfrac{1}{1-f(z)-h(z)-\cfrac{f(z)g(z)}{1-h(z)-\cfrac{f(z)g(z)}{1-h(z)-\cfrac{f(z)g(z)}{\ddots}}}}, \tag{14}$$

where $f(z), g(z)$ *and* $h(z)$ *are transitions in parallel and* $B(z)$ *is the GF in Corollary* 3.2. *Moreover, if* $f(z) = g(z)$ *and* $h(z) \neq 0$, *then we obtain the GF*

$$G(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{n+1}{n+k+1}\binom{n+2k+l}{k,l,k+n}f^{2k+n}(z)h^{l}(z). \tag{15}$$

Theorem 3.10 *The generating function for the prefix k-Fibonacci paths according to the their length is*

$$\begin{aligned}PT_k(z) &= \sum_{i=0}^{\infty}|\mathcal{PM}_{F_{k,i}}|z^i\\ &= \frac{(1-2z)(1-kz-z^2)-z-\sqrt{(1-z(k+1)-z^2)^2+4z^2(1-kz-z^2)^2}}{2z((1-kz-z^2)(2z-1)+z)}\end{aligned}$$

and

$$\left[z^t\right] PT_k(z) = \sum_{n=0}^{t}\sum_{m=0}^{t}\sum_{l=0}^{t-2m-n} \frac{n+1}{n+m+1}\binom{n+2m+l}{m,l,m+n} F^{(l)}_{k,t-2m-n-l+1},\ t \geqslant 0.$$

Proof The proof is analogous to the proof of Theorem 3.3 and 3.7. □

In Table 4 we show the first terms of the sequence $|\mathcal{PM}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 2, 6, 19, 62, 205, 684, 2298, 7764, 26355, 89820, $\cdots$
2	1, 2, 7, 26, 101, 396, 1564, 6203, 24693, 98605, 394853, $\cdots$
3	1, 2, 8, 35, 162, 757, 3558, 16766, 79176, 374579, 1775082, $\cdots$
4	1, 2, 9, 46, 251, 1384, 7668, 42555, 236463, 1315281, 7322967, $\cdots$

Table 4 Sequences $|\mathcal{PM}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$

In Figure 11 we show the set $\mathcal{MP}_{F_{2,3}}$.

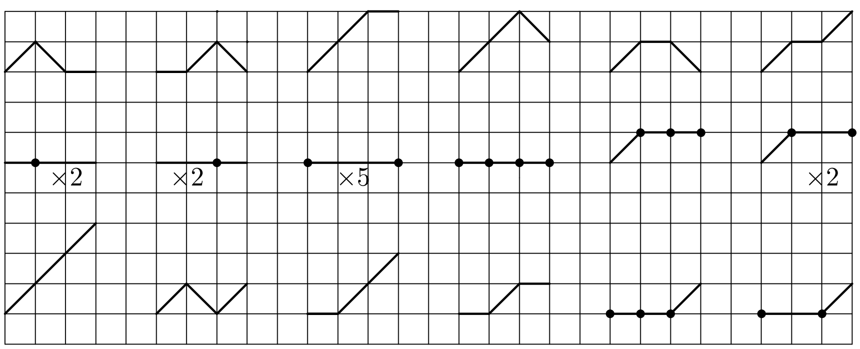


Figure 11 Prefix k-Fibonacci paths of length 3, $|\mathcal{PM}_{F_{2,3}}| = 26$

Lemma 3.11 *The GF of the automaton* $\textsc{Fin}_{\mathbb{Z}}(\mathcal{M}_{BLin})$*, see Figure* 12*, is*

$$\begin{aligned} H(z) &= \frac{EE'}{E+E'-EE'(1-h_0)}\left(1+\sum_{j=1}^{\infty}\prod_{k=1}^{j-1} f_k E_k f_0 E_j + \sum_{j=1}^{\infty}\prod_{k=1}^{j-1} g'_k E'_k g'_0 E'_j\right)\\ &= \frac{E'(z)G(z)+E(z)G'(z)-E(z)E'(z)}{E(z)+E'(z)-E(z)E'(z)(1-h_0(z))}, \end{aligned}$$

where $G(z)$ *is the GF in Lemma* 3.8 *and* $G'(z), E'(z)$ *are the GFs obtained from* $G(z)$ *and* $E(z)$ *changing* $f(z)$ *to* $g'(z)$ *and* $g(z)$ *to* $f'(z)$.

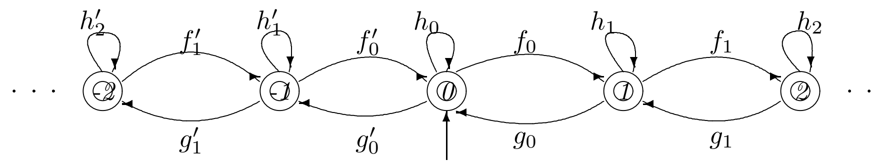


Figure 12 *Linear infinite counting automaton* $\mathrm{FIN}_{\mathbb{Z}}(\mathcal{M}_{BLin})$

Moreover, if for all integer $i \geqslant 0$, $f_i(z) = f(z) = f'_i(z), g_i(z) = g(z) = g'_i(z)$ *and* $h_i(z) = h(z) = h'_i(z)$ *in* $\mathrm{FIN}_{\mathbb{Z}}(\mathcal{M}_{BLin})$*, then the GF is*

$$H(z) = \frac{1}{1 - f(z) - g(z) - h(z)}. \tag{16}$$

Theorem 3.12 *The generating function for the prefix grand k-Fibonacci paths according to the their length is*

$$PT_k^*(z) = \sum_{i=0}^{\infty} |\mathcal{PM}_{F^*_{k,i}}|z^i = \frac{1 - kz - z^2}{1 - (k+3)z - (1-2k)z^2 + 2z^3}.$$

it Proof The proof is analogous to the proof of Theorem 3.3 and 3.7. □

In Table 5 we show the first terms of the sequence $|\mathcal{PM}^*_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 3, 10, 35, 124, 441, 1570, 5591, 19912, 70917, 252574, ...
2	1, 3, 11, 44, 181, 751, 3124, 13005, 54151, 225492, 938997, ...
3	1, 3, 12, 55, 264, 1285, 6280, 30727, 150392, 736157, 3603528, ...
4	1, 3, 13, 68, 379, 2151, 12268, 70061, 400249, 2286780, 13065595 ...

Table 4 Sequences $|\mathcal{PM}^*_{F_{k,i}}|$ for $k = 1, 2, 3, 4$

Acknowledgments

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One Modulo N Gracefullness Of Arbitrary Supersubdivisions of Graphs

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Abstract: A function f is called a graceful labelling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$in such a way that (i) ϕ is $1 - 1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$. In this paper we prove that the arbitrary supersubdivisions of paths, disconnected paths, cycles and stars are one modulo N graceful for all positive integers N.

Key Words: Modulo graceful graph, Smarandache modulo graceful graph, supersubdivisions of graphs, paths, disconnected paths, cycles and stars.

AMS(2010): 05C78

§1. Introduction

S.W.Golomb introduced graceful labelling ([1]). The odd gracefulness was introduced by R.B.Gnanajothi in [2]. C.Sekar introduced one modulo three graceful labelling ([8]) recently. V.Ramachandran and C.Sekar ([6]) introduced the concept of one modulo N graceful where N is any positive integer.In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful.We prove that the the arbitrary supersubdivisions of paths, disconnected paths, cycles and stars are one modulo N graceful for all positive integers N.

§2. Main Results

Definition 2.1 *A graph G is said to be one Smarandache modulo N graceful on subgraph $H < G$ with q edges (where N is a positive integer) if there is a function ϕ from the vertex set*

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of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of H to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, and $E(G)\setminus E(h)$ to $\{1, 2, \cdots, |E(G)|-q\}$, where $\phi^*(uv)=|\phi(u)-\phi(v)|$. Particularly, if $H = G$ such a graph is said to be one modulo N graceful graph.

Definition 2.2([9]) *In the complete bipartite graph $K_{2,m}$ we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of m vertices the m-vertices part of $K_{2,m}$.Let G be a graph with p vertices and q edges. A graph H is said to be a supersubdivision of G if H is obtained by replacing every edge e_i of G by the complete bipartite graph $K_{2,m}$ for some positive integer m in such a way that the ends of e_i are merged with the two vertices part of $K_{2,m}$ after removing the edge e_i from G.H is denoted by $SS(G)$.*

Definition 2.3([9]) *A supersubdivision H of a graph G is said to be an arbitrary supersubdivision of the graph G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). H is denoted by $ASS(G)$.*

Definition 2.4 *A graph G is said to be connected if any two vertices of G are joined by a path. Otherwise it is called disconnected graph.*

Definition 2.5 *A star S_n with n spokes is given by (V, E) where $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ and $E(S_n) = \{v_0v_i/i = 1, 2 \ldots, n\}$. v_0 is called the centre of the star.*

Definition 2.6 *A cycle C_n with n points is a graph given by (V, E) where $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.*

Theorem 2.7 *Arbitrary supersubdivisions of paths are one modulo N graceful for every positive integer N.*

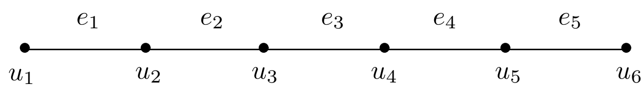
Proof Let P_n be a path with successive vertices $u_1, u_2, u_3, \cdots, u_n$ and let e_i $(1 \le i \le n-1)$ denote the edge u_iu_{i+1} of P_n. Let H be an arbitrary supersubdivision of the path P_n where each edge e_i of P_n is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer,such as those shown in Fig.1 for P_6. We observe that H has $M = 2(m_1+m_2+\cdots+m_{n-1})$ edges.

Define $\phi(u_i) = N(i-1), i = 1, 2, 3, \cdots, n$. For $k = 1, 2, 3, \cdots, m_i$, let

$$\phi(u^{(k)}_{i,i+1}) = \begin{cases} N(M-2k+1)+1 & \text{if } i = 1, \\ N(M-2k+i) - 2N(m_1+m_2+\cdots+m_{i-1})+1 & \text{if } i = 2, 3, \cdots n-1. \end{cases}$$

It is clear from the above labelling that the m_i+2 vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels for $1 \le i \le n-1$. Therefore, the vertices of each K_{2,m_i}, $1 \le i \le n-1$ in the arbitrary supersubdivision H of P_n have distinct labels and also the edges of each K_{2,m_i}, $1 \le i \le n-1$ in the arbitrary supersubdivision graph H of P_n have distinct labels. Also the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$, and (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $\phi^*(uv) = |\phi(u)-\phi(v)|$.

Hence H is one modulo N graceful.



Path P_6

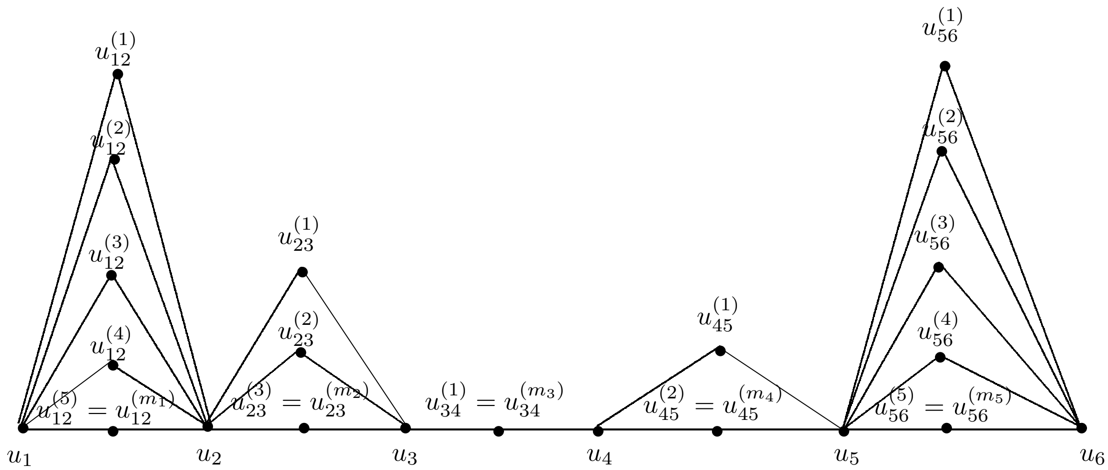


Fig.1 An arbitrary supersubdivision of P_6

Clearly, ϕ defines a one modulo N graceful labelling of arbitrary supersubdivision of the path P_n. □

Example 2.8 An odd graceful labelling of $ASS(P_5)$ is shown in Fig.2.

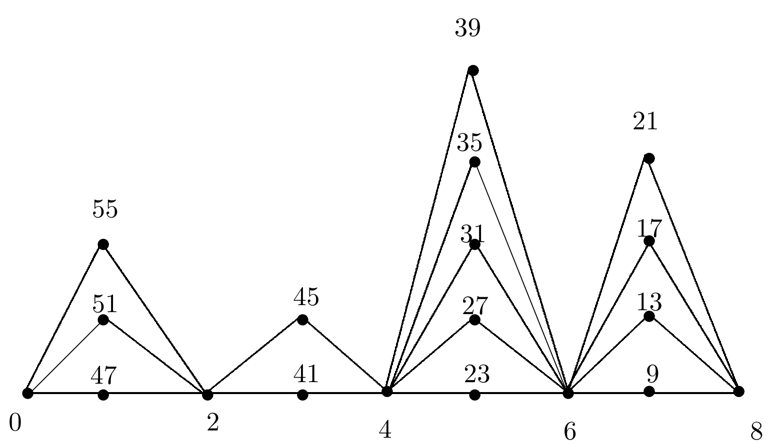


Fig.2

Example 2.9 A graceful labelling of $ASS(P_6)$ is shown in Fig.3.

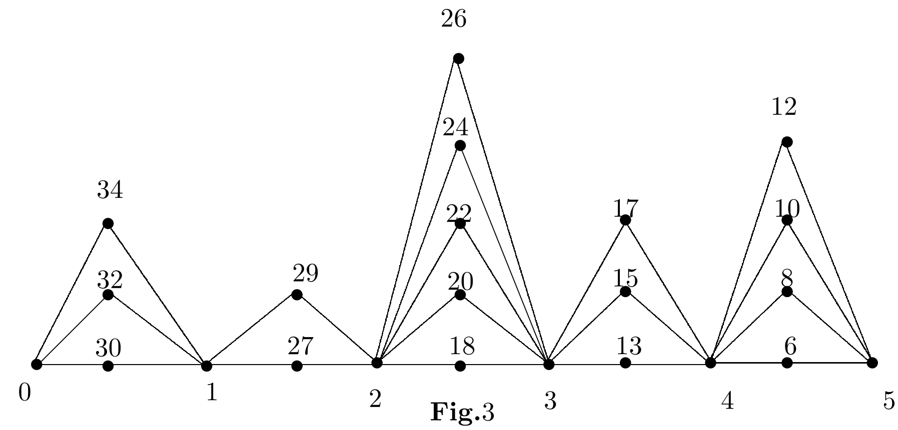


Fig.3

Example 2.10 A one modulo 7 graceful labelling of $ASS(P_6)$ is shown in Fig.4.

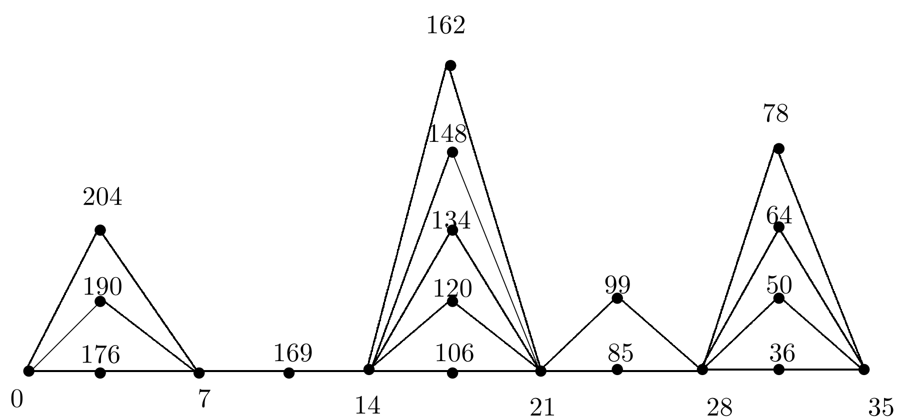


Fig.4

Theorem 2.11 *Arbitrary supersubdivision of disconnecte paths $P_n \cup P_r$ are one modulo N graceful provided the arbitrary supersubdivision is obtained by replacing each edge of G by $K_{2,m}$ with $m \geqslant 2$.*

Proof Let P_n be a path with successive vertices $v_1, v_2, \cdots, v_n$ and let e_i $(1 \leq i \leq n-1)$ denote the edge v_iv_{i+1} of P_n. Let P_r be a path with successive vertices $v_{n+1}, v_{n+2}, \cdots, v_{n+r}$ and let $e_i(n+1 \leq i \leq n+r-1)$ denote the edge v_iv_{i+1}.
Let H be an arbitrary supersubdivision of the disconnected graph $P_n \cup P_r$ where each edge e_i of $P_n \cup P_r$ is replaced by a complete bipartite graph K_{2,m_i} with $m_i \geqslant 2$ for $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$. We observe that H has $M = 2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+\cdots+m_{n+r-1})$ edges.

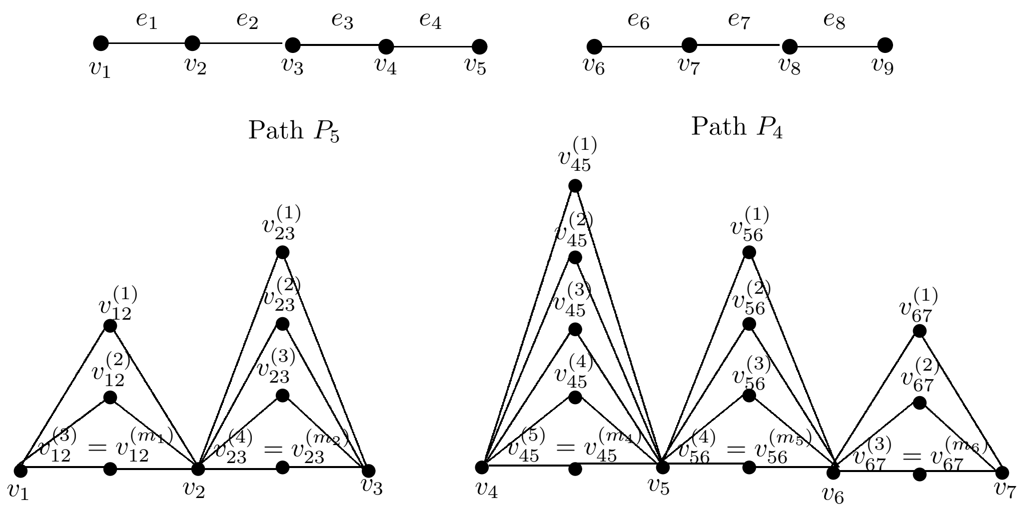


Fig.5 An arbitrary supersubdivision of $P_3 \cup P_4$

Define $\phi(v_i) = N(i-1), i = 1, 2, 3, \cdots, n$, $\phi(v_i) = N(i), i = n+1, n+2, n+3, \cdots, n+r$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 \quad \text{if } i=1, \\ N(M-2+i)+1-2N(m_1+m_2+\cdots+m_{i-1}+k-1) \quad \text{if } i=2,3,\cdots n-1, \\ N(M-1+i)+1-2N(m_1+m_2+\cdots+m_{n-1}+k-1) \quad \text{if } i=n+1, \\ N(M-1+i)+1-2N[(m_1+m_2+\cdots+m_{n-1})+ \\ (m_{n+1}+\cdots+m_{i-1})+k-1] \quad \text{if } i=n+2,n+3,\cdots n+r-1. \end{cases}$$

It is clear from the above labelling that the m_i+2 vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels for $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$.Therefore the vertices of each K_{2,m_i}, $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$ in the arbitrary supersubdivision H of $P_n \cup P_r$ have distinct labels and also the edges of each K_{2,m_i}, $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$ in the arbitrary supersubdivision graph H of $P_n \cup P_r$ have distinct labels. Also the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$, and (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $\phi^*(uv)$=$|\phi(u)-\phi(v)|$. Hence H is one modulo N graceful.

Clearly, ϕ defines a one modulo N graceful labelling of arbitrary supersubdivisions of disconnected paths $P_n \cup P_r$. □

Example 2.12 An odd graceful labelling of $ASS(P_6 \cup P_3)$ is shown in Fig.6.

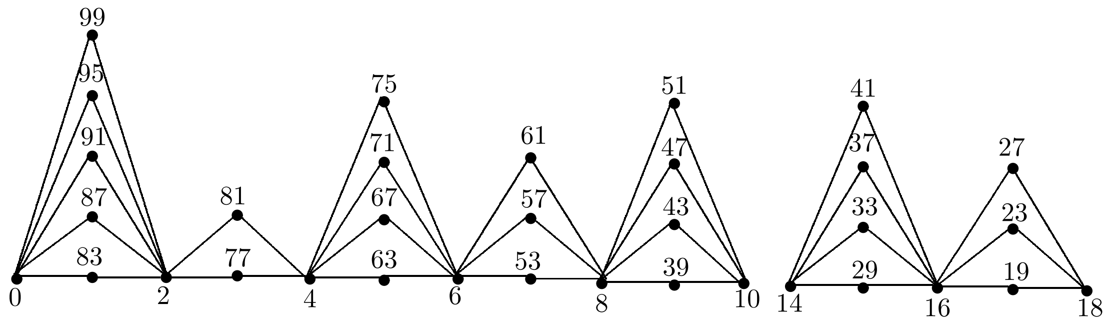


Fig.6

Example 2.13 A graceful labelling of $ASS(P_3 \cup P_4)$ is shown in Fig.7.

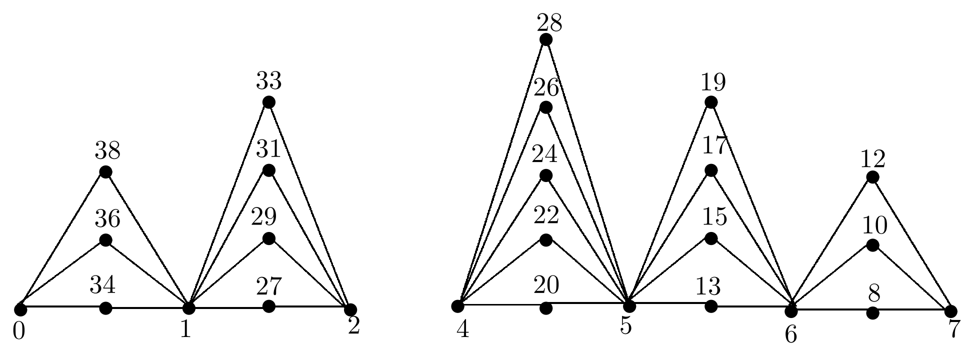


Fig.7

Example 2.14 A one modulo 4 graceful labelling of $ASS(P_4 \cup P_3)$ is shown in Fig.8.

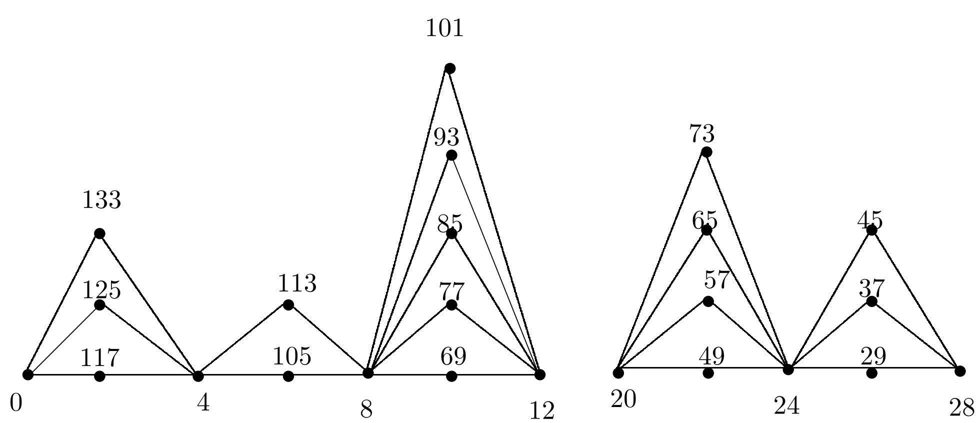


Fig.8

Theorem 2.15 *For any any $n \geq 3$, there exists an arbitrary supersubdivision of C_n which is* *one modulo N graceful for every positive integer N.*

Proof Let C_n be a cycle with consecutive vertices $v_1, v_2, v_3, \cdots, v_n$. Let G be a supersubdivision of a cycle C_n where each edge e_i of C_n is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer for $1 \leq i \leq n-1$ and $m_n = (n-1)$. It is clear that G has $M = 2(m_1 + m_2 + \cdots + m_n)$ edges. Here the edge $v_{n-1}v_1$ is replaced by $K_{2,n-1}$ for the construction of arbitrary supersubdivision of C_n.

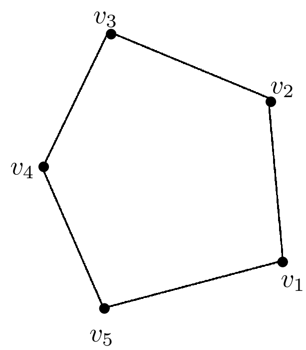


Fig.9 Cycle C_n

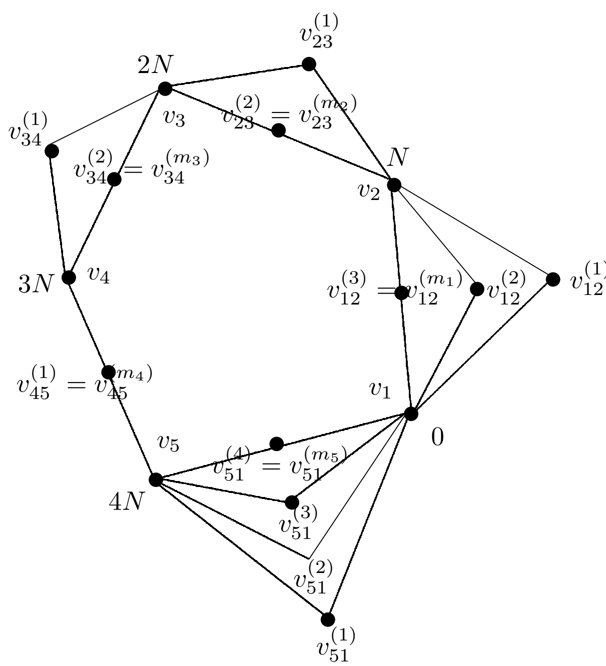


Fig.10 An arbitrary Supersubdivision of C_5

Define $\phi(v_i) = N(i-1), i = 1, 2, 3, \cdots, n$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 & \text{if } i = 1, \\ N(M-2k+i)+1-2N(m_1+m_2+\cdots+m_{i-1}) & \text{if } i = 2, 3, \cdots n-1. \end{cases}$$

and $\phi(v_{n,1}^{(k)}) = N(n-k+m_n-1)+1$.

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u)-\phi(v)|$. Hence, H is one modulo N graceful. Clearly, ϕ defines a modulo N graceful labelling of arbitrary supersubdivision of cycle C_n. □

Example 2.16 An odd graceful labelling of $ASS(C_5)$ is shown in Fig.11.

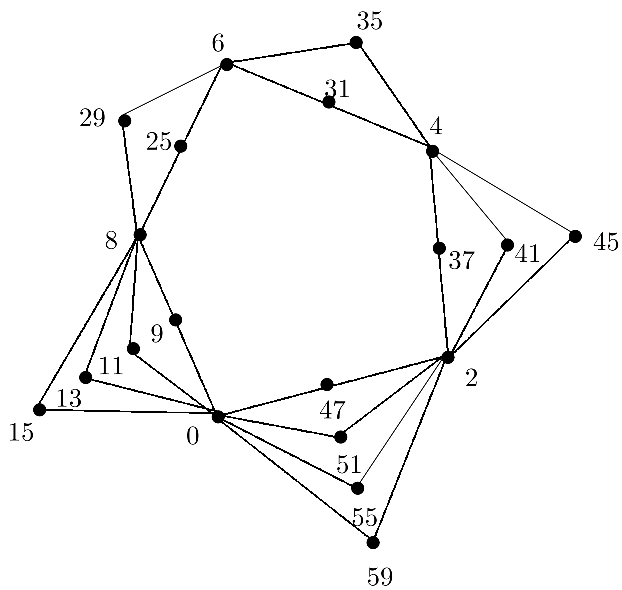


Fig.11

Example 2.17 A graceful labelling of $ASS(C_5)$ is shown in Fig.12.

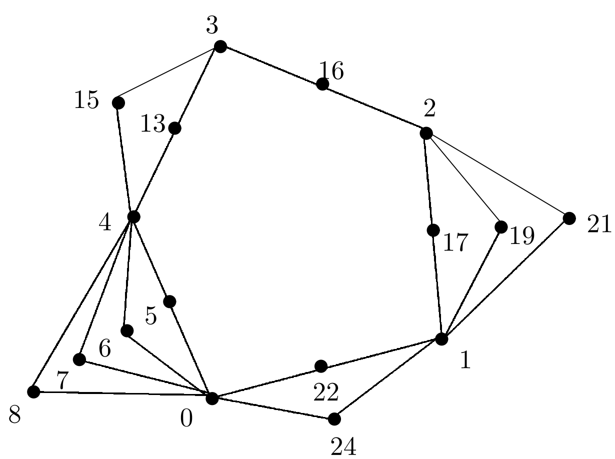


Fig.12

Example 2.18 A one modulo 3 graceful labelling of $ASS(C_4)$ is shown in Fig.13.

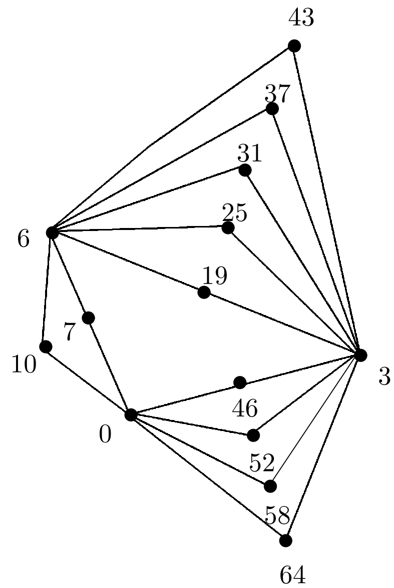


Fig.13

Theorem 2.19 *Arbitrary supersubdivision of any star is one modulo N graceful for every positive integer N.*

Proof The proof is divided into 2 cases.

Case 1 $N = 1$

It has been proved in [4] that arbitrary supersubdivision of any star is graceful.

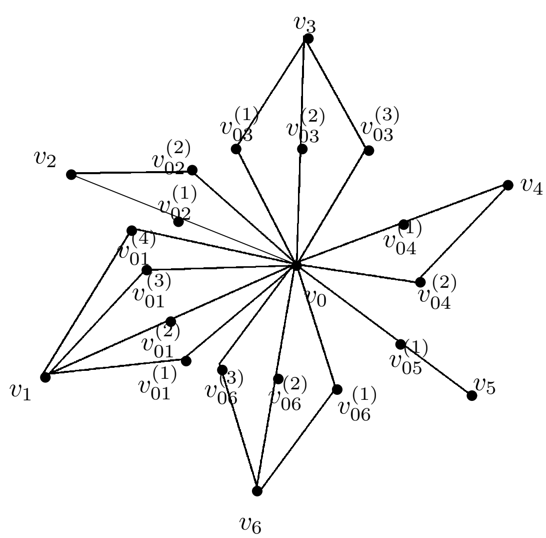


Fig.14 An arbitrary supersubdivision of S_6

Case 2 $N > 1$.

Let S_n be a star with vertices $v_0, v_1, v_2, \cdots, v_n$ and let e_i denote the edge $v_0 v_i$ of S_n for $1 \leq$

$i \leq n$. Let H be an arbitrary supersubdivision of S_n. That is for $1 \leq i \leq n$ each edge e_i of S_n is replaced by a complete bipartite graph K_{2,m_i} with m_i is any positive integer for $1 \leq i \leq n-1$ and $m_n = (n-1)$. It is clear that H has $M = 2(m_1+m_2+\cdots+m_n)$ edges. The vertex set and edge set of H are given by $V(H) = \{v_0, v_1, v_2 \cdots, v_n, v_{01}^{(1)}, v_{01}^{(2)} \cdots, v_{01}^{(m_1)}, v_{02}^{(1)}, v_{02}^{(2)}, \cdots, v_{02}^{(m_2)}, \cdots, v_{0n}^{(1)}, v_{0n}^{(2)}, \cdots, v_{0n}^{(m_n)}\}$.

Define $\phi : V(H) \to \{0, 1, 2, \cdots 2\sum_{i=1}^{n} m_i\}$ as follows:

let $\phi(v_0) = 0$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{0i}^{(k)}) = \begin{cases} N(M-k)+1 & \text{if } i = 1, \\ N(M-k)+1-N(m_1+m_2+\cdots+m_{i-1}) & \text{if} i = 2, 3, \cdots n. \end{cases}$$

$$\phi(v_i) = \begin{cases} N(M-m_1) & \text{if } i = 1, \\ NM - N(2m_1+2m_2+\cdots+2m_{i-1}+m_i) & \text{if } i = 2, 3, \cdots n. \end{cases}$$

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$. Hence H is one modulo N graceful.

Clearly, ϕ defines a one modulo N graceful labelling of arbitrary supersubdivision of star S_n. □

Example 2.20 A one modulo 5 graceful labelling of $ASS(S_4)$ is shown in Fig.14.

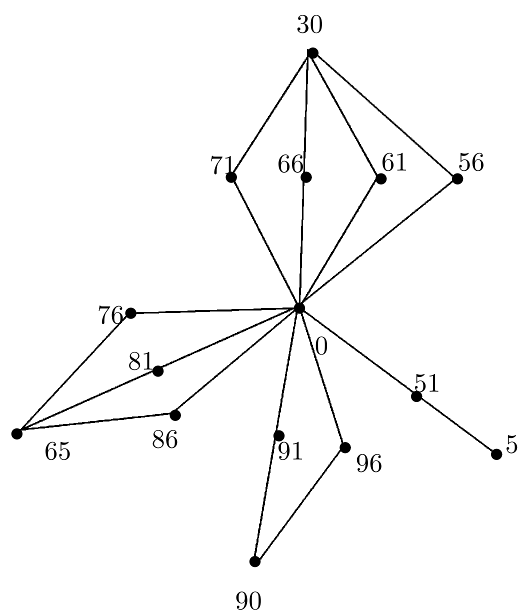


Fig.14

Example 2.21 An odd graceful labelling of $ASS(S_6)$ is shown in Fig.15.

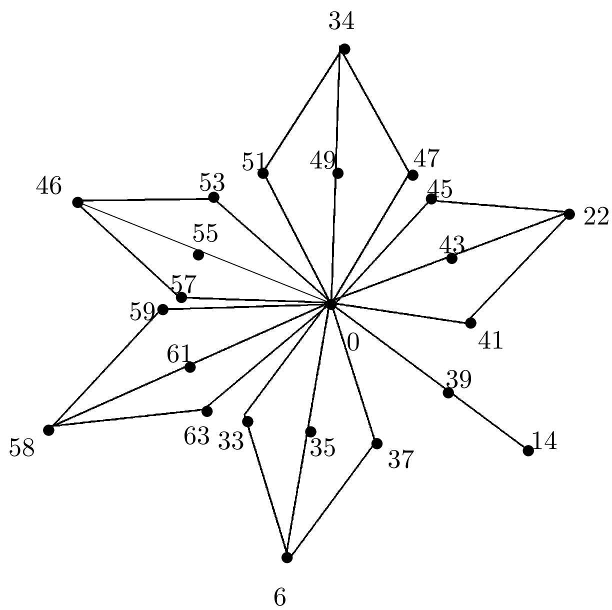


Fig.15

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The Natural Lift Curves and Geodesic Curvatures of the Spherical Indicatrices of The Spacelike-Timelike Bertrand Curve Pair

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Abstract: In this paper, when (α, α^*) spacelike-timelike Bertrand curve pair is given, the geodesic curves and the arc-lenghts of the curvatures (T^*), (N^*), (B^*) and the fixed pole curve (C^*) which are generated over the S_1^2 Lorentz sphere or the H_0^2 hyperbolic sphere by the Frenet vectors $\{T^*,\ N^*,\ B^*\}$ and the unit Darboux vector C^* have been obtained. The condition being the naturel lifts of the spherical indicatrix of the α^* is an integral curve of the geodesic spray has expressed.

Key Words: Lorentz space, spacelike-timelike Bertrand curve pair, naturel lift, geodesic spray.

AMS(2010): 53A04, 53B30, 51B20

§1. Introduction

It is well known that many studies related to the differential geometry of curves have been made. Especially, by establishing relations between the Frenet Frames in mutual points of two curves several theories have been obtained. The best known of these: Firstly, Bertrand Curves discovered by J. Bertrand in 1850 are one of the important and interesting topics of classical special curve theory. A Bertrand curve is defined as a special curve which shares its principal normals with another special curve, called Bertrand mate or Bertrand curve Partner. Secondly, involute–evolute curves discovered by C. Huygens in 1658, who is also known for his work in topics, discovered involutes while trying to build a more accurate clock. The curve α is called evolute of α^* if the tangent vectors are orthogonal at the corresponding points for each $s \in I$: In this case, α^* is called involute of the curve α and the pair of (α, α^*) is called a involute-evolute curve pair. Thirdly, Mannheim curve discovered by A. Mannheim in 1878. Liu and Wang have given a new definition of the curves as known Mannheim curves [8] and [15]. According to the definition given by Liu and Wang, the principal normal vector field of α is linearly dependent on the binormal vector field of α^*. Then α is called a Mannheim curve and α^* a Mannheim

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Partner Curve of α. The pair (α, α^*) is said to be a Mannheim pair. Furthermore, they showed that the curve is a Mannheim Curve α if and only if its curvature and torsion satisfy the formula $\kappa = \lambda\left(\kappa^2 + \tau^2\right)$, where $\lambda$is a nonzero contant [8], [9] and [15].

In three dimensional Euclidean space E^3 and three dimensional Minkowski space IR_1^3 the spherical indicatrices of any space curve with the natural lifts and the geodesic sprays of fixed pole curve of any space curve have computed and accordingly, some results related to the curve α for the geodesic spray on the tangent bundle of the natural lifts to be an integral curve have been obtained [5], [11]. On the other hand, the natural lifts and the curvatures of the spherical indicatrices of the Mannheim Pair and the Involute-Evolute curves have been investigated and accordingly, some results related to the curve α for the geodesic spray on the tangent bundle of the natural lifts to be an integral curve have been obtained [2], [4], [5], [7] and [12].

In this paper, arc-lengths and geodesic curvatures of the spherical indicatrix curves with the fixed pole curve of the (α, α^*) spacelike-timelike Bertrand curve pair have been obtained with respect to IR_1^3 Lorent space and S_1^2 Lorentz sphere or H_0^2 Hyperbolic sphere. In addition, the relations among the geodesic curvatures and arc-lengths are given. Finally, the condition being the natural lifts of the spherical indicatrix curves of the α^* timelike curve are an integral curve of the geodesic spray has expressed depending on α spacelike curve.

§2. Preliminaries

Let Minkowski 3-space $\mathbb{R}_1^3$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g\left(X, X\right) = x_1^2 + x_2^2 - x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g\left(X, X\right) < 0$, spacelike if $g\left(X, X\right) > 0$ and lightlike (or null) if $g\left(X, X\right) = 0$. Similarly, an arbitrary curve $\alpha = \alpha\left(s\right)$ in $\mathbb{R}_1^3$ where s is an arc-length parameter, can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors, $\alpha'\left(s\right)$ are respectively timelike, spacelike or null (lightlike) for every $s \in \mathbb{R}$. The norm of a vector $X \in \mathbb{R}_1^3$ is defined by [10]

$$\|X\| = \sqrt{\left|g\left(X, X\right)\right|}.$$

We denote by $\{T\left(s\right),\ N\left(s\right),\ B\left(s\right)\}$ the moving Frenet frame along the curve α. Let α be a timelike curve with curvature κ and torsion τ. Let frenet vector fields of α be $\{T,\ N,\ B\}$. In this trihedron, T is a timelike vector field, N and B are spacelike vector fields. Then Frenet formulas are given by ([16])

$$\begin{cases} T' = \kappa N \\ N' = \kappa T - \tau B \\ B' = \tau N. \end{cases} \tag{2.1}$$

Let α be a timelike vector, the frenet vectors T timelike, N and B are spacelike vector, respectively, such that

$$T \times N = -B,\ \mathrm{N} \times B = T,\ B \times T = -N$$

and the frenet instantaneous rotation vector is given by ([14])

$$W = \tau T - \kappa B, \quad \|W\| = \sqrt{\ |\kappa^2 - \tau^2|}.$$

Let φ be the angle between W and -B vectors and if W is a spacelike vector, then we can write

$$\begin{cases} \kappa = \|W\| \cosh\varphi, \ \tau = \|W\| \sinh\varphi, \\ \mathrm{C} = \sinh\varphi T - \cosh\varphi B \end{cases} \tag{2.2}$$

and if W is a timelike vector, then we can write

$$\begin{cases} \kappa = \|W\| \sinh\varphi, \ \tau = \|W\| \cosh\varphi, \\ C = \cosh\varphi T - \sinh\varphi B. \end{cases} \tag{2.3}$$

The frenet formulas of spacelike with timelike binormal curve, $\alpha : I \to \mathbb{R}^3_1$ are as followings:

$$\begin{cases} T' = \kappa N \\ N' = \kappa T - \tau B \\ B' = \tau N. \end{cases} \tag{2.4}$$

(see [5] for details), and the frenet instantaneous rotation vector is defined by ([10])

$$W = \tau T - \kappa B, \qquad \|W\| = \sqrt{|\tau^2 - \kappa^2|}.$$

Here, $T \times N = B$, $N \times B = -T$, $B \times T = -N$. Let φ be the angle between W and $-B$ vectors and if W is taken as spacelike, then the unit Darboux vector can be stated by

$$\begin{cases} \kappa = \|W\| \sinh\varphi, \ \tau = \|W\| \cosh\varphi, \\ C = \cosh\varphi T - \sinh\varphi B. \end{cases} \tag{2.5}$$

and if W is taken as timelike, then it is described by

$$\begin{cases} \kappa = \|W\| \cosh\varphi, \ \tau = \|W\| \sinh\varphi, \\ \mathrm{C} = \sinh\varphi T - \cosh\varphi B \end{cases} \tag{2.6}$$

Let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ be the vectors in $\mathbb{R}^3_1$. The cross product of X and Y is defined by ([1])

$$X \wedge Y = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1).$$

The Lorentzian sphere and hyperbolic sphere of radius r and center 0 in $\mathbb{R}^3_1$ are given by

$$S^2_1 = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}^3_1 \middle| \ g(X, X) = r^2, \ r \in \mathbb{R} \right\}$$

and

$$H_0^2 = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \middle|\ g(X, X) = -r^2,\ r \in \mathbb{R} \right\}$$

respectively. Let M be a hypersurface in $\mathbb{R}_1^3$. A curve $\alpha : I \to M$ is an integral curve of $X \in \chi(M)$ provided $\alpha' = X_\alpha$; that is

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \text{ for all } s \in I \qquad ([10]).$$

For any parameterized curve $\alpha : I \to M$, the parameterized curve, $\overline{\alpha} : I \to TM$ given by $\overline{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)\big|_{\alpha(s)}$ is called the natural lift of α on TM ([13]). Thus we can write

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}(\alpha'(s))\big|_{\alpha(s)} = D_{\alpha'(s)}\alpha'(s)$$

where D is standard connection on $\mathbb{R}_1^3$. For $v \in TM$ the smooth vector field $X \in \chi(M)$ defined by

$$X(v) = \varepsilon g(v, S(v))\big|_{\alpha(s)}\ ,\ \varepsilon = g(\xi, \xi)\ [\textbf{?}]$$

is called the geodesic spray on the manifold TM, where ξ is the unit normal vector field of M and S is shape operator of M.

Let $\alpha : I \to \mathbb{R}_1^3$ be a spacelike with timelike binormal curve. Let us consider the Frenet frame $\{T,\ N,\ B\}$ and the vector C. Accordingly, arc-lengths and the geodesic curvatures of the spherical indicatrix curves (T), (N) and (B) with the fixed pole curve (C) with respect to $\mathbb{R}_1^3$, respectively generated by the vectors T, N and B with the unit Darboux vector C are as follows:

$$\begin{cases} s_T = \displaystyle\int_0^s |\kappa| ds, & s_N = \displaystyle\int_0^s \|W\| ds, \\ s_B = \displaystyle\int_0^s |\tau| ds, & s_C = \displaystyle\int_0^s |\varphi'| ds. \end{cases} \tag{2.7}$$

if W is a spacelike vector, then we can write

$$\begin{cases} k_T = \dfrac{1}{\sinh\varphi}, & k_N = \sqrt{\left|1 + (\dfrac{\varphi'}{\|W\|})^2\right|}, \\ k_B = \dfrac{1}{\cosh\varphi}, & k_C = \sqrt{\left|1 + (\dfrac{\|W\|}{\varphi'})^2\right|}. \end{cases} \tag{2.8}$$

if W is a timelike vector, then we have

$$\begin{cases} k_T = \dfrac{1}{\cosh\varphi}, & k_N = \sqrt{\left|1 - (\dfrac{\varphi'}{\|W\|})^2\right|}, \\ k_B = \dfrac{1}{\sinh\varphi}, & k_C = \sqrt{\left|-1 + (\dfrac{\|W\|}{\varphi'})^2\right|} \end{cases} \tag{2.9}$$

(see [3] for details).

Definition 2.1([6]) *Let* α*be spacelike with timelike binormal curve and* α^* *be timelike curve* *in $\mathbb{R}_1^3$. $\{T,\ N,\ B\}$ and $\{T^*,\ N^*,\ B^*\}$ are Frenet frames, respectively, on these curves. $\alpha(s)$ and $\alpha^*(s)$ are called Bertrand curves if the principal normal vectors N and N^* are linearly dependent, and the pair (α,α^*) is said to be spacelike-timelike Bertrand curve pair.*

Theorem 2.1([6]) *Let (α,α^*) be spacelike-timelike Bertrand curve pair. For corresponding $\alpha(s)$ and $\alpha^*(s)$ points*

$$d\left(\alpha(s),\alpha^*(s)\right) = constant,\ \forall s\in I\ .$$

Theorem 2.2([6]) *Let $(\alpha.\alpha^*)$ be spacelike-timelike Bertrand curve pair.. The measure of the angle between the vector fields of Bertrand curve pair is constant.*

§3. The Natural Lift Curves And Geodesic Curvatures Of The Spherical Indicatrices Of The Spacelike-Timelike Bertrand Curve Pair

Theorem 3.1 *Let (α,α^*) be spacelike-timelike Bertrand curve pair. The relations between the Frenet vectors of the curve pair are as follows*

$$\begin{cases} T^* = sinh\theta T - \cosh\theta B \\ N^* = N \\ B^* = \cosh\theta T - \sinh\theta B \end{cases}$$

Here, the angle θ is the angle between T and T^*.

Proof By taking the derivative of $\alpha^*(s) = \alpha(s) + \lambda N(s)$ with respect to arc-lenght s and using the equation (2.4), we get

$$T^*\frac{ds^*}{ds} = T(1-\lambda\kappa) - \lambda\tau B. \tag{3.1}$$

The inner products of the above equation with respect to T and B are respectively defined as

$$\begin{cases} \sinh\theta\dfrac{ds^*}{ds} = 1-\lambda\kappa, \\ \cosh\theta\dfrac{ds^*}{ds} = -\lambda\tau. \end{cases} \tag{3.2}$$

and substituting these present equations in (3.1) we obtain

$$T^* = \sinh\theta T - \cosh\theta B. \tag{3.3}$$

Here, by taking derivative and using the equation (2.4) we get

$$N^* = N. \tag{3.4}$$

We can write

$$B^* = \cosh\theta T - \sinh\theta B \tag{3.5}$$

by availing the equation $B^* = -(T^* \times N^*)$. □

Corollary 3.1 *Let (α, α^*) be a spacelike-timelike Bertrand curve pair. Between the curvature κ and the torsion τ of the α, there is relationship*

$$\mu\tau + \lambda\kappa = 1 \quad \text{and} \quad \mu = -\lambda \tanh\theta, \tag{3.6}$$

where λ and μ are nonzero real numbers.

Proof From equation (3.2), we obtain

$$\frac{\sinh\theta}{1-\lambda\kappa} = \frac{\cosh\theta}{-\lambda\tau},$$

And by arranging this equation, we get

$$\tanh\theta = \frac{1-\lambda\kappa}{-\lambda\tau}$$

and if we choose $\mu = -\lambda\tanh\theta$ for brevity, then we obtain

$$\mu\tau + \lambda\kappa = 1.$$ □

Theorem 3.2 *There are connections between the curvatures κ and κ^* and the torsions τ and τ^* of the spacelike-timelike Bertrand curve pair (α, α^*), which are shown as follows*

$$\begin{cases} \kappa^* = \dfrac{\cosh^2\theta - \lambda\kappa}{\lambda(1-\lambda\kappa)}, \\ \tau^* = -\dfrac{\cosh^2\theta}{\lambda^2\tau}. \end{cases} \tag{3.7}$$

Proof If (α, α^*) be a spacelike-timelike Bertrand curve pair, we can write $\alpha(s) = \alpha^*(s) - \lambda N^*(s)$. By taking the derivative of this equation with respect to s^* and using equation (2.1) we obtain

$$T = T^* \frac{ds^*}{ds}(1-\lambda\kappa^*) + \lambda\tau^* B^* \frac{ds^*}{ds}.$$

The inner products of the above equation with respect to T^* and B^* are as following

$$\begin{cases} \sinh\theta = -(1-\lambda\kappa^*)\dfrac{ds^*}{ds}, \\ \cosh\theta = \lambda\tau^*\dfrac{ds^*}{ds}. \end{cases} \tag{3.8}$$

respectively. The proof can easily be completed by using and rearranging the equations (3.2) and (3.8). □

Corollary 3.2 *Let (α, α^*) be a spacelike-timelike Bertrand curve pair.*

$$\kappa^* = \frac{\lambda\kappa - \cosh^2\theta}{\lambda^2\tau\tanh\theta}. \tag{3.9}$$

Proof By using the equations (3.6) and with substitution of them in 3.7 we get the desired result. □

Theorem 3.3 *Let (α, α^*) be a spacelike-timelike Bertrand curve pair. There are following relations between Darboux vector W of curve α and Darboux vector W^* of curve α^**

$$W^* = -\frac{\cosh\theta}{\lambda\tau} W. \tag{3.10}$$

Proof For the Darboux vector W^* of timelike curve α^*, we can write

$$W^* = \tau^* T^* - \kappa^* B^*.$$

By substituting (3.3), (3.5), (3.7) and (3.9) into the last equation, we obtain

$$W^* = \frac{\cosh\theta}{\lambda\tau}\left[\frac{1}{\lambda}\coth\theta\,(1-\lambda\kappa)\,T + \kappa B\right].$$

By substituting (3.6) into above equation, we get

$$W^* = -\frac{\cosh\theta}{\lambda\tau} W.$$

This completes the proof. □

Now, let compute arc-lengths with the geodesic curvatures of spherical indicatrix curves with the (T^*), (N^*) and (B^*) with the fixed pole curve (C^*) with respect to $\mathbb{R}^3_1$ and H^2_0 or S^2_1.

Firstly, for the arc-length s_{T^*} of tangents indicatrix (T^*) of the curve α^*, we can write

$$s_{T^*} = \int_0^s \left\| \frac{dT^*}{ds} \right\| ds.$$

By taking the derivative of equation (3.3), we have

$$s_{T^*} \leqslant |\sinh\theta| \int_0^s |\kappa|\, ds + |\cosh\theta| \int_0^s |\tau|\, ds.$$

By using equation (2.7) we obtain

$$s_{T^*} \leqslant |\sinh\theta|\, s_T + |\cosh\theta|\, s_B.$$

For the arc-length s_{N^*} of principal normals indicatrix (N^*) of the curve α^*, we can write

$$s_{N^*} = \int_0^s \left\| \frac{dN^*}{ds} \right\| ds.$$

By substituting (3.4) into above equation, we get

$$s_{N^*} = s_N.$$

Similarly, for the arc-length s_{B^*} of binormals indicatrix (B^*) of the curve α^*, we can write

$$s_{B^*} = \int_0^s \left\| \frac{dB^*}{ds} \right\| ds.$$

By taking the derivative of equation (3.5), we have

$$s_{B^*} \leqslant |\cosh\theta| \int_0^s |\kappa|\, ds + |\sinh\theta| \int_0^s |\tau|\, ds.$$

By using equation (2.7), we obtain

$$s_{B^*} \leqslant |\cosh\theta|\, s_T + |\sinh\theta|\, s_B.$$

Finally, for the arc-length s_{C^*} of the fixed pole curve (C^*), we can write

$$s_{C^*} = \int_0^s \left\| \frac{dC^*}{ds} \right\| ds.$$

If W^* is a spacelike vector, we can write

$$C^* = \sinh\varphi^* T^* - \cosh\varphi^* B^*$$

from the equation (2.2). By taking the derivative of this equation, we obtain

$$s_{C^*} = \int_0^s \left| (\varphi^*)' \right| ds. \tag{3.11}$$

On the other hand, from equation (2.2) and by using

$$\cosh\varphi^* = \frac{\kappa^*}{\|W^*\|} \quad \text{ve} \quad \sinh\varphi^* = \frac{\tau^*}{\|W^*\|}$$

we can set

$$\tanh\varphi^* = \frac{\tau^*}{\kappa^*}.$$

By substituting (3.7) and (3.9) into the last equation and differentiating, we obtain

$$(\varphi^*)' = \frac{\lambda\kappa' \sinh\theta\cosh\theta}{\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta}. \tag{3.12}$$

By substituting (3.12) into (3.11), we have

$$s_{C^*} = \int_0^s \left| \frac{\lambda\kappa' \sinh\theta\cosh\theta}{\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta} \right| ds.$$

If W^* is a timelike vector, we have the same result. Thus the following corollary can be drawn.

Corollary 3.3 *Let (α, α^*) be a spacelike-timelike Bertrand curve pair and $\{T^*,\ N^*,\ B^*\}$ be the Frenet frame of the curve α^*. For the arc-lengths of the spherical indicatrix curves (T^*), (N^*) and (B^*) with the fixed pole curve (C^*) with respect to $\mathbb{R}^3_1$, we have*

(1) $s_{T^*}|\sinh\theta|s_T + |\cosh\theta|s_B$;

(2) $s_{N^*} = s_N$;

(3) $s_{B^*} \leqslant |\cosh\theta|\, s_T + |\sinh\theta|\, s_B$;

(4) $s_{C^*} = \displaystyle\int_0^s \left| \frac{\lambda\kappa' \sinh\theta\cosh\theta}{\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta} \right| ds.$

Now, let us compute the geodesic curvatures of the spherical indicatrix curves (T^*), (N^*) and (B^*) with the fixed pole curve (C^*) with respect to $\mathbb{R}^3_1$. For the geodesic curvature k_{T^*} of the tangents indicatrix (T^*) of the curve α^*, we can write

$$k_{T^*} = \|D_{T_{T^*}} T_{T^*}\|. \tag{3.13}$$

By differentiating the curve $\alpha_{T^*}(s_{T^*}) = T^*(s)$ with the respect to s_{T^*} and normalizing, we obtain

$$T_{T^*} = N.$$

By taking derivative of the last equation we get

$$D_{T_{T^*}} T_{T^*} = \frac{-\kappa T + \tau B}{|\kappa sinh\theta - \tau\cosh\theta|}. \tag{3.14}$$

By substituting (3.14) into (3.13) we have

$$k_{T^*} = \frac{\|W\|}{|\kappa\sinh\theta - \tau\cosh\theta|}.$$

Here, if W is a spacelike vector, by substituting 2.5 and 2.8 into the last equation we have

$$k_{T^*} = \left| \frac{k_T \cdot k_B}{k_B \cdot \sinh\theta - k_T \cdot \cosh\theta} \right|,$$

if W is a timelike vector, then by substituting 2.6 and 2.9 we have the same result.

Similarly, by differentiating the curve $\alpha_{N^*}(s_{N^*}) = N^*(s)$ with the respect to $s_{N^*}$and by normalizing we obtain

$$T_{N^*} = -\frac{\kappa}{\|W\|}T + \frac{\tau}{\|W\|}B.$$

If W is a spacelike vector, then by using equation (2.5) and (2.8) we have

$$T_{N^*} = -\sinh\varphi T + \cosh\varphi B,$$

$$D_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}(-\cosh\varphi T + \sinh\varphi B) + N, \tag{3.15}$$

$$k_{N^*} = k_N = \sqrt{\left|\left(\frac{\varphi'}{\|W\|}\right)^2 + 1\right|}.$$

If W is a timelike vector, then by using of the equations (2.5) and (2.9) we have

$$D_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}\left(\cosh\varphi T - \sinh\varphi B\right) - \mathrm{N}, \tag{3.16}$$

$$k_{N^*} = k_N = \sqrt{\left|1 - \left(\frac{\varphi'}{\|W\|}\right)^2\right|}.$$

By differentiating the curve $\alpha_{B^*}(s_{B^*}) = B^*(s)$ with the respect to s_{B^*} and by normalizing, we obtain

$$T_{B^*} = N.$$

By taking the derivative of the last equation we get

$$D_{T_{B^*}} T_{B^*} = \frac{-\kappa T + \tau B}{|\kappa\cosh\theta - \tau\sinh\theta|} \tag{3.17}$$

or by taking the norm of equation (3.17), we obtain

$$k_{B^*} = \frac{\|W\|}{|\kappa\cosh\theta - \tau\sinh\theta|}.$$

If W is a spacelike vector, then by substituting (2.5) and (2.8) we have

$$k_{B^*} = \left|\frac{k_T \cdot k_B}{k_B \cdot \cosh\theta - k_T \cdot \sinh\theta}\right|,$$

if W is a timelike vector, then by substituting (2.6) and (2.9) we have the same result. By differentiating the curve $\alpha_{C^*}(s_{C^*}) = C^*(s)$ with the respect to s_{C^*} and normalizing, if W^* is a spacelike vector, then by substituting (2.2) we obtain

$$T_{C^*} = \cosh\varphi^* T^* - \sinh\varphi^* B^*,$$

$$D_{T_{C^*}} T_{C^*} = \left(\sinh\varphi^* T^* - \cosh\varphi^* B^*\right) + \frac{\|W^*\|}{(\varphi^*)'} N^*, \tag{3.18}$$

$$k_{C^*} = \sqrt{1 + \left(\frac{\|W^*\|}{(\varphi^*)'}\right)^2}. \tag{3.19}$$

By substituting (3.10) and (3.12) into (3.19) and rearranging we have

$$k_{C^*} = \sqrt{\left|\frac{(\tau^2 - \kappa^2)\left[\lambda^2\kappa^2 + (1 - 2\lambda\kappa)\cosh^2\theta\right]^2}{(\lambda^2\tau\kappa')^2\sinh^2\theta} + 1\right|}.$$

If W^* is a timelike vector, then by substituting (2.1) and (2.3) we get

$$T_{C^*} = \sinh\varphi^* T^* - \cosh\varphi^* B^*,$$

$$D_{T_{C^*}} T_{C^*} = (\cosh\varphi^* T^* - \sinh\varphi^* B^*) - \frac{\|W^*\|}{(\varphi^*)'} N^*, \tag{3.20}$$

$$k_{C^*} = \sqrt{\left| -1 + \left(\frac{\|W^*\|}{(\varphi^*)'} \right)^2 \right|}. \tag{3.21}$$

By substituting (3.10) and (3.12) into (3.21) we have

$$k_{C^*} = \sqrt{\left| \frac{(\kappa^2 - \tau^2)\left[\lambda^2\kappa^2 + (1 - 2\lambda\kappa)\cosh^2\theta\right]^2}{(\lambda^2\tau\kappa')^2 \sinh^2\theta} - 1 \right|}.$$

Then the following corollary can be given.

Corollary 3.4 *Let (α, α^*) be a spacelike-timelike Bertrand curve cuople and $\{T^*,\ N^*,\ B^*\}$ be Frenet frame of the curve α^*. For the geodesic curvatures of the spherical indicatrix curves (T^*), (N^*) and (B^*) with the fixed pole curve (C^*) with the respect to $\mathbb{R}^3_1$ we have*

(1)

$$k_{T^*} = \left| \frac{k_T \cdot k_B}{k_B \cdot \sinh\theta - k_T \cdot \cosh\theta} \right|;$$

(2)

$$\begin{cases} k_{N^*} = k_N = \sqrt{\left| \left(\frac{\varphi'}{\|W\|}\right)^2 + 1 \right|}, & W \textit{ spacelike ise} \\ k_{N^*} = k_N = \sqrt{\left| 1 - \left(\frac{\varphi'}{\|W\|}\right)^2 \right|}, & W \textit{ timelike ise} \end{cases}$$

(3)

$$k_{B^*} = \left| \frac{k_T \cdot k_B}{k_B \cdot \cosh\theta - k_T \cdot \sinh\theta} \right|.$$

(4)

$$\begin{cases} k_{C^*} = \sqrt{\left| \frac{(\tau^2-\kappa^2)[\lambda^2\kappa^2+(1-2\lambda\kappa)\cosh^2\theta]^2}{(\lambda^2\tau\kappa')^2\sinh^2\theta} + 1 \right|}, & W^* \textit{ spacelike ise} \\ k_{C^*} = \sqrt{\left| \frac{(\kappa^2-\tau^2)[\lambda^2\kappa^2+(1-2\lambda\kappa)\cosh^2\theta]^2}{(\lambda^2\tau\kappa')^2\sinh^2\theta} - 1 \right|}, & W^* \textit{ timelike} \end{cases}.$$

Now let us compute the geodesic curvatures (T^*), (N^*) and (B^*) with the fixed pole curve (C^*) with respect to H_0^2 or S_1^2.

For the geodesic curvature γ_{T^*} of the tangents indicatrix curve (T^*) of the curve α^* with respect to H_0^2, we can write

$$\gamma_{T^*} = \left\| \overline{\overline{D}}_{T_{T^*}} T_{T^*} \right\|. \tag{3.22}$$

Here, $\overline{\overline{D}}$ become a covariant derivative operator. By (3.3) and (3.14) we obtain

$$D_{T_{T^*}} T_{T^*} = \overline{\overline{D}}_{T_{T^*}} T_{T^*} + \varepsilon g \left(S \left(T_{T^*} \right), T_{T^*} \right) T^*,$$

$$\overline{\overline{D}}_{T_{T^*}} T_{T^*} = \left(\frac{-\kappa}{|\kappa \sinh \theta - \tau \cosh \theta|} - \sinh \theta \right) \cdot T + \left(\frac{\tau}{|\kappa \sinh \theta - \tau \cosh \theta|} + \cosh \theta \right) \cdot B. \quad (3.23)$$

By substituting (3.23) into (3.22) we get

$$\gamma_{T^*} = \sqrt{\left| \frac{\tau^2 - \kappa^2}{\left(\kappa \sinh \theta - \tau \cosh \theta \right)^2} - 1 \right|}.$$

If W is a spacelike vector, then by using of the equations (2.5) and (2.8) we have

$$\gamma_{T^*} = \sqrt{\left| \left(\frac{k_T k_B}{k_B \sinh \theta - k_T \cosh \theta} \right)^2 - 1 \right|},$$

if W is a timelike vector, then by using of the equations (2.6) and (2.9) we have

$$\gamma_{T^*} = \sqrt{\left| - \left(\frac{k_T k_B}{k_B \sinh \theta - k_T \cosh \theta} \right)^2 - 1 \right|}.$$

If the curve $\left(\overline{T^*} \right)$ is an integral curve of the geodesic spray, then $\overline{\overline{D}}_{T_{T^*}} T_{T^*} = 0$. Thus, by (3.23) we can write

$$\begin{cases} \frac{-\kappa}{|\kappa \sinh \theta - \tau \cosh \theta|} - \sinh \theta = 0 \\ \\ \frac{\tau}{|\kappa \sinh \theta - \tau \cosh \theta|} + \cosh \theta = 0 \end{cases}$$

and here, we obtain $\kappa = 0$, $\tau \neq 0$ and $\theta = 0$. So, we can give following corollary.

Corollary 3.5 *Let (α, α^*) be a spacelike-timelike Bertrand curve pair. The natural lift $\left(\overline{T^*} \right)$ of the tangent indicatrix (T^*) is never an integral curve of the geodesic spray.*

For the geodesic curvature γ_{N^*} of the principal normals indicatrix curve (N^*) of the curve α^* with respect to S_1^2 we can write

$$\gamma_{N^*} = \left\| \overline{D}_{T_{N^*}} T_{N^*} \right\|. \quad (3.24)$$

Here, $\overline{D}$ become a covariant derivative operator. If W is a spacelike vector, by using of the equation (3.15) we obtain

$$\overline{D}_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|} \left(-\cosh \varphi T + \sinh \varphi B \right). \quad (3.25)$$

By substituting (3.25) into (3.24) we get

$$\gamma_{N^*} = \frac{\varphi'}{\|W\|}. \tag{3.26}$$

On the other hand, from the equation (2.5) by using

$$\sinh\varphi = \frac{\kappa}{\|W\|} \text{ ve } \cosh\varphi = \frac{\tau}{\|W\|}$$

we can set

$$\tanh\varphi = \frac{\kappa}{\tau}.$$

By taking the derivative of the last equation we get

$$\varphi' = \frac{\kappa'\tau - \tau'\kappa}{\|W\|^2}.$$

By substituting above the equation into (3.26) we have

$$\gamma_{N^*} = \gamma_N = \frac{\kappa'\tau - \tau'\kappa}{\|W\|^3}.$$

If W is a timelike vector, by using of the equation (3.16) we obtain

$$\overline{D}_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}\left(-\sinh\varphi T + \cosh\varphi B\right)\gamma_{N^*} = \frac{\varphi'}{\|W\|}. \tag{3.27}$$

On the other hand, from equation (2.6) by using

$$\cosh\varphi = \frac{\kappa}{\|W\|} \text{ and } \sinh\varphi = \frac{\tau}{\|W\|}$$

we can set

$$\tanh\varphi = \frac{\kappa}{\tau}.$$

By taking the derivative of the last equation we get

$$\varphi' = \frac{\tau'\kappa - \kappa'\tau}{\|W\|^2}$$

or

$$\gamma_{N^*} = \gamma_N = \frac{\tau'\kappa - \kappa'\tau}{\|W\|^3}.$$

If the curve $\left(\overline{N^*}\right)$ is an integral curve of the geodesic spray, then $\overline{D}_{T_{N^*}} T_{N^*} = 0$. Thus, by (3.25) and (3.27) we can write $\varphi' = 0$ and here, we obtain $\frac{\kappa}{\tau}$ =constant. So, we can give following corollary.

Corollary 3.6 *Let (α, α^*) be a spacelike-timelike Bertrand curve pair. If the curve α is a helix curve, the natural lift $\left(\overline{N^*}\right)$ of the pirincipal normals indicatrix (N^*) is an integral curve of the* *geodesic spray.*

For the geodesic curvature γ_{B^*} of the binormals indicatrix curve (B^*) of the curve $\alpha^*$with respect to S_1^2 and substituting (3.5) and (3.17) we obtain

$$D_{T_{B^*}}T_{B^*} = \overline{D}_{T_{B^*}}T_{B^*} + \varepsilon g\left(S\left(T_{B^*}\right), T_{B^*}\right)B^*,$$

$$\overline{D}_{T_{B^*}}T_{B^*} = \left(\frac{-\kappa}{\left|\kappa\cosh\theta - \tau\sinh\theta\right|} + \cosh\theta\right)T + \left(\frac{\tau}{\left|\kappa\cosh\theta - \tau\sinh\theta\right|} - \sinh\theta\right)B, \quad (3.28)$$

$$\gamma_{B^*} = \sqrt{\left|-1 + \frac{\tau^2 - \kappa^2}{\left(-\kappa\sinh\theta + \tau\cosh\theta\right)^2}\right|}.$$

If W is a spacelike vector, then by using of the equations (2.5) and (2.8) we have

$$\gamma_{B^*} = \sqrt{\left|-1 - \left(\frac{k_T k_B}{k_B.\cosh\theta - k_T.\sinh\theta}\right)^2\right|},$$

if W is a timelike vector, then by using of the equations (2.6) and (2.9) we get

$$\gamma_{B^*} = \sqrt{\left|-1 + \left(\frac{k_T k_B}{k_B.\cosh\theta - k_T.\sinh\theta}\right)^2\right|}.$$

If the curve $\left(\overline{B^*}\right)$ is an integral curve of the geodesic spray, then $\overline{D}_{T_{B^*}}T_{B^*} = 0$. Thus, by (3.28) we can write

$$\begin{cases} \dfrac{-\kappa}{\left|\kappa\cosh\theta - \tau\sinh\theta\right|} + \cosh\theta = 0 \\ \dfrac{\tau}{\left|\kappa\cosh\theta - \tau\sinh\theta\right|} - \sinh\theta = 0 \end{cases}$$

and here, we obtain $\kappa > 0$, $\tau = 0$ and $\theta = 0$. So, we can give following corollary.

Corollary 3.7 *Let (α, α^*) be a spacelike-timelike Bertrand curve pair. If the curve $\alpha$is a planary curve and frames are equivalent, the natural lift $\left(\overline{B^*}\right)$ of the binormals indicatrix (B^*) is an integral curve of the geodesic spray.*

If W^* is a spacelike vector, for the geodesic curvature γ_{C^*} of the fixed pole curve (C^*) of the curve α^* with respect to S_1^2 and by using of the equations (2.2) and (3.18) we obtain

$$D_{T_{C^*}}T_{C^*} = \overline{D}_{T_{C^*}}T_{C^*} + \varepsilon g\left(S\left(T_{C^*}\right), T_{C^*}\right)C^*,$$

$$\overline{D}_{T_{C^*}}T_{C^*} = \frac{\|W^*\|}{\left(\varphi^*\right)'}N^*, \quad (3.29)$$

$$\gamma_{C^*} = \left\|\frac{\|W^*\|}{\left(\varphi^*\right)'}\right\|.$$

By substituting (3.10) and (3.12) into the last equation we have

$$\gamma_{c^*} = \frac{\|W\| \left[\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta\right]}{\lambda^2\tau\kappa' \sinh\theta}.$$

If W^* is a timelike vector, for the geodesic curvature γ_{C^*} of the fixed pole curve (C^*) with respect to H_0^2 and by using of the equations (2.3) and (3.20) we have the same result. If the curve $\left(\overline{C^*}\right)$ is an integral curve of the geodesic spray, then $\overline{\overline{D}}_{T_{C^*}} T_{C^*} = 0$. Thus by (3.29) we can write $\|W^*\| = 0$ and here, we get $\kappa^* = \tau^* = 0$ or $\kappa^* = \tau^*$. Thus, by using of the equation (3.7) and (3.9) we obtain

$$\kappa = \frac{\cosh^2\theta - \sinh\theta\cosh\theta}{\lambda}.$$

So, we can give following corollary.

Corollary 3.8 *Let (α, α^*) be a spacelike-timelike Bertrand curve pair. If the curve α is a curve that provides the requirement $\kappa = \dfrac{\cosh^2\theta - \sinh\theta\cosh\theta}{\lambda}$, the natural lift $\left(\overline{C^*}\right)$ of the fixed pole curve (C^*) is an integral curve of the geodesic spray.*

Corollary 3.9 *Let (α, α^*) be a spacelike-timelike Bertrand curve pair and $\{T^*,\ N^*,\ B^*\}$ be Frenet frame of the curve α^*. For the geodesic curvatures of the spherical indicatrix curves (T^*), (N^*) and (B^*) with the fixed pole curve (C^*) with respect to H_0^2 or S_1^2, we have*

(1)

$$\begin{cases} \gamma_{T^*} = \sqrt{\left|\left(\dfrac{k_T k_B}{k_B\sinh\theta - k_T\cosh\theta}\right)^2 - 1\right|}, & W \text{ spacelike} \\ \gamma_{T^*} = \sqrt{\left|-1 - \left(\dfrac{k_T k_B}{k_B\sinh\theta - k_T\cosh\theta}\right)^2\right|}, & W \text{ timelike} \end{cases}$$

(2)

$$\begin{cases} \gamma_{N^*} = \gamma_N = \dfrac{\kappa'\tau - \tau'\kappa}{\|W\|^3}, & W \text{ spacelike} \\ \gamma_{N^*} = \gamma_N = \dfrac{\tau'\kappa - \kappa'\tau}{\|W\|^3}, & W \text{ timelike} \end{cases}$$

(3)

$$\begin{cases} \gamma_{B^*} = \sqrt{\left|-1 - \left(\dfrac{k_T k_B}{k_B\cosh\theta - k_T\sinh\theta}\right)^2\right|}, & W \text{ spacelike} \\ \gamma_{B^*} = \sqrt{\left|-1 + \left(\dfrac{k_T k_B}{k_B\cosh\theta - k_T\sinh\theta}\right)^2\right|}, & W \text{ timelike} \end{cases}$$

(4)

$$\gamma_{c^*} = \frac{\|W\| \left[\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta\right]}{\lambda^2\tau\kappa' \sinh\theta}.$$

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Antisotropic Cosmological Models of Finsler Space With (γ, β)-Metric

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Abstract: In this paper we have studied an anisotropic model of space time with Finslerian spaces with (γ, β)-metrics as suggested by one of the co-author in his paper [21] with an extra requirement of $\gamma^3 = a_{ijk}(x)y^iy^jy^k$. Here γ, is a cubic metric and $\beta = b_i(x)y^i$, is a one form metric. The observed anisotropy of the microwave background radiation is incorporated in the Finslerian metric of space time.

Key Words: Cosmology, Finsler space with (γ, β)-metric, cubic metric and one form metric.

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§1. Introduction

The concept of cubic metric on a differentiable manifold with the local co-ordinate x^i, defined by

$$L(x, y) = \{a_{ijk}(x)y^iy^jy^k\}^{\frac{1}{3}}$$

was introduced by M. Matsumoto in the year 1979 ([1]), where,$a_{ijk}(x)$ are components of a symmetric tensor field of (0, 3)-type, depending on the position x alone, and a Finsler space with a cubic metric (called the cubic Finsler space). During investigation of some interesting results we have gone through papers/research outcomes regarding cubic Finsler spaces as referred in the papers [3, 4, 5, 6, 7, 8]. It has been observed that there are various interesting results on geometry of spaces with a cubic metric as a generalization of Euclidean or Riemannian geometry have been published in recent years. It is further noticed that one of the paper published by one of the coauthor of this paper [21] in the year 2011, define the concept of (γ, β)-metric considering γ is a cubic metric and β is a one-form and discussed various important results in stand-point of the Finsler Geometry in this paper. Here we wish to mention that in paper [2] concerned with the unified field theory of gravitation and electromagnetism Randers wrote that

[1]Received February 8, 2013, Accepted May 28, 2014.

Perhaps the most characteristic property of the physical world is the uni direction of time-like intervals. Since there is no obvious reason why this asymmetry should disappear in the mathematical description it is of interest to consider the possibility of a metric with asymmetrical property.

It is also noticed that many researchers are interested to investigate something new result in physics and those possible application in modern cosmology and other reference of the same kind [9, 10, 11].We are fully agreed with the theory expressed in the above referred publication regarding, as it is based on tangent bundle on space time manifold are positively with local Lorentz violations which may be related with dark energy and dark matter models with variable in cosmology. Certainly this may also be one of the most recent hidden connections between Finsler geometry and cosmology. Recently many researchers are constructing suitable cosmological models with variable Lambda term including our own research group [12, 13, 14, 15, 16]. So in extension of our research work, we have decided to study Cosmological model of General Theory of Relativity based on the frame work of Finsler geometry in this communication.

§2. Results

In the present paper we try to generalize the above changes by defining a Lagrangian which expresses this anisotropy as such

$$L = L(\gamma, \beta), \tag{2.1}$$

where $\gamma = \{a_{ijk}(x)y^i y^j y^k\}^{\frac{1}{3}}$ is a cubic metric and $\beta = \phi(x)\hat{b}_i y^i$is a one-form and for this metric. The purpose of the present study is to obtain the relationship between the anisotropic cosmological models of space time with above generalized Finslerian metric motivated by the work of Stavrions and Diakogiannis [16].

Let us consider an n- dimensional Finsler space (M^n, L) and an adaptable 1-form on M^n we shall use a Lagrangian function on M^n , given by the equation:

$$L = L\{(a_{ijk}(x)y^i y^j y^k)^{\frac{1}{3}}, \varphi(x)\hat{b}_i y^i\}, \tag{2.2}$$

where $b_i(x) = \phi(x)\hat{b}_i$, the vector $\hat{b}_i$ represents the observed an isotropic of the microwave background radiation. A coordinate transformation on the total space TM may be expressed as

$$\bar{x^i} = \bar{x^i}(x^0, x^1, x^2, x^3), \tag{2.3}$$

A fundamental function or a Finsler metric is a scalar field L(x, y) which satisfies the following three conditions:

(1) It is defined and differentiable for any point of $TM^n - (0)$;

(2) It is positively homogeneous of first degree in y^i, that is, $L(x, py) = pL(x, y)$ for any positive number p;

(3) It is regular, that is, $g_{ij}(x)\dfrac{\partial^2}{\partial y^i \partial y^j}\left(\dfrac{L^2}{2}\right)$ constitute the regular matrix (g_{ij}).The inverse matrix of g^{ij} is indicated by (g_{ij}). The homogeneity condition (2) enables us to consider the

integral $s = \int_b^a L(\frac{dx}{dt})dt$ along an arc , independently of the choice of parameter t except the orientation. The manifold M^n equipped with a fundamental function $L(x, y)$ is called a Finsler space $F^n = (M^n, L)$ and the s is called the length of the arc. Thus the following two conditions are desirable for $L(x, y)$ from the geometrical point of view.

(4) It is positive-valued for any point $TM^n - (0)$;

(5) $g_{ij}(x, y)$ define a positive-definite quadratic form.

Here we have to remark that there are some cases where the conditions (1), (4) and (5) should be restricted to some domain of $TM^n - (0)$.The value $L(x, y)$ is called the length of the tangent vector y at a point x. We get $L^2 = g_{ij}(x, y)y^i y^j$. The set $\left(\frac{y}{L(x,y)} = 1\right)$ in the tangent space at x or geometric-cally the set of all the end points of such y is called the indicatrix at x. If we have an equation f(x, y) =0 of the indicatrix at x,then the fundamental function L is defined by $\frac{(f(x,y))}{L} = 0$. The tensor g_{ij} is called the fundamental tensor. From L we get two other important tensors

$$l_i = \dot{\partial}_i L h_{ij} = L\dot{\dot{\partial}}_i \dot{\dot{\partial}}_j.$$

The former is called the normalized supporting element, because $l^i = g^{ir} l_r$ is written as $\frac{y^i}{L(x,y)}$ and satisfies $L(x, y) = 1$. The latter is called angular metric tensor. It satisfies $h_{ij} y^j = 0$ and the rank of h_{ij} is equal to $(n - 1)$.

The Cartan torsion coefficients C_{ijk} are given by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}. \tag{2.4}$$

The torsions and curvatures which we use are given by [17, 18, 19, 20]

$$P_{ijk} = C_{ijk|l} y^l, \tag{2.5}$$

$$S_{jikh} = C_{iks} C^s_{jh} - C_{ihs} C^s_{jk}, \tag{2.6}$$

$$P_{ihkj} = C_{ijk|h} - C_{hjk|i} + C^r_{hj} C_{rik|l} y^l - C^r_{ij} C_{rkh|l} y^l, \tag{2.7}$$

$$S^l_{ikh} = g^{lj} S_{jikh}, \tag{2.8}$$

$$P^l_{ikh} = g^{lj} P_{jikh}, \tag{2.9}$$

Differentiating equation (2.1) with respect to y^i, the normalized supporting elements $l_i = \dot{\partial}_i L$is given by

$$l_i = \dot{\partial}_i L = \frac{\partial L}{\partial y^i}, \tag{2.10}$$

$$l_i = \dot{\partial}_i L = \frac{\partial L}{\partial y}\left(\frac{a_{ijk} y^j y^k}{\gamma^2}\right) + \frac{\partial L}{\partial y}\phi(x)\hat{b_i},$$

$$l_i = \dot{\partial}_i L = L_\gamma \left(\frac{a_i(x,y)}{\gamma^2}\right) + L_\beta \phi(x)\hat{b^i}, \tag{2.11}$$

where

$$(a_{ijk}y^j y^k) = a_i(x,y). \tag{2.12}$$

Again differentiating equation (2.11) with respect to y^j, we have

$$\dot\partial_i\dot\partial_j L = \dot\partial_j\dot\partial_i L = \dot\partial_j\left\{\frac{L_\gamma}{\gamma^2}a_i(x,y)\right\} + \dot\partial_j\{L_\beta\phi(x)\hat\beta_i\},$$

$$\dot\partial_i\dot\partial_j L = \left[\frac{L_\gamma}{\gamma^2}a_{ij} + L_{\beta\beta}\phi^2\hat{b_i}\hat{b_j} + \frac{L_{\gamma\beta}}{\gamma^2}\phi(a_i\hat{b_j}a_j\hat{b_i} + a_ia_j)\right], \tag{2.13}$$

$$\dot\partial_j\phi(x)\hat{b}_i = 0,$$

where

$$2a_{ijk}y^k = a_{ij}(x,y). \tag{2.14}$$

The angular metric tensor $h_{ij} = L\dot\partial_i\dot\partial_j L$ as

$$h_{ij} = L\dot\partial_i\dot\partial_j L = L\left[\frac{L_\gamma}{\gamma^2}a_{ij} + L_{\beta\beta}\phi^2\hat{b_i}\hat{b_j} + \frac{L_{\gamma\beta}}{\gamma^2}\phi(a_i\hat{b_j} + a_j\hat{b_i}) + \frac{(L_{\gamma\gamma} - \frac{2L_\gamma}{\gamma})}{\gamma^4}a_ia_j\right], \tag{2.15}$$

$$h_{ij} = L\dot\partial_i\dot\partial_i L = \left[u_{-1}a_{ij} + u_0\phi^2\hat{b_i}\hat{b_j} + u_{-2}\phi(a_i\hat{b_j} + a_j\hat{b_i}) + u_{-4}a_ia_j\right], \tag{2.16}$$

where

$$u_{-1} = \frac{LL_\gamma}{L^2}, u_0 = LL_{\beta\beta}, u_{-2} = \frac{LL_{\gamma\beta}}{L^2}, u_{-4} = \frac{L(L_{\gamma\gamma} - \frac{2L_\gamma}{\gamma})}{\gamma}. \tag{2.17}$$

§3. Anisotropic Cosmological Model with Finsler Space of $(\gamma\beta)$ Metric

The Lagrangian function on M_n, given by the equation $L = L(\gamma, \phi(x)\hat{b_i}y^i)$, where $\gamma = (a_{ijk}y^iy^jy^k)^{\frac{1}{3}}$. For the anisotropy, we must insert an additional term to the cubic metric line element. This additional term fulfills the following requirements:

(1) It must give absolute maximum contribution for the direction of movement parallel to the anisotropy axis;

(2) The new line element must coincide with the cubic metric one for the direction vertical to the anisotropy axis;

(3) It must not symmetric with respect to replacement $y^i = -y^i$;

(4) We see that a term which satisfies the above conditions is $\beta = \phi(x)b^i$, where $b_i(x)$ reveals this anisotropic axis.

Now let $b_i(x) = \phi(x)\hat{b^i}$ where $\hat{b^i}$ the unit vector in the direction is $b_i(x)$. Then $\phi(x)$ plays the role of length of the vector $b_i(x)$, $\phi(x) \in R$. β is the Finslerian line element and γ is cubic one.

We have

$$\gamma = cd\tau = \mu d(ct) = \mu dx^0, \tag{3.1}$$

where $\mu\sqrt{1-\frac{v^2}{c^2}}$ and v is the 3- velocity in cubic space-time. One possible explanation of the anisotropy axis could be that it represents the resultant of spin densities of the angular momenta of galaxies in a restricted area of space ($b_i(x)$ space like).

The Finsler metric tensor g_{ij} is

$$g_{ij} = \dot{\partial}_i \dot{\partial}_j \frac{L^2}{2} = h_{ij} + l_i l_j, \tag{3.2}$$

Thus,

$$\begin{aligned} g_{ij} &= h_{ij} + l_i l_j \\ &= L\left[\frac{L_\gamma}{\gamma^2} a_{ij} + L_{\beta\beta}\phi^2 \hat{b_i}\hat{b_j} + \frac{L_{\gamma\beta}}{\gamma^2}\phi(a_i\hat{b_j} + a_j\hat{b_i}) + \frac{(L_{\gamma\gamma} - \frac{2L_\gamma}{\gamma})}{\gamma^4} a_i a_j\right] \\ &\quad + \left(\frac{L_\gamma}{\gamma^2} a_i + L_\beta \phi b^i\right)\left(\frac{L_\gamma}{\gamma^2}) a_j + L_\beta \phi \hat{b_j}\right) \end{aligned}$$

$$g_{ij} = h_{ij} + l_i l_j = [u_{-1} a_{ij} + m_0 \phi^2 \hat{b_i}\hat{b_j} + m_{-2}\phi(a_i\hat{b_j} + a_j\hat{b_i}) + m_{-4} a_i a_j, \tag{3.3}$$

where

$$m_0 = LL_{\beta\beta} + (L_\beta^2), m_(-2) = \left(\frac{LL_{\gamma\beta}}{\gamma^2}\right) + \frac{L_\gamma L_\beta}{\gamma^2}, m_{-4} = \frac{L}{\gamma^4}\left(L_{\gamma\gamma} - \frac{(2L_\gamma)}{\gamma}\right) + \frac{L_\gamma^2}{\gamma^4}, \tag{3.4}$$

where, we put $y^i = a_{ij} y^j$ and a_{ij} is the fundamental tensor for the Finsler space F^n. It will be easy to see that the determinant $\parallel g_{ij} \parallel$ does not vanish, and the reciprocal tensor with components g^{ij} is given by

$$g^{ij} = \left[\frac{1}{(u_{-1}} a^i j - z_2 \phi^2 \hat{B}_i \hat{B}_j - z_0 \phi(a^i \hat{b^j} + a^j \hat{b^i}) - z_{-2} a^i a^j\right], \tag{3.5}$$

where

$$\begin{aligned} z_2 &= \frac{\phi_0 u_{-1}^2 \phi^2 (\eta_{-2} + \phi^2 m_{-4} \bar{a}^2 - 2m_{-2}\bar{a})}{\eta_{-2} u_{-1}(u_{-1} + c^2)} \\ z_0 &= \frac{m_{-2} u_{-1} - \phi^2 m_{-4}\bar{a}}{\eta_{-2} u_{-1}}, \\ z_{-2} &= \frac{m_{-4} u_{-1} - c^2 m_{-4}}{\eta_{-2} u_{-1}}. \end{aligned} \tag{3.6}$$

As

$$\begin{aligned} &c^2 = \phi^2 b^2, \bar{a} = a_i B^j = a^{im} a_i b_m = a^i b_j, \\ &b^2 = B^i b_i = a^{im} b_m b_i, \\ &\phi \hat{B^i} = a^{ij} \phi \hat{b_j}, \phi \hat{a^i} = a^{\hat{i}j} \phi \hat{a^j}, \end{aligned} \tag{3.7}$$

where g^{ij} is the reciprocal tensor of g_{ij} and a^{ij} is the inverse matrix of a_{ij} as it may be verified by direct calculation, where $b^2 = \hat{b_i}\hat{b^i} = 0, \pm 1$ according whether $\hat{b^i}$ is null, space like or time like. It is interesting to observe that, that if y^i represents the velocity of a particle (time like) then $\hat{b^i}$ is bound to be space like. This follows from the fact that one possible value of $\hat{b^i}y^i$ is zero. Therefore we have decided to calculate Cartan covariant tensor C.

§4. The Cartan Covariant Tensor C

The Cartan covariant tensor C with the components C_{ijk} is obtained by Differentiating equation (3.3) with respect to y^k. We get

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij},$$

$$C_{ijk} = \frac{1}{2}\left[2u_{-1}a_{ijk} + m_{o\beta}\phi^3 b_i b_j b_k + \prod_{(ijk)}\left(K_i a_{jk} + m_{-2}\phi^2 a_i \hat{b_j}\hat{b_k} + \frac{m_{-2\gamma}}{\gamma^2}\phi a_i a_j \hat{b_k}\right) + \frac{m_{-4\gamma}}{\gamma^2}a_i a_j a_k\right], \tag{4.1}$$

where $\prod_{(ijk)}$ represent the sum of cyclic permutation of i, j, k.

$$K_i = m_{-4}a_i + m_{-2}\phi\hat{b_i}. \tag{4.2}$$

If $\phi = 0$, i.e., absence of anisotrophy, then $K_i = m_{-4}a_i$.

From equations (4.1)and (4.2), we have

$$\begin{aligned} C_{ijk} \quad &= \quad \frac{1}{2u_{-1}}[2u_{-1}^2 a_{ijk} + d_{-2}\phi^3 b_i b_j b_k + \prod_{(ijk)}(K_i h_{jk} + d_{-4}\phi^2 a_i b_j b_k \\ &\quad + d_{-6}\phi a_i a_j a_k) + d_{-8}a_i a_j a_k], \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} &d_{-2} = u_{-1}m_{0\beta} - 3m_{-2}u_0,\ d_{-4} = u_{-1}m_{-2\beta} - u_0 m_{-4} - 2m_{-2}u_{-2} \\ &d_{-6} = u_{-1}m_{-4\beta} - 2m_{-4}u_{-2} - m_{-2}u_{-4},\ d_{-8} = u_{-1}\frac{m_{-4\gamma}}{\gamma^2} - 3m_{-4}u_{-4}. \end{aligned} \tag{4.4}$$

After simplification, we have

$$C_{ijk} = \frac{1}{2u_(-1)}\left[2^2_{u_{-1}}a_{ijk} + \prod_{(ijk)}(H_{jk}K_i)\right], \tag{4.5}$$

where

$$H_{ij} = h_{ij} + \frac{d_{-2}}{3(m_{-2})^3}K_i K_j. \tag{4.6}$$

In equation (4.6), we replace the covariant indices j by k and k by s, we have

$$C_{iks} = \frac{1}{2u_{-1}^2}\left[a_{iks} + \prod_{(iks)}(H_{ks}K_i)\right]. \tag{4.7}$$

Now $C_{ijk}g^{jh} = C^h_{ik}$ Multiplying equation (3.5) and equation (4.1) and after simplification, we have

$$\begin{aligned} C^h_{ik} &= \frac{1}{2u^2_{-1}}[2u_(- 1^2 a^h_{ik} + (\delta^h_i K_k - l^h l_i K_k) + (\delta^h_k K_i - l^h l_k K_i) + \\ &\quad \frac{d_{(-2)}}{(m_{-2})^3} K^h K_i K_k + h_{ik} K^h], \end{aligned} \tag{4.8}$$

where $K^i = K_h g^{hk}$, $l^h = g^{hi} l_i$, $a^h_{ik} = a_{jik} g^{jh}$.

Now in equation (4.8) interchange the covariant indices h by s, i by j and k by h, we have

$$C^s_{jh} = \frac{1}{2u_{-1}}\left[2u^2_{-1}a^s_{jh} + (\delta^s_j K_h - l^s l_j K_h) + (\delta^s_h K_j - l^s l_h K_j) + \frac{d_{-2}}{(m_(- 2)^3} K^s K_j K_h + h_{jh} K^s\right]. \tag{4.9}$$

Therefore, $S_{jikh} = C_{iks}C^s_{jh} - C_{ihs}C^s_{jk}$ yields

$$\begin{aligned} S_{jikh} &= \frac{1}{4(u_{-1})^2}\theta_{(kh)}[(4(u_{-1})^4 a^s_{jh} a_{sik} + 2(u_{-1})^2 (a^s_{ik} K_s H_{jh} \\ &\quad + a^s_{jh} K_s H_{ik}) - (l_j K_h + l_h K_j) A_{ik} - (l_i K_k + l_k K_i) A_{jh} \\ &\quad + H^{'}_{ik} K_j K_h + H^{'}_{jh} K_i K_k)], \end{aligned} \tag{4.10}$$

where $A_{ik} = 2(u_{-1})^2 a_{ik} - \frac{\bar{K}}{L^2} h_{ik}$, $H^{'}_{ik} = 2(u_{-1})^2 \frac{2d_{-2}}{3(m^3_{-2}} a_{sik} K_s + (1 + \frac{K^2}{L^4}) h_{ik}$ and

$$\frac{K^2}{L^4} = K^s K_s, \quad \frac{\bar{K}}{L^2} = K^s l_s, \quad K^s g_{is} = K_i, \quad a_{isk} l^s = \frac{a_{ik}}{L}.$$

Thus S-curvature as defined in the equation (2.6) above represents the anisotropy of matter.

If $b_{i|h} = 0$ then for $L(\gamma, \beta)$- metric, we have $a_{i|j} = 0$, $a_{ij|k} = 0$. Because of $l_{i|j} = 0 h_{ij|k} = 0$, differentiating covariant derivative of equation (4.6) with respect to h, we get $C_{ijk|h} = u_{-1} a_{ijk|h}$. Therefore the v torsion tensor P_{ijk} is written as

$$P_{ijk} = C_{ij}k|\ hy^h = C_{ijk|0} = u_{-1} a_{ijk|0}. \tag{4.11}$$

Therefore $S_{jikh} = C_{iks}C^s_{jh} - C_{ihs}C^s_{jk}$ yields

$$\begin{aligned} S_{jikh} &= \frac{1}{4(u_{-1})^2}\theta_{(kh)}[4(u_{-1})^4 a^s_{jh} a_{sik} + 2(u_{-1})^2 (a^s_{ik} K_s H_{jh} \\ &\quad + a^s_{jh} K_s H_{ik} - (l_j K_h + l_h K_j) A_{ik} - (l_i K_k + l_k K_i) A_{jh} \\ &\quad + H^{'}_{ik} K_j K_h + H^{'}_{jh} K_i K_k)], \end{aligned} \tag{4.12}$$

where $A_{ik} = 2(u_{-1})^2 a_{ik} - \frac{\bar{K}}{L^2} h_{ik}$ and

$$H^{'}_{ik} = 2(u_{-1})^2 \frac{2d_{-2}}{3m^3_{-2}} a_{sik} K_s + \left(1 + \frac{K^2}{L^4}\right) h_{ik},$$

$$\frac{K^2}{L^4} = K^s K_s, \quad \frac{\bar{K}}{L^2} = K^s l_s, \quad K^s g_{is} = K_i, \quad a_{isk} l^s = \frac{a_{ik}}{L}.$$

Thus S-curvature as defined in the equation (2.6) above represents the anisotropy of matter. If $b_{i|h} = 0$ then for $L(\gamma, \beta)$- metric we have

$$a_{i|j} = 0, a_{ij|k} = 0. \tag{4.13}$$

Because of $l_{i|j} = 0 h_{ij|k} = 0$, differentiating covariant derivative of equation (4.6) with respect to h we get

$$C_{ijk|h} = u_{-1} a_{ijk|h}. \tag{4.14}$$

Therefore, the v torsion tensor P_{ijk} is written as

$$P_{ijk} = C_{ijk|h} y^h = C_{ijk|0} = u_{-1} a_{ijk|0}. \tag{4.15}$$

Now the v curvature tensor P_{hijk} ([19, 20]) is written as

$$P_{hijk} = \theta_{(hi)}(C_{ijk|h} + C^r_{hj} C_{rik|0}),$$

$$\begin{aligned} C^r_{hj} C_{rik|0} &= (u_{-1})^2 a^r_{hj} a_{rik|0} + \frac{1}{2} a_{hik|0} K_j - \frac{1}{2} L a_{ik|0}(l_h K_j \\ &\quad + l_j K_h) + \frac{1}{2} a_{jik|0} K_h + \frac{d_{-2}}{2(m_{-2})^3} a_{rik|0} K^r K_j K_h + \frac{1}{2} h_{jh} K^r a_{rik}. \end{aligned} \tag{4.16}$$

Thus

$$\begin{aligned} P_{hijk} &= \theta_{(hi)}[a_{ijk|h} + \frac{1}{2} a_{ijk|0} K_h \\ &\quad - \frac{1}{2L} a_{ik|0}(l_h K_j + l_j K_h) + a_{rik|0} K^r H_{jh} + a_{rik|0} A^r_{hj}], \end{aligned} \tag{4.17}$$

where $A^r_{hj} = (u_{-1})^2 a^r_{hj} + \frac{d_{-2}}{b(m_{-2})^3} a_{rik|0} K^r K_j K_h$

$$\begin{aligned} S_{jikh} &= \frac{1}{4(u_{-1})^2} \theta_{(kh)}[4(u_{-1})^4 a^s_{jh} a_{sik} + 2(u_{-1})^2 (a_{sik} K_s H_{jh} \\ &\quad + a^s_{jh} K_s H_{ik} - (l_j K_h + l_h K_j) A_{ik} - (l_i K_k + l_k K_i) A_{jh} \\ &\quad + H'_{ik} K_j K_h + H'_{jh} K_i K_k)]. \end{aligned} \tag{4.18}$$

From equation (3.5) and equation (4.12), we have

$$\begin{aligned} S^h_{ijk} &= S_{sijk} g^{sh} = \frac{1}{4u^3_{-1}} \theta_{(jk)}\{4u^4_{-1} a^r_{hk} a_{rij} + 2u^2_{-1}(H^h_k a_{rij} K^r + a^{rh}_k K_r H_{ij}) \\ &\quad - (l^h K_k + l_k K^h) A_{ij} - (l_i K_j + l_j K_i) A^h_k + H'_{ij} K_k K^h + H'^h_k K_i K_j\} \\ &\quad - [\frac{1}{4u^2_{-1}\{z_2 \phi^2} \hat{B}^s \hat{B}^h + z_0 \phi(\hat{B}^s a^h + \hat{B}^h a^s) \\ &\quad + z_2 a^s a_h\} \theta_{(jk)}[4u^4_{-1} a^r_{sk} a_{rij} + 2u^2_{-1}(a_{rij} K_r H_{sk} + a^r_{sk} K_r H_{ij}) \\ &\quad - (l_s K_k + l_k K_s) A_{ij} - (l_i K_j + l_j K_i) A_{sk} \\ &\quad + H'_{sk} K_s K_k + H'_{hk} K_i K_j]\}], \end{aligned} \tag{4.19}$$

$$
\begin{aligned}
S^h_{ijk} &= \frac{1}{(4(u_{-1})^3)}\theta_{(jk)}4(u_{-1})^4 a^r_{hk}a_{rij} + 2(u_{-1})^2(H^h_k a_{rij}K^r \\
&\quad + a^{rh}_k K_r H_{ij}) - (l^h K_k + l_k K^h)A_{ij} - (l_i K_j + l_j K_i A^h_k + H^{'}_{ij}K_k K^h \\
&\quad + H^{'h}_k K_i K_j - \frac{1}{(4(u_(- 1)^2)}M^h_{ijk}, \qquad (4.20)
\end{aligned}
$$

where

$$
\begin{aligned}
M^h_{ijk} &= [\theta_{(jk)}\{z_2\phi^2\hat{B}^s\hat{B}^h + z_0\phi(\hat{B}^s a^h + \hat{B}^h a^s) \\
&\quad + z_2 a^s a_h\}\{4(u_{-1})^4 a^r_{sk}a_{rij} + 2(u_{-1})^2(a_{rij}K_r H_{sk} \\
&\quad + a^r_{sk}K_r H_{ij}) - (l_s K_k + l_k K_s)A_{ij} - (l_i K_j + l_j K_i)A_{sk} \\
&\quad + H^{'}_{sk}K_s K_k + H^{'}_{hk}K_i K_j\}], \qquad (4.21)
\end{aligned}
$$

$$
K_s a^{sh} = K^h, l_s a^{sh} = l^h, H_{sk}a^{sh} = H^h_k, A_{sk}a^{hs} = A^h_k. \qquad (4.22)
$$

From equation (4.17) and equation (3.5), we have

$$
\begin{aligned}
P^h_{ijk} &= P_{sijk}g^{sh} = \frac{1}{u_{(-1)}}\theta_{(hi)}[a_{ijk|s}a^{sh} + \frac{1}{2}a_{ijk|0}K^h - \frac{1}{2}La_{ik|0}(l^h K_j \\
&\quad + K_j K^h) + a_{rik|0}K^r H^h_j + a_{rik|0}A^{rh}_j] - [\theta_{(hi)}\{z_2\phi^2\hat{B}^s\hat{B}^h + z_0\phi(\hat{B}^s a^h \\
&\quad + \hat{B}^h a^s) + z_2 a^s a_h\}\{a_{ijk|s} + \frac{1}{2}a_{ijk|0}K_s - \frac{1}{2}La_{ik|0}(l_s K_j + l_j K_s \\
&\quad + a_{ris|0}K^r H_{js} + a_{rik|0}A^r_{hj}\}], \qquad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
P^h_{ijk} &= \frac{1}{u_{-1}}\theta_{(hi)}[a_{ijk|s}a^{sh} + \frac{1}{2}a_{ijk|0}K^h - \frac{1}{2L}a_{ik|0}(l^h K_j + K_j K^h) \\
&\quad + a_{rik|0}K^r H^h_j + a_{rik|0}A^{rh}_j] - N^h_{ijk}, \qquad (4.24)
\end{aligned}
$$

where,

$$
\begin{aligned}
N^h_{ijk} &= [\theta_{(hi)}z_2\phi^2\hat{B}^s\hat{B}^h + z_0\phi(\hat{B}^s a^h + \hat{B}^h a^s) + z_2 a^s a_h a_{ijk|s} \\
&\quad + \frac{1}{2}a_{ijk|0}K_s - \frac{1}{2L}a_{ik|0}(l_s K_j + l_j K_s) + a_{ris|0}K^r H_{js} + a_{rik|0}A^r_{hj}]. \qquad (4.25)
\end{aligned}
$$

§5. Concluding Remarks

The above discussed applications may be considered as Finslerian extension of the Cubic root structure of space-time. The important results and properties associated with Cartan's tensor have been presented in section 4. Here we may observe that when C_{ijk} is equal to zero then the metric tensor g_{ij} is reduced to the Reimannian one. Historically we may say that y-dependent as discussed in the above sections has been combined with the concept of anisotropy. As we know that the cosmological constant problem of general relativity can be extended to locally anisotropic spaces with Finslerian structure. According to S. Weinberg [22] everything that contributes to the energy density at the vacuum acts just like a cosmological constant. In the Finslerian framework of space-time the anisotropic form of the microwave background radiation may contribute to that content, if we consider a metric of the form of Eq. (2.1).

The field equations in a Finslerian space-time are to be obtained from a variational principle. We observed that for the similar metric Stavrions and Diakogiannis [16] have also obtained the relationship between the anisotropic cosmological models of space time and Randers Finslerian metric. Here it is further mentioned that the Finslerian geodesics satisfy the Euler-Lagrange equations of geodesics

$$\frac{d^2x^b}{ds^2} + \Gamma^b_{ij} y^i y^j + \sigma(x) r^{ae}(\partial_j \hat{b}_e - \partial_e \hat{b}_j) y^j = 0.$$

In this equation we observe the additional term $r^{ae}(\partial_j \hat{b}_e - \partial_e \hat{b}_j) y^j = 0$.
where $\sigma = \sqrt{r_{if} y^i y^j}$ and Γ^b_{ij} are the cubic Christoffel symbols. This term expresses a rotation of the anisotropy. We may say that the equations of geodesics of the cubic space-time may be generalized as shown in the above calculations. It is also mentioned that if y^i represents the velocity of a particle (time like) then $\hat{b}^i$ is bound to be space like. This follows from the fact that one possible value of $\hat{b}^i y^i$is zero. All the above connections, in which the matter density is hidden, can be considered as a property of the field itself. A weak F^n space-time is proposed for the study and detection of gravitational waves, in virtue of the equation of deviation of geodesics. We have already considered an interesting class of F^n.

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Characterizations of Space Curves According to Bishop Darboux Vector in Euclidean 3-Space E^3

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Abstract: In this paper, we obtained some characterizations of space curves according to Bihop frame in Euclidean 3-space E^3 by using Laplacian operator and Levi-Civita connection. Furthermore, we gave the general differential equations which characterize the space curves according to the Bishop Darboux vector and the normal Bishop Darboux vector.

Key Words: Bishop frame, Darboux vector, Euclidean 3-Space, Laplacian operator.

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§1. Introduction

It is well-known that a curve of constant slope or general helix is defined by the property that the tangent of the curve makes a constant angle with a fixed straight line which is called the axis of the general helix. A necessary and sufficient condition for a curve to be a general helix is that the ratio of curvature to torsion be constant ([10]). The study of these curves in has been given by many mathematicians. Moreover, İlarslan studied the characterizations of helices in Minkowski 3-space E_1^3 and found differential equations according to Frenet vectors characterizing the helices in E_1^3 ([15]). Then, Kocayiğit obtained general differential equations which characterize the Frenet curves in Euclidean 3-space E^3 and Minkowski 3-space E_1^3 ([11]).

Analogue to the helix curve, Izumiya and Takeuchi have defined a new special curve called the slant helix in Euclidean 3-space E^3 by the property that the principal normal of a space curve γ makes a constant angle with a fixed direction ([19]). The spherical images of tangent indicatrix and binormal indicatrix of a slant helix have been studied by Kula and Yaylı ([16]). They obtained that the spherical images of a slant helix are spherical helices. Moreover, Kula et al. studied the relations between a general helix and a slant helix ([17]). They have found some differential equations which characterize the slant helix.

Position vectors of slant helices have been studied by Ali and Turgut ([3]). Also, they have given the generalization of the concept of a slant helix in the Euclidean n-space E^n ([4]).

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Furthermore, Chen and Ishikawa classified biharmonic curves, the curves for which $\Delta H = 0$ holds in semi-Euclidean space E_v^n where Δ is Laplacian operator and H is mean curvature vector field of a Frenet curve ([9]). Later, Kocayiğit and Hacısalihoğlu studied biharmonic curves and 1-type curves i.e., the curves for which $\Delta H = \lambda H$ holds, where λ is constant, in Euclidean 3-space E^3 ([12]) and Minkowski 3-space E_1^3 ([13]). They showed the relations between 1-type curves and circular helix and the relations between biharmonic curves and geodesics. Moreover, slant helices have been studied by Bükçü and Karacan according to Bishop frame in Euclidean 3-space ([5]) and Minkowski space ([6,7]). Characterizations of space curves according to Bishop frame in Euclidean 3-space E^3 have been given in [14].

In this paper, we gave some characterizations of space curves according to Bishop Frame in Euclidean 3-space E^3 by using Laplacian operator. We found the differential equations characterizing space curves according to the Bishop Darboux vector and the normal Bishop Darboux vector.

§2. Preliminaries

Let $\alpha : I \subset \mathbb{R}$ be an arbitrary curve in Euclidean 3-space E^3. Recall that the curve α is said to be of unit speed (or parameterized by arc length function s) if $\left\langle \overrightarrow{\alpha'}, \overrightarrow{\alpha'} \right\rangle = 1$, where $\langle , \rangle$ is the standard scalar (inner) product of E^3 given by $\langle \overrightarrow{x}, \overrightarrow{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3$ for each $\overrightarrow{x} = (x_1, x_2, x_3)$, $\overrightarrow{y} = (y_1, y_2, y_3) \in E^3$. In particular, the norm of a vector $\overrightarrow{x} \in E^3$ is given by $\|\overrightarrow{x}\| = \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle}$. Denote by $\left\{ \overrightarrow{T}(s), \overrightarrow{N}(s), \overrightarrow{B}(s) \right\}$ the moving Frenet frame along the unit speed curve α. Then the Frenet formulas are given by

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}' \\ \overrightarrow{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix},$$

where $\overrightarrow{T}$, $\overrightarrow{N}$ and $\overrightarrow{B}$ are called tangent, principal normal and binormal vector fields of the curve, respectively. $\kappa(s)$ and $\tau(s)$ are called curvature and torsion of the curve α, respectively ([20]).

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $\overrightarrow{T}(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $\left(\overrightarrow{N_1}(s), \overrightarrow{N_2}(s)\right)$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $\overrightarrow{T}(s)$ at each point. If the derivatives of $\left(\overrightarrow{N_1}(s), \overrightarrow{N_2}(s)\right)$ depend only on $\overrightarrow{T}(s)$ and not each other we can make $\overrightarrow{N_1}(s)$ and $\overrightarrow{N_2}(s)$ vary smoothly throughout the path regardless of the curvature ([18,1,2]).

In addition, suppose the curve α is an arclength-parameterized C^2 curve. Suppose we have

C^1 unit vector fields $\overrightarrow{N_1}$ and $\overrightarrow{N_2} = \overrightarrow{T} \wedge \overrightarrow{N_1}$ along the curve α so that

$$\left\langle \overrightarrow{T}, \overrightarrow{N_1} \right\rangle = \left\langle \overrightarrow{T}, \overrightarrow{N_2} \right\rangle = \left\langle \overrightarrow{N_1}, \overrightarrow{N_2} \right\rangle = 0$$

i.e., $\overrightarrow{T}$, $\overrightarrow{N_1}$, $\overrightarrow{N_2}$ will be a smoothly varying right-handed orthonormal frame as we move along the curve. (To this point, the Frenet frame would work just fine if the curve were C^3 with $\kappa \neq 0$) But now we want to impose the extra condition that $\left\langle \overrightarrow{N_1'}, \overrightarrow{N_2} \right\rangle = 0$. We say the unit first normal vector field $\overrightarrow{N_1}$ is parallel along the curve α. This means that the only change of $\overrightarrow{N_1}$ is in the direction of $\overrightarrow{T}$. A Bishop frame can be defined even when a Frenet frame cannot (e.g., when there are points with $\kappa = 0$). Therefore, we have the alternative frame equations

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}_1' \\ \overrightarrow{N}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N_1} \\ \overrightarrow{N_2} \end{bmatrix}.$$

One can show that

$$\kappa(s) = \sqrt{k_1^2 + k_2^2},\ \theta(s) = \arctan\left(\frac{k_2}{k_1}\right),\ k_1 \neq 0,\ \tau(s) = -\frac{d\theta(s)}{ds}$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(s)ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation ([18,1,2]).

Let $\alpha : I \to E^3$ be a unit speed space curve with nonzero nature curvatures k_1, k_2. Then α is a slant helix if and only if $\frac{k_1}{k_2}$ is constant ([5]).

Let $\bigtriangledown$ denotes the Levi-Civita connection given by $\bigtriangledown_{\alpha'} = \frac{d}{ds}$ where s is the arclenght parameter of the space curve α. The Laplacian operator of α is defined by ([13])

$$\triangle = -\bigtriangledown_{\alpha'}^2 = -\bigtriangledown_{\alpha'} \bigtriangledown_{\alpha'}.$$

§3. Characterizations of Space Curves

In this section we gave the characterizations of the space curves according to Bishop frame in Euclidean 3-space E^3. Furthermore, we obtained the general differential equations which characterize the space curves according to the Bishop Darboux vector $\overrightarrow{W}$ and the normal Bishop Darboux vector $\overrightarrow{W}^{\perp}$ in E^3.

Theorem 3.1([8]) *Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The Bishop Darboux vector $\overrightarrow{W}$ of the curve α is given by*

$$\overrightarrow{W} = -k_2\overrightarrow{N_1} + k_1\overrightarrow{N_2}. \tag{3.1}$$

Definition 3.1 *A regular space curve α in E^3 said to has harmonic Darboux vector $\overrightarrow{W}$ if*

$$\Delta\overrightarrow{W} = 0.$$

Definition 3.2 *A regular space curve α in E^3 said to has harmonic 1-type Darboux vector $\overrightarrow{W}$ if*

$$\Delta\overrightarrow{W} = \lambda\overrightarrow{W}, \qquad \lambda \in \mathbb{R}. \tag{3.2}$$

Theorem 3.2 *Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing α according to the Bishop Darboux vector $\overrightarrow{W}$ is given by*

$$\lambda_4 \nabla^3_{\alpha'}\overrightarrow{W} + \lambda_3 \nabla^2_{\alpha'}\overrightarrow{W} + \lambda_2 \nabla_{\alpha'}\overrightarrow{W} + \lambda_1 \overrightarrow{W} = 0, \tag{3.3}$$

where

$$\begin{aligned}
\lambda_4 &= f^2 \\
\lambda_3 &= -f\left(f' + g\right) \\
\lambda_2 &= -\left[\left(f' + g\right) g - k_1\left(k_2''' + k_1 f\right) f + k_2\left(k_1''' - k_2 f\right) f\right] \\
\lambda_1 &= -\left[\left(f' + g\right)\left(\frac{k_1'}{k_2'}\right)\left(k_2'\right)^2 + k_1'\left(k_2''' + k_1 f\right) f - k_2'\left(k_1''' - k_2 f\right) f\right]
\end{aligned}$$

and

$$f = \left(\frac{k_1}{k_2}\right)'\left(k_2\right)^2, \quad g = k_1 k_2'' - k_1'' k_2.$$

Proof Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. By differentiating $\overrightarrow{W}$ three times with respect to s, we obtain the followings.

$$\nabla_{\alpha'}\overrightarrow{W} = -k_2'\overrightarrow{N_1} + k_1'\overrightarrow{N}_2, \tag{3.4}$$

$$\nabla^2_{\alpha'}\overrightarrow{W} = -\left(k_1' k_2 - k_1 k_2'\right)\overrightarrow{T} - k_2''\overrightarrow{N_1} + k_1''\overrightarrow{N_2}, \tag{3.5}$$

$$\begin{aligned}
\nabla^3_{\alpha'}\overrightarrow{W} \;=\; & -\left[\left(k_1' k_2 - k_1 k_2'\right)' + k_1 k_2'' - k_1'' k_2\right]\overrightarrow{T} \\
& - \left[k_2''' + k_1\left(k_1' k_2 - k_1 k_2'\right)\right]\overrightarrow{N_1} \\
& + \left[k_1''' - k_2\left(k_1' k_2 - k_1 k_2'\right)\right]\overrightarrow{N_2}
\end{aligned} \tag{3.6}$$

From (3.1) and (3.4) we get

$$\overrightarrow{N_1} = \frac{k_1}{k_1' k_2 - k_1 k_2'}\nabla_{\alpha'}\overrightarrow{W} - \frac{k_1'}{k_1' k_2 - k_1 k_2'}\overrightarrow{W} \tag{3.7}$$

and

$$\overrightarrow{N_2} = \frac{k_2}{k_1' k_2 - k_1 k_2'} \nabla_{\alpha'} \overrightarrow{W} - \frac{k_2'}{k_1' k_2 - k_1 k_2'} \overrightarrow{W}. \tag{3.8}$$

By substituting (3.7) and (3.8) in (3.5) we have

$$\overrightarrow{T} = -\frac{1}{k_1' k_2 - k_1 k_2'} \nabla^2_{\alpha'} \overrightarrow{W} - \frac{k_1 k_2'' - k_1'' k_2}{\left(k_1' k_2 - k_1 k_2'\right)^2} \nabla_{\alpha'} \overrightarrow{W} - \frac{k_1'' k_2' - k_1' k_2''}{\left(k_1' k_2 - k_1 k_2'\right)^2} \overrightarrow{W}. \tag{3.9}$$

By substituting (3.7), (3.8) and (3.9) in (3.6) we obtain

$$\lambda_4 \nabla^3_{\alpha'} \overrightarrow{W} + \lambda_3 \nabla^2_{\alpha'} \overrightarrow{W} + \lambda_2 \nabla_{\alpha'} \overrightarrow{W} + \lambda_1 \overrightarrow{W} = 0,$$

where

$$\begin{aligned} \lambda_4 &= f^2 \\ \lambda_3 &= -f\left(f' + g\right) \\ \lambda_2 &= -\left[\left(f' + g\right) g - k_1\left(k_2''' + k_1 f\right) f + k_2\left(k_1''' - k_2 f\right) f\right] \\ \lambda_1 &= -\left[\left(f' + g\right)\left(\frac{k_1'}{k_2'}\right)\left(k_2'\right)^2 + k_1'\left(k_2''' + k_1 f\right) f - k_2'\left(k_1''' - k_2 f\right) f\right] \end{aligned}$$

and

$$f = \left(\frac{k_1}{k_2}\right)' \left(k_2\right)^2, \quad g = k_1 k_2'' - k_1'' k_2. \qquad \square$$

Corollary 3.1 *Let $\alpha(s)$ be a general helix in E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing α according to the Bishop Darboux vector $\overrightarrow{W}$ is given by*

$$g \nabla_{\alpha'} \overrightarrow{W} - \left(\frac{k_1'}{k_2'}\right)' \left(k_2'\right)^2 \overrightarrow{W} = 0.$$

Theorem 3.3 *Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing α according to the normal Bishop Darboux vector $\overrightarrow{W^\perp}$ is given by*

$$\lambda_3 \nabla^2_{\alpha'} \overrightarrow{W^\perp} + \lambda_2 \nabla_{\alpha'} \overrightarrow{W^\perp} + \lambda_1 \overrightarrow{W^\perp} = 0, \tag{3.10}$$

where

$$\begin{aligned} \lambda_3 &= f \\ \lambda_2 &= g \\ \lambda_1 &= \left(\frac{k_1'}{k_2'}\right)\left(k_2'\right)^2 \end{aligned}$$

and

$$f = \left(\frac{k_1}{k_2}\right)' \left(k_2\right)^2, \qquad g = k_1 k_2'' - k_1'' k_2.$$

Proof Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. By differentiating $\overrightarrow{W^\perp}$ two times with respect to s, we obtain the followings.

$$\overrightarrow{W^\perp} = -k_2\overrightarrow{N_1} + k_1\overrightarrow{N_2}, \tag{3.11}$$

$$\nabla_{\alpha'}\overrightarrow{W^\perp} = -k_2'\overrightarrow{N_1} + k_1'\overrightarrow{N}_2, \tag{3.12}$$

$$\nabla^2_{\alpha'}\overrightarrow{W^\perp} = -k_2''\overrightarrow{N_1} + k_1''\overrightarrow{N_2}. \tag{3.13}$$

From (3.11) and (3.12) we get

$$\overrightarrow{N_1} = \frac{k_1}{k_1'k_2 - k_1k_2'}\nabla_{\alpha'}\overrightarrow{W^\perp} - \frac{k_1'}{k_1'k_2 - k_1k_2'}\overrightarrow{W^\perp} \tag{3.14}$$

and

$$\overrightarrow{N_2} = \frac{k_2}{k_1'k_2 - k_1k_2'}\nabla_{\alpha'}\overrightarrow{W^\perp} - \frac{k_2'}{k_1'k_2 - k_1k_2'}\overrightarrow{W^\perp}. \tag{3.15}$$

By substituting (3.14) and (3.15) in (3.13) we obtain

$$f\nabla^2_{\alpha'}\overrightarrow{W^\perp} + g\nabla_{\alpha'}\overrightarrow{W^\perp} + \left(\frac{k_1'}{k_2'}\right)'(k_2')^2\,\overrightarrow{W^\perp} = 0. \tag{3.16}$$

This completes the proof. □

Corollary 3.2 *Let $\alpha(s)$ be a slant helix in E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing α according to the normal Bishop Darboux vector $\overrightarrow{W^\perp}$ is given by*

$$g\nabla_{\alpha'}\overrightarrow{W^\perp} + \left(\frac{k_1'}{k_2'}\right)'(k_2')^2\,\overrightarrow{W^\perp} = 0.$$

Theorem 3.4 *Let α be a unit speed space curve in E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$. Then, α is of harmonic 1-type Darboux vector if and only if the curvature k_1 and the torsion k_2 of the curve α satisfy the followings.*

$$-k_1'' = \lambda k_1, \qquad k_1'k_2 - k_1k_2' = 0, \qquad -k_2'' = \lambda k_2. \tag{3.17}$$

Proof Let α be a unit speed space curve and let Δ be the Laplacian associated with ∇. From (3.4) and (3.5) we can obtain following.

$$\Delta\overrightarrow{W} = (k_1'k_2 - k_1k_2')\,\overrightarrow{T} + k_2''\overrightarrow{N_1} - k_1''\overrightarrow{N_2}. \tag{3.18}$$

We assume that the space curve α is of harmonic 1-type Darboux vector $\overrightarrow{W}$. Substituting (3.18) in (3.2) we get (3.17). □

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Magic Properties of Special Class of Trees

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Abstract: In this paper, we consider special class of trees called uniform k-distant trees, which have many interesting properties. We show that they have an edge-magic total labeling, a super edge-magic total labeling, a (a, d)-edge-antimagic vertex labeling, an (a, d)-edge-antimagic total labeling, a super (a, d)- edge-antimagic total labeling. Also we introduce a new labeling called edge bi-magic vertex labeling and prove that every uniform k-distant tree has edge bi-magic vertex labeling.

Key Words: k-distant tree, magic labeling, anti-magic labeling, total labeling, Smarandache anti-magic labeling.

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§1. Introduction

For graph theory terminology and notation, we follow either Bondy and Murty [3] or Murugan [8]. In this paper, we consider a graph to be finite and without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, whereas the edge set of G is denoted by $E(G)$.

A labeling of a graph is a function that sends some set of graph element to a set of positive integers. If the domain is $V(G)$ or $E(G)$ or $V(G) \cup E(G)$, then the labeling is called vertex labeling or edge labeling or total labeling respectively. The edge-weight of an edge uv under a vertex labeling is the sum of the vertex labels at its ends; under a total labeling, we also add the label of uv.

Trees are important family of graphs and posses many interesting properties. The famous Graceful Tree Conjecture, also known as Ringel-Kotzig or Rosa's or even Ringel-Kotzig-Rosa Conjecture, says that all trees have a graceful labeling was mentioned in [11]. Yao et al. [5] have conjectured that every tree is (k, d)-graceful for $k > 1$ and $d > 1$. Hedge [6] has conjectured that all trees are (k, d)-balanced for some values of k and d. A caterpillar is a tree with the property that the removal of its endpoints leaves a path. A lobster is a tree with the property that the removal of the endpoints leaves a caterpillar. Bermond [2] conjectured that lobsters are graceful and this is still open.

The conjecture, *All Trees are Harmonious* is still open and is unsettled for many years. Gallian in his survey [5] of graph labeling, has mentioned that no attention has been given to

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analyze the harmonious property of lobsters. It is clear that uniform 2-distant trees are special lobsters. Also, Atif Abueida and Dan Roberts [1] have proved that uniform k-distant trees admit a harmonious labeling, when they have even number of vertices. Murugan [9] has proved that all uniform k-distant trees are harmonious. In this paper, we analyze some interesting properties of uniform k-distant trees.

§2. k-Distant Trees

A k-distant tree consists of a main path called the *spine*, such that each vertex on the spine is joined by an edge to at most one path on k-vertices. Those paths are called *tails* (i.e. each tail must be incident with a vertex on the spine). When every vertex on the spine has exactly one incident tail of length k, we call the tree a uniform k-distant tree.

A uniform k-distant tree with odd number of vertices is called a uniform k-distant odd tree. A uniform k-distant tree with even number of vertices is called a uniform k-distant even tree.

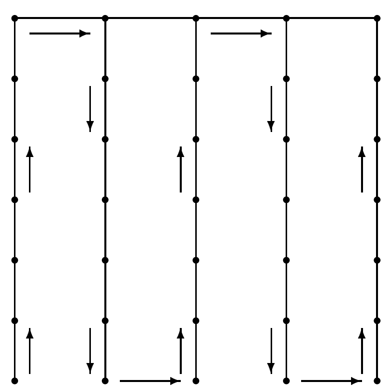

Figure 1 Order to name the vertices

To prove our results, we name the vertices and edges of any uniform k-distant tree as in Figure 2 with the help of Figure 1. The arrows on the Figure 1 show the order of naming the vertices and edges.

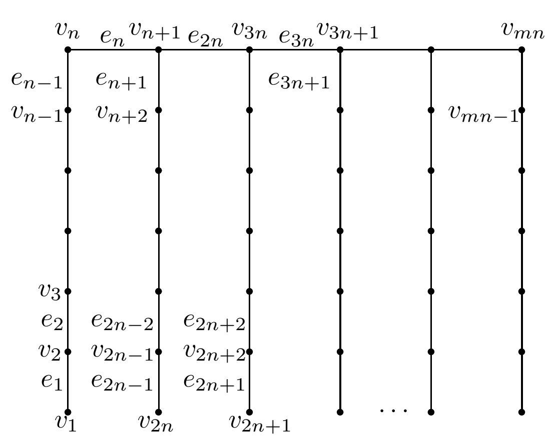

Figure 2 Uniform k-distant tree

§3. Variations of Magic Labelings

In this section, we list a few existing labelings which are useful for the development of this paper and we introduce a new labeling called *edge bi-magic vertex labeling*. Let G be a graph with vertex set V and edge set E.

Definition 3.1 (Edge-Magic Total Labeling) *An edge-magic total labeling of a graph $G(V, E)$ is a bijection f from $V \cup E$ to $\{1, 2, \ldots, |V \cup E|\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is constant.*

This was introduced by Kotzig and Rosa [7] and rediscovered by Ringel and Llado [10].

Definition 3.2(Super Edge-Magic Total Labeling) *A super edge-magic total labeling of a graph $G(V, E)$ is an edge-magic total labeling with the additional property that the vertex labels are 1 to $|V|$.*

This was introduced by Enomoto et al. [4].

Definition 3.3((a, d)-Edge Antimagic Vertex Labeling) *An (a, d)-edge antimagic vertex labeling is a bijection from $V(G)$ onto $\{1, 2, \ldots, |V(G)|\}$ such that the set of edge-weights of all edges in G is*

$$\{a, a + d, \ldots, a + (|E(G)| - 1)d\}$$

where $a > 0$ and $d \geqslant 0$ are two fixed integers.

This was introduced by Simanjuntak et al. [12].

Definition 3.4((a, d)-Edge Antimagic Total Labeling) *An (a, d)-edge antimagic total labeling is a bijection from $V(G) \cup E(G)$ onto the set $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ so that the set of edge-weights of all edges in G is equal to $\{a, a + d, \ldots, a + (|E(G)| - 1)d\}$, for two integers $a > 0$ and $d \geqslant 0$.*

This was introduced by Simanjuntak et al. [12].

Definition 3.5(Super (a, d)-Edge-Antimagic Total Labeling) *An (a, d)-edge-antimagic total labeling will be called super if it has the property that the vertex-labels are the integers $1, 2, \cdots, |V(G)|$.*

Definition 3.6(Edge Bi-Magic Total Labeling) *An edge bi-magic total labeling of a graph $G(V, E)$ is a bijection f from $V \cup E$ to $\{1, 2, \ldots, |V \cup E|\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is k_1 or k_2 where k_1 and k_2 are constants.*

This was introduced by Vishnupriya et al. [13]. Now we introduce *edge bi- magic vertex labeling.*

Definition 3.7(Edge Bi-Magic Vertex Labeling) *An edge bi-magic vertex labeling of a graph $G(V, E)$ is a bijection f from V to $\{1, 2, \ldots, |V(G)|\}$ such that for all edges xy, $f(x) + f(y)$ is*

k_1 or k_2 where k_1 and k_2 are constants.

Definition 3.8(Smarandache anti-Magic Labeling) *Let G be a graph and $H < G$. A Smarandache antimagic labeling on H is a bijection from $V(H)\cup E(H)$ onto the set $\{1,2,\ldots,|V(H)|+|E(H)|\}$ so that the set of edge-weights of all edges in H is equal to $\{a,a+d,\cdots,a+(|E(H)|-1)d\}$ for two given integers $a>0$ and $d\geqslant 0$, and $f(x)+f(y)+f(xy)$ is constant for all edges xy in $E(G)\setminus E(H)$. Clearly, a Smarandache antimagic labeling on G is nothing else but an (a,d)-edge antimagic total labeling.*

§4. Results

Theorem 4.1 *Every uniform k-distant tree has an edge-magic total labeling.*

Proof Consider a uniform k-distant tree T with q edges. Since it is a tree, $q=p-1$, where p is the number of vertices of T.

Define a labeling f from $V(T)\cup E(T)$ into $\{1,2,\ldots,p+q\}$ such that

$$\begin{aligned} f(v_i) &= \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd} \\ \left\lceil\dfrac{p}{2}\right\rceil+\dfrac{i}{2} & \text{if } i \text{ is even} \end{cases} \\ f(e_i) &= 2p-i \end{aligned}$$

We note that the sum of the labels of two consecutive vertices on the spine (that is, labels on the edges of the spine) is equal to the sum of the labels at the end vertices of the corresponding tail (for example, sum of the labels of v_n and v_{n+1} is equal to sum of the labels of v_1 and v_{2n}), by construction and labeling.

Case 1 i is odd.

Consider

$$\begin{aligned} f(v_i)+f(v_{i+1})+f(e_i) &= \frac{i+1}{2}+\left\lceil\frac{p}{2}\right\rceil+\frac{i+1}{2}+2p-i \\ &= \left\lceil\frac{p}{2}\right\rceil+2p+i+1-i \\ &= 2p+\left\lceil\frac{p}{2}\right\rceil+1 \end{aligned}$$

Case 2 i is even.

Consider

$$\begin{aligned} f(v_i)+f(v_{i+1})+f(e_i) &= \left\lceil\frac{p}{2}\right\rceil+\frac{i}{2}+\frac{i+2}{2}+2p-i \\ &= \left\lceil\frac{p}{2}\right\rceil+2p+i+1-i \\ &= 2p+\left\lceil\frac{p}{2}\right\rceil+1 \end{aligned}$$

Since $f(v_i)+f(v_{i+1})+f(e_i)=2p+\left\lceil\frac{p}{2}\right\rceil+1$, T has an edge-magic total labeling. □

Theorem 4.2 *Every uniform k-distant tree has a super edge-magic total labeling.*

Proof Consider the edge-magic total labeling of an uniform k-distant tree as in Theorem 4.1. Since the vertex labels are 1 to $|V|$, T has a super edge-magic total labeling. □

Theorem 4.3 *Every uniform k-distant tree has a (a,d)-edge-antimagic vertex labeling.*

Proof Consider a uniform k-distant tree T with q edges. Since it is a tree, $q=p-1$, where p is the number of vertices of T. Define a labeling f from $V(T)$ into $\{1,2,\ldots,p\}$ such that

$$f(v_i)=\begin{cases}\dfrac{i+1}{2} & \text{if } i \text{ is odd}\\ \left\lceil\dfrac{p}{2}\right\rceil+\dfrac{i}{2} & \text{if } i \text{ is even}\end{cases}$$

We note that the sum of the labels of two consecutive vertices on the spine (that is, labels on the edges of the spine) is equal to the sum of the labels at the end vertices of the corresponding tail (for example, sum of the labels of v_n and v_{n+1} is equal to sum of the labels of v_1 and v_{2n}), by construction and labeling.

Case 1 i is odd.

Consider

$$f(v_i)+f(v_{i+1})=\frac{i+1}{2}+\left\lceil\frac{p}{2}\right\rceil+\frac{i+1}{2}=\left\lceil\frac{p}{2}\right\rceil+i+1$$

Now

$$f(v_{i+1})+f(v_{i+2})=\left\lceil\frac{p}{2}\right\rceil+\frac{i+1}{2}+\frac{i+3}{2}=\left\lceil\frac{p}{2}\right\rceil+i+2.$$

Therefore, each $f(v_i)+f(v_{i+1})$ is distinct and differ by 1.

Case 2 i is even.

Consider

$$f(v_i)+f(v_{i+1})=\left\lceil\frac{p}{2}\right\rceil+\frac{i}{2}+\frac{i+2}{2}=\left\lceil\frac{p}{2}\right\rceil+i+1$$

Now

$$f(v_{i+1})+f(v_{i+2})=\frac{i+2}{2}+\left\lceil\frac{p}{2}\right\rceil+\frac{i+2}{2}=\left\lceil\frac{p}{2}\right\rceil+i+2.$$

Therefore, each $f(v_i)+f(v_{i+1})$ is distinct and differ by 1. Hence, T is (a,d)-edge antimagic vertex labeling, where $a=f(v_1)+f(v_2)=1+\left\lceil\frac{p}{2}\right\rceil+1=\left\lceil\frac{p}{2}\right\rceil+2$ and $d=1$. Hence, every uniform k-distant tree has a (a,d) edge-antimagic vertex labeling. □

Theorem 4.4 *Every uniform k-distant tree has a (a,d)-edge-antimagic total labeling.*

Proof Consider a uniform k-distant tree T with q edges. Since it is a tree, $q=p-1$, where p is the number of vertices of T.

Define a labeling f from $V(T)\cup E(T)$ into $\{1,2,\ldots,p+q\}$ such that

$$\begin{aligned} f(v_i) &= \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd} \\ \left\lceil\dfrac{p}{2}\right\rceil+\dfrac{i}{2} & \text{if } i \text{ is even} \end{cases} \\ f(e_i) &= p+i \end{aligned}$$

We note that the sum of the labels of two consecutive vertices on the spine (that is, labels on the edges of the spine) is equal to the sum of the labels at the end vertices of the corresponding tail (for example, sum of the labels of v_n and v_{n+1} is equal to sum of the labels of v_1 and v_{2n}), by construction and labeling.

Case 1 i is odd.

Consider

$$\begin{aligned} f(v_i)+f(v_{i+1})+f(e_i) &= \frac{i+1}{2}+\left\lceil\frac{p}{2}\right\rceil+\frac{i+1}{2}+p+i \\ &= \left\lceil\frac{p}{2}\right\rceil+p+i+1+i \\ &= p+\left\lceil\frac{p}{2}\right\rceil+2i+1. \end{aligned}$$

Now

$$\begin{aligned} f(v_{i+1})+f(v_{i+2})+f(e_{i+1}) &= \left\lceil\frac{p}{2}\right\rceil+\frac{i+1}{2}+\frac{i+3}{2}+p+i+1 \\ &= \left\lceil\frac{p}{2}\right\rceil+p+i+2+i+1 \\ &= p+\left\lceil\frac{p}{2}\right\rceil+2i+3. \end{aligned}$$

Case 2 i is even.

Consider

$$\begin{aligned} f(v_i)+f(v_{i+1})+f(e_i) &= \left\lceil\frac{p}{2}\right\rceil+\frac{i}{2}+\frac{i+2}{2}+p+i \\ &= \left\lceil\frac{p}{2}\right\rceil+p+i+1+i \\ &= p+\left\lceil\frac{p}{2}\right\rceil+2i+1 \end{aligned}$$

Now

$$\begin{aligned} f(v_{i+1})+f(v_{i+2})+f(e_{i+1}) &= \frac{i+2}{2}+\left\lceil\frac{p}{2}\right\rceil+\frac{i+2}{2}+p+i+1 \\ &= \left\lceil\frac{p}{2}\right\rceil+p+i+2+i+1 \\ &= p+\left\lceil\frac{p}{2}\right\rceil+2i+3. \end{aligned}$$

Therefore, each $f(v_i)+f(v_{i+1})+f(e_i)$ is distinct and the edge labels increase by 2. Hence,

T is (a,d)-edge antimagic total labeling, where $a = f(v_1)+f(v_2)+f(e_1) = 1+\left\lceil \frac{p}{2} \right\rceil+1+p+1 = p+\left\lceil \frac{p}{2} \right\rceil+3$ and $d = 2$. Hence, every uniform k-distant tree has a (a,d) edge-antimagic total labeling. □

Theorem 4.5 *Every uniform k-distant tree has a super (a,d)-edge-antimagic total labeling.*

Proof Consider the edge-magic total labeling of an uniform k-distant tree as in Theorem 4.3. Since the vertex labels are 1 to $|V|$, T has a super (a,d)-edge-antimagic total labeling. □

Theorem 4.6 *Every uniform k-distant tree has a edge bi-magic vertex labeling.*

Proof Consider a uniform k-distant tree T with q edges. Since it is a tree, $q = p-1$, where p is the number of vertices of T. Define a labeling f from $V(T)$ into $\{1,2,\ldots,p\}$ such that

$$f(v_i) = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd} \\ p-\dfrac{i-2}{2} & \text{if } i \text{ is even} \end{cases}$$

We note that the sum of the labels of two consecutive vertices on the spine (that is, labels on the edges of the spine) is equal to the sum of the labels at the end vertices of the corresponding tail (for example, sum of the labels of v_n and v_{n+1} is equal to sum of the labels of v_1 and v_{2n}), by construction and labeling.

Case 1 i is odd.

Consider

$$f(v_i)+f(v_{i+1}) = \frac{i+1}{2}+p-\frac{i-1}{2} = p+1$$

Now

$$f(v_{i+1})+f(v_{i+2}) = p-\frac{i-1}{2}+\frac{i+3}{2} = p+2$$

Case 2 i is even.

Consider

$$f(v_i)+f(v_{i+1}) = p-\frac{i-2}{2}+\frac{i+2}{2} = p+2$$

Now

$$f(v_{i+1})+f(v_{i+2}) = \frac{i+2}{2}+p-\frac{i}{2} = p+1$$

Therefore, each edge has either $p+1$ or $p+2$ as edge weight. Hence every uniform k-distant tree has a edge bi-magic vertex labeling. □

§5. Conclusion

Uniform k-distant trees are special class of trees which have many interesting properties. In this paper we have proved that every uniform k-distant tree has an edge-magic total labeling, a super edge-magic total labeling, a (a,d)-edge-antimagic vertex labeling, a (a,d)-edge-antimagic total labeling, a super (a, d)- edge-antimagic total labeling and a edge bi-magic vertex labeling.

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On Pathos Adjacency Cut Vertex Jump Graph of a Tree

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Abstract: In this paper the concept of pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is introduced. We also present a characterization of graphs whose pathos adjacency cut vertex jump graphs are planar, outerplanar, minimally non-outerplanar, Eulerian and Hamiltonian.

Key Words: Jump graph $J(G)$, pathos, Smarandache pathos-cut jump graph, crossing number cr(G), outerplanar, minimally non-outerplanar, inner vertex number $i(G)$.

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§1. Introduction

For standard terminology and notation in graph theory, not specifically defined in this paper, the reader is referred to Harary [2]. The concept of *pathos* of a graph G was introduced by Harary [3], as a collection of minimum number of edge disjoint open paths whose union is G. The *path number* of a graph G is the number of paths in any pathos. The path number of a tree T is equal to k, where $2k$ is the number of odd degree vertices of T. A *pathos vertex* is a vertex corresponding to a path P in any pathos of T.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

The *jump graph* of a graph G ([1]), written $J(G)$, is the graph whose vertices are the edges of G, with two vertices of $J(G)$ adjacent whenever the corresponding edges of G are not adjacent. Clearly, the jump graph $J(G)$ is the complement of the line graph $L(G)$ of G.

The *pathos jump graph* of a tree T [5], written $J_P(T)$, is the graph whose vertices are the edges and paths of pathos of T, with two vertices of $J_P(T)$ adjacent whenever the corresponding edges of T are not adjacent and the edges that lie on the corresponding path P_i of pathos of T.

The *cut vertex jump graph* of a graph G ([6]), written $JC(G)$, is the graph whose vertices are the edges and cut vertices of G, with two vertices of $JC(G)$ adjacent whenever the corresponding edges of G are not adjacent and the edges incident to the cut vertex of G.

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The *edge degree* of an edge pq of a tree T is the sum of the degrees of p and q. A graph G is *planar* if it can be drawn on the plane in such a way that no two of its edges intersect. If all the vertices of a planar graph G lie in the exterior region, then G is said to be an outerplanar.

An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing its outer planarity. For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be *minimally non-outerplanar* if the inner vertex number $i(G)=1$ ([4]).

The least number of edge-crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by $\mathrm{cr}(G)$.

A *wheel graph* W_n is a graph obtained by taking the join of a cycle and a single vertex. The *Dutch windmill* graph $D_3^{(m)}$, also called a *friendship graph*, is the graph obtained by taking m copies of the cycle graph C_3 with a vertex in common, and therefore corresponds to the usual *windmill graph* $W_n^{(m)}$. It is therefore natural to extend the definition to $D_n^{(m)}$, consisting of m copies of C_n.

A *Smarandache pathos-cut jump graph* of a tree T on subtree $T_1 < T$, written $SPJC(T_1)$, is the graph whose vertices are the edges, paths of pathos and cut vertices of T_1 and vertices $V(T)-V(T_1)$, with two vertices of $SPJC(T_1)$ adjacent whenever the corresponding edges of T_1 are not adjacent, edges that lie on the corresponding path P_i of pathos, the edges incident to the cut vertex of T_1 and edges in $E(T)\setminus E(T_1)$. Particularly, if $T_1 = T$, such a graph is called *pathos adjacency cut vertex jump graph* and denoted by $PJC(T)$. Two distinct pathos vertices P_m and P_n are adjacent in $PJC(T)$ whenever the corresponding paths of pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex, say v_c in T.

Since the pattern of pathos for a tree is not unique, the corresponding pathos adjacency cut vertex jump graph is also not unique.

In the following, Fig.1 shows a tree T and Fig.2 is its corresponding $PJC(T)$.

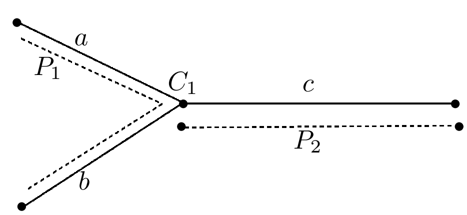

Fig.1 Tree T

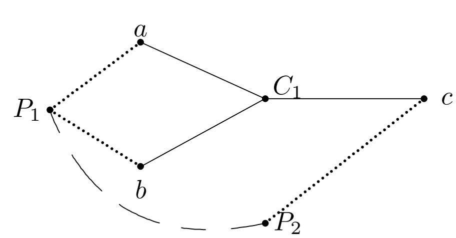

Fig.2 $PJC(T)$

The following existing result is required to prove further results.

Theorem A([2]) *A connected graph G is Eulerian if and only if each vertex in G has even degree.*

Some preliminary results which satisfies for any $PJC(T)$ are listed following.

Remark 1 For any tree T with $n \geq 3$ vertices, $J(T) \subseteq J_P(T)$ and
$J(T) \subseteq JC(T) \subseteq PJC(T)$. Here $\subseteq$ is the subgraph notation.

Remark 2 If the edge degree of an edge pq in a tree T is even(odd) and p and q are the cut vertices, then the degree of the corresponding vertex pq in $PJC(T)$ is even.

Remark 3 If the edge degree of a pendant edge pq in T is even(odd), then the degree of the corresponding vertex pq in $PJC(T)$ is even.

Remark 4 If T is a tree with p vertices and q edges, then the number of edges in $J(T)$ is

$$\frac{q(q+1) - \sum_{i=1}^{p} d_i^2}{2},$$

where d_i is the degree of vertices of T.

Remark 5 Let T be a tree(except star graph). Then the number of edges whose end vertices are the pathos vertices in $PJC(T)$ is $(k-1)$, where k is the path number of T.

Remark 6 If T is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the number of edges whose end vertices are the pathos vertices in $PJC(T)$ is $\frac{k(k-1)}{2}$, where k is the path number of T. For example, the edge P_1P_2 in Fig.2.

§2. Calculations

In this section, we determine the number of vertices and edges in $PJC(T)$.

Lemma 2.1 *Let T be a tree(except star graph) on p vertices and q edges such that d_i and C_j are the degrees of vertices and cut vertices C of T, respectively. Then $PJC(T)$ has $(q+k+C)$ vertices and*

$$\frac{q(q+1) - \sum_{i=1}^{p} d_i^2}{2} + \sum_{j=1}^{C} C_j + q + (k-1)$$

edges, where k is the path number of T.

Proof Let T be a tree(except star graph) on p vertices and q edges. The number vertices of $PJC(T)$ equals the sum of edges, paths of pathos and cut vertices C of T. Hence $PJC(T)$ has $(q+k+C)$ vertices. The number of edges of $PJC(T)$ equals the sum of edges in $J(T)$, degree of cut vertices, edges that lie on the corresponding path P_i of pathos of T and the number of edges whose end vertices are the pathos vertices. By Remark 4 and 5, the number of edges in $PJC(T)$ is given by

$$\frac{q(q+1)-\sum_{i=1}^{p} d_i^2}{2}+\sum_{j=1}^{C} C_j+q+(k-1). \qquad \square$$

Lemma 2.2 *If T is a star graph $K_{1,n}$ on $n \geq 3$ vertices and m edges, then $PJC(T)$ has $(m+k+1)$ vertices and $\frac{4m+k(k-1)}{2}$ edges, where k is the path number of T.*

Proof Let T be a star graph $K_{1,n}$ on $n \geq 3$ vertices and m edges. By definition, $PJC(T)$ has $(m+k+1)$ vertices. Also, for a star graph, the number of edges of $PJC(T)$ equals the sum of edges in $J(T)$, i.e., zero, twice the number of edges of T and the number of edges whose end vertices are the pathos vertices. By Remark 6, the number of edges in $PJC(T)$ is given by

$$2m+\frac{k(k-1)}{2} \Rightarrow \frac{4m+k(k-1)}{2}. \qquad \square$$

§3. Main Results

Theorem 3.1 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is planar if and only if the following conditions hold:*

(i) T is a path P_n on n=3 and 4 vertices;

(ii) T is a star graph $K_{1,n}$, on n =3,4,5 and 6 vertices.

Proof (i) Suppose $PJC(T)$ is planar. Assume that T is a path P_n on $n \geq 5$ vertices. Let T be a path P_5 and let the edge set $E(P_5)=\{e_1,e_2,e_3,e_4\}$. Then the jump graph $J(T)$ is the path $P_4=\{e_3,e_1,e_4,e_2\}$. Since the path number of T is exactly one, $J_P(T)$ is W_n-e, where W_n is the join of a cycle with the vertices corresponding to edges of T and a single vertex corresponding to pathos vertex P, and e is an edge between any two vertices corresponding to arcs of T in W_n. Let $\{C_1,C_2,C_3\}$ be the cut vertex set of T. Then the edges joining to $J(T)$ from the corresponding cut vertices gives $PJC(T)$ such that the crossing number of $PJC(T)$ is one, i.e., $\mathrm{cr}(PJC(T))=1$, a contradiction.

For sufficiency, we consider the following two cases.

Case 1 If T is a path P_3, then $PJC(T)$ is cycle C_4, which is planar.

Case 2 Let T be a path P_4 and let $E(P_4)=\{e_1,e_2,e_3\}$. Also, the path number of T is exactly one, i.e., P. Then $J_P(T)$ is $K_{1,3}+e$, where P is the vertex of degree three, and e is an edge between any two vertices corresponding to edges of T in $K_{1,3}$. Let $\{C_1,C_2\}$ be the cut vertex set of T. Then the edges joining to $J(T)$ from the corresponding cut vertices gives $PJC(T)=W_n-\{a,b\}$, where W_n is join of a cycle with the vertices corresponding to edges and cut vertices of T and a single vertex corresponding to pathos vertex P, and $\{a,b\}$ are the edges between pathos vertex P and cut vertices C_1 and C_2 of W_n. Clearly, $\mathrm{cr}(PJC(T))=0$. Hence $PJC(T)$ is planar.

(ii) Suppose that $PJC(T)$ is planar. Let T be a star graph $K_{1,n}$ on $n \geq 7$ vertices. If T is $K_{1,7}$, then $J(T)$ is a null graph of order seven. Since each edge in T lies on exactly one cut vertex C, $JC(T)$ is a star graph $K_{1,7}$. Furthermore, the path number of T is exactly four. Hence $PJC(T)$ is $D_4^{(4)} - v$, where v is a vertex at distance one from the common vertex in $D_4^{(4)}$. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T, by Remark 6, $\mathrm{cr}(PJC(T)) = 1$, a contradiction.

Conversely, suppose that T is a star graph $K_{1,n}$ on n=3,4,5 and 6 vertices. For n=3,4,5 and 6 vertices, $J(T)$ is a null graph of order n. Since each edge in T lies on exactly one cut vertex C, $JC(T)$ is a star graph of order $n + 1$. The path number of T is at most 3. Now, for n=4, $PJC(T)$ is the join of two copies of cycle C_4 with a common vertex and for n=6, $PJC(T)$ is the join of three copies of cycle C_4 with a common vertex. Next, for n=3, $PJC(T)$ is $D_4^{(2)} - v$, and n=5, $PJC(T)$ is $D_4^{(3)}$, respectively, where v is the vertex at distance one from the common vertex. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T, by Remark 6, $\mathrm{cr}(PJC(T)) = 0$. Hence $PJC(T)$ is planar. □

Theorem 3.2 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is an outerplanar if and only if T is a path P_3.*

Proof Suppose that $PJC(T)$ is an outerplanar. By Theorem 3.1, $PJC(T)$ is planar if and only if T is a path P_3 and P_4. Hence it is enough to verify the necessary part of the Theorem for a path P_4. Assume that T is a path P_4 and the edge set $E(P_4)$= e_i, where e_i=(v_i, v_{i+1}), for all $i = 1, 2, 3$. Then the jump graph $J(T)$ is a disconnected graph with two connected components, namely K_1 and K_2, where $K_1 = e_2$ and $K_2 = (e_1, e_3)$. Let $\{C_1, C_2\}$ be the cut vertex set of T. Hence $JC(T)$ is the cycle $C_5 = \{C_1, e_1, e_3, C_2, e_2, C_1\}$. Furthermore, the path number of T is exactly one. Then the edges joining to $J(T)$ from the corresponding pathos vertex gives $PJC(T)$ such that the inner vertex number of $PJC(T)$ is non-zero, i.e., $i(PJC(T)) \neq 0$, a contradiction.

Conversely, if T is a path P_3, then $PJC(T)$ is a cycle C_4, which is an outerplanar. □

Theorem 3.3 *For any tree T, $PJC(T)$ is not maximal outerplanar.*

Proof By Theorem 3.2, $PJC(T)$ is an outerplanar if and only if T is a path P_3. Moreover, for a path P_3, $PJC(T)$ is a cycle C_4, which is not maximal outerplanar, since the addition of an edge between any two vertices of cycle C_4 does not affect the outerplanarity of C_4. Hence for any tree T, $PJC(T)$ is not maximal outerplanar. □

Theorem 3.4 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is minimally non-outerplanar if and only if T is (i) a star graph $K_{1,3}$, and (ii) a path P_4.*

Proof (i) Suppose that $PJC(T)$ is minimally non-outerplanar. If T is a star graph $K_{1,n}$ on $n \geq 7$ vertices, by Theorem 3.1, $PJC(T)$ is nonplanar, a contadiction. Let T be a star graph $K_{1,n}$ on n=4,5 and 6 vertices. Now, for n=4, $PJC(T)$ is the join of two copies of cycle C_4 with a common vertex and for n=6, $PJC(T)$ is the join of three copies of cycle C_4 with a common vertex. For n=5, $PJC(T)$ is $D_4^{(3)} - v$. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T, the inner vertex number of $PJC(T)$ is more than one, i.e., $i(PJC(T)) > 1$, a contradiction.

Conversely, suppose that T is a star graph $K_{1,3}$. Then $J(T)$ is a null graph of order three. Since edge in T lies on exactly one cut vertex C, $JC(T)$ is a star graph $K_{1,3}$. The path number of T is exactly two. By definition, $PJC(T)$ is $D_4^{(2)} - v$. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T, the inner vertex number of $PJC(T)$ is exactly one, i.e., $i(PJC(T)) = 1$. Hence $PJC(T)$ is minimally non-outerplanar.

(ii) Suppose $PJC(T)$ is minimally non-outerplanar. Assume that T is a path on $n \geq 5$ vertices. If T is a path P_5, by Theorem 3.1, $PJC(T)$ is nonplanar, a contradiction.

Conversely, if T is a path P_4, by Case 2 of sufficiency part of Theorem 3.1, $PJC(T)$ is $W_n - \{a, b\}$. Clearly, $i(PJC(T)) = 1$. Hence $PJC(T)$ is minimally non-outerplanar. □

Theorem 3.5 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is Eulerian if and only if the following conditions hold:*

(i) T is a path P_n on $n = 2i + 1$ vertices, for all $i = 1, 2, \cdots$;

(ii) T is a star graph $K_{1,n}$ on $n = 4j + 2$ vertices, for all $j = 0, 1, 2, \cdots$.

Proof (i) Suppose that $PJC(T)$ is Eulerian. If T is a path P_n on $n = 2(i+1)$ vertices, for all $i = 1, 2, \cdots$, then the number of vertices in $J(T)$ is $(2i + 1)$, which is always odd. Since the path number of T is exactly one, by definition, the degree of the corresponding pathos vertex in $PJC(T)$ is odd. By Theorem [A], $PJC(T)$ is non-Eulerian, a contradiction.

For sufficiency, we consider the following two cases.

Case 1 If T is a path P_3, then $PJC(T)$ is a cycle C_4, which is Eulerian.

Case 2 Suppose that T is a path P_n on $n = 2i + 1$ vertices, for all $i = 2, 3, \cdots$. Let $\{e_1, e_2, \cdots e_{n-1}\}$ be the edge set of T. Then $d(e_1)$ and $d(e_{n-1})$ in $J(T)$ is even and degree of the remaining vertices $e_2, e_3, \cdots, e_{n-2}$ is odd. The number of cut vertices in T is $(n - 2)$. By definition, in $JC(T)$ the degree of even and odd degree vertices of $J(T)$ will be incremented by one and two, respectively. Hence the degree of every vertex of $JC(T)$ except cut vertices is odd. Furthermore, the path number of T is exactly one and the corresponding pathos vertex is adjacent to every vertex of $J(T)$. Clearly, every vertex of $PJC(T)$ has an even degree. By Theorem A, $PJC(T)$ is Eulerian.

(ii) Suppose that $PJC(T)$ is Eulerian. We consider the following two cases.

Case 1 Suppose that T is a star graph $K_{1,n}$ on $n = 2j + 1$ vertices, for all $j = 1, 2, \cdots$. Then $J(T)$ is a null graph of order n. Since each edge in T lies on exactly one cut vertex C, $JC(T)$ is a star graph $K_{1,n}$ in which $d(C)$ is odd. Moreover, since the degree of a cut vertex C does not change in $PJC(T)$, it is easy to observe that the vertex C remains as an odd degree vertex in $PJC(T)$. By Theorem A, $PJC(T)$ is non-Eulerian, a contradiction.

Case 2 Suppose that T is a star graph $K_{1,n}$ on $n = 4j$ vertices, for all $j = 1, 2, \cdots$. Then $J(T)$ is a null graph of order n. Since each edge in T lies on exactly one cut vertex C, $JC(T)$

is a star graph $K_{1,n}$ in which $d(C)$ is even. Since the path number of T is $[\frac{n}{2}]$, by definition, $PJC(T)$ is the join of at least two copies of cycle C_4 with a common vertex. Hence for every $v \in PJC(T)$, $d(v)$ is even. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T, there exists at least one pathos vertex, say P_m of odd degree in $PJC(T)$. By Theorem [A], $PJC(T)$ is non-Eulerian, a contradiction.

For sufficiency, we consider the following two cases.

Case 1 For a star graph $K_{1,2}$, T is a path P_3. Then $PJC(T)$ is a cycle C_4, which is Eulerian.

Case 2 Suppose that T is a star graph $K_{1,n}$ on $n = 4j + 2$ vertices, for all $j = 1, 2, \cdots$. Then the jump graph $J(T)$ is a null graph of order n. Since each edge in T lies on exactly one cut vertex C, $JC(T)$ is a star graph $K_{1,n}$ in which $d(C)$ is even. The path number of T is $[\frac{n}{2}]$. By definition, $PJC(T)$ is the join of at least three copies of cycle C_4 with a common vertex. Hence for every $v \in PJC(T)$, $d(v)$ is even. Finally, on embedding $PJC(T)$ in any plane for the the adjacency of pathos vertices corresponding to paths of pathos in T, degree of every vertex of $PJC(T)$ is also even. By Theorem A, $PJC(T)$ is Eulerian. □

Theorem 3.6 *For any path P_n on $n \geq 3$ vertices, $PJC(T)$ is Hamiltonian.*

Proof Suppose that T is a path P_n on $n \geq 3$ vertices with $\{v_1, v_2, \cdots v_n\} \in V(T)$ and $\{e_1, e_2, \cdots e_{n-1}\} \in E(T)$. Let $\{C_1, C_2, \cdots C_{n-2}\}$ be the cut vertex set of T. Also, the path number of T is exactly one and let it be P.

By definition $\{e_1, e_2, \cdots e_{n-1}\} \cup \{C_1, C_2, \cdots C_{n-2}\} \cup P$ form the vertex set in $PJC(T)$. In forming $PJC(T)$, the pathos P becomes a vertex adjacent to every vertex of $\{e_1, e_2, \cdots, e_{n-1}\}$ in $J(T)$. Also, the cut vertices C_j, for all $j = 1, 2, \cdots, (n-2)$ are adjacent to (e_i, e_{i+1}) for all $i = 1, 2, \cdots, (n-1)$ of $J_P(T)$. Clearly, there exist a cycle $(P, e_1, C_1, e_2, C_2, \cdots e_{n-1}, C_{n-2}, e_{n-1}, P)$ containing all the vertices of $PJC(T)$. Hence $PJC(T)$ is Hamiltonian. □

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Just nr-Excellent Graphs

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Abstract: Given an k-tuple of vectors, $S = (v_1, v_2, \cdots, v_k)$, the neighborhood adjacency code of a vertex v with respect to S, denoted by $nc_S(v)$ and defined by $(a_1, a_2, \cdots, a_k)$ where a_i is 1 if v and v_i are adjacent and 0 otherwise. S is called a Smarandachely neighborhood resolving set on subset $V' \subset V(G)$ if $nc_S(u) \neq nc_S(v)$ for any $u, v \in V'$. Particularly, if $V' = V(G)$, such a S is called a neighborhood resolving set or a neighborhood r-set. The least(maximum) cardinality of a minimal neighborhood resloving set of G is called the neighborhood(upper neighborhood) resolving number of G and is denoted by $nr(G)$ $(NR(G))$. A study of this new concept has been elaborately studied by S. Suganthi and V. Swaminathan. Fircke et al, in 2002 made a beginning of the study of graphs which are excellent with respect to a graph parameters. For example, a graph is domination excellent if every vertex is contained in a minimum dominating set. A graph G is said to be just nr-excellent if for each $u \in V$, there exists a unique nr-set of G containing u. In this paper, the study of just nr-excellent graphs is initiated.

Key Words: Locating sets, locating number, Smarandachely neighborhood resolving set, neighborhood resolving set, neighborhood resolving number, just nr-excellent.

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§1. Introduction

In the case of finite dimensional vector spaces, every ordered basis induces a scalar coding of the vectors where the scalars are from the base field. While finite dimensional vector spaces have rich structures, graphs have only one structure namely adjacency. If a graph is connected, the adjacency gives rise to a metric. This metric can be used to define a code for the vertices. P. J. Slater [20] defined the code of a vertex v with respect to a k-tuple of vertices $S = (v_1, v_2, \cdots, v_k)$ as $(d(v, v_1), d(v, v_2), \cdots, d(v, v_k))$ where $d(v, v_j)$ denotes the distance of the vertex v from the vertex v_j. Thus, entries in the code of a vertex may vary from 0 to diameter of G. If the codes of the vertices are to be distinct, then the number of vertices in G is less than or equal to

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$(diam(G)+1)^k$. If it is required to extend this concept to disconnected graphs, it is not possible to use the distance property. One can use adjacency to define binary codes, the motivation for this having come from finite dimensional vector spaces over Z_2. There is an advantage as well as demerit in this type of codes. The advantage is that the codes of the vertices can be defined even in disconnected graphs. The drawback is that not all graphs will allow resolution using this type of codes.

Given an k-tuple of vectors, $S = (v_1, v_2, \cdots, v_k)$, the neighborhood adjacency code of a vertex v with respect to S is defined as $(a_1, a_2, \cdots, a_k)$ where a_i is 1 if v and v_i are adjacent and 0 otherwise. Whereas in a connected graph $G = (V, E)$, V is always a resolving set, the same is not true if we consider neighborhood resolvability. If u and v are two vertices which are non-adjacent and $N(u) = N(v)$, u and v will have the same binary code with respect to any subset of V, including V. The least(maximum) cardinality of a minimal neighborhood resloving set of G is called the neighborhood(upper neighborhood) resolving number of G and is denoted by $nr(G)$ $(NR(G))$. This concept has been done in [31], [32], [33], [34], [35], [36] and [37].

Suk J. Seo and P. Slater [27] defined the same type of problem as an open neighborhood locating dominating set (OLD-set), is a minimum cardinality vertex set S with the property that for each vertex v its open neighborhood $N(v)$ has a unique non-empty intersection with S. But in Neighborhood resolving sets $N(v)$ may have the empty intersection with S. Clearly every OLD-set of a graph G is a neighborhood resolving set of G, but the converse need not be true.

M.G. Karpovsky, K. Chakrabarty, L.B. Levitin [15] introduced the concept of identifying sets using closed neighborhoods to resolve vertices of G. This concept was elaborately studied by A. Lobestein [16].

Let μ be a parameter of a graph. A vertex $v \in V(G)$ is said to be μ-good if v belongs to a μ-minimum (μ-maximum) set of G according as μ is a super hereditary (hereditary) parameter. v is said to be μ-bad if it is not μ-good. A graph G is said to be μ-excellent if every vertex of G is μ-good. Excellence with respect to domination and total domination were studied in [8], [12], [23],[24], [25], [26]. N. Sridharan and Yamuna [24], [25], [26], have defined various types of excellence.

A simple graph $G = (V, E)$ is nr- excellent if every vertex is contained in a nr-set of G. A graph G is said to be just nr-excellent if for each $u \in V$, there exists a unique nr-set of G containing u. This paper is devoted to this concept. In this paper, definition, examples and properties of just nr-excellent graphs is discussed.

§2. Neighborhood Resolving Sets in Graphs

Definition 2.1 *Let G be any graph. Let $S \subset V(G)$. Consider the k-tuple $(u_1, u_2, \cdots, u_k)$ where $S = \{u_1, u_2, \cdots, u_k\}$, $k \geq 1$. Let $v \in V(G)$. Define a binary neighborhood code of v with* *respect to the k-tuple* $(u_1, u_2, \cdots, u_k)$, *denoted by* $nc_S(v)$ *as a k-tuple* $(r_1, r_2, \cdots, r_k)$, *where*

$$r_i = \begin{cases} 1, & if\ v \in N(u_i), 1 \leq i \leq k \\ 0, & otherwise. \end{cases}$$

Then, S is called a neighborhood resolving set or a neighborhood r-set if $nc_S(u) \neq nc_S(v)$ *for any* $u, v \in V(G)$.

The least cardinality of a minimal neighborhood resloving set of G is called the neighborhood resolving number of G and is denoted by $nr(G)$. *The maximum cardinality of a minimal neighborhood resolving set of G is called the upper neighborhood resolving number of G and is denoted by* $NR(G)$.

Clearly $nr(G) \leq NR(G)$. *A neighborhood resolving set S of G is called a minimum neighborhood resolving set or nr-set if S is a neighborhood resolving set with cardinality* $nr(G)$.

Example 2.2 Let G be a graph shown in Fig.1.

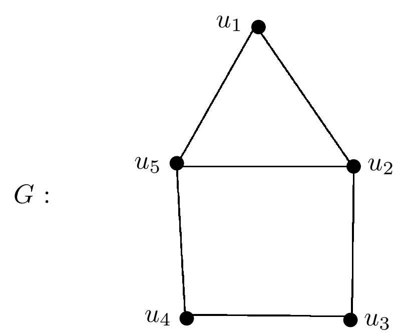

Fig.1

Then, $S_1 = \{u_1, u_2, u_5\}$ is a neighborhood resolving set of G since $nc_S(u_1) = (0, 1, 1)$, $nc_S(u_2) = (1, 0, 1)$, $nc_S(u_3) = (0, 1, 0)$, $nc_S(u_4) = (0, 0, 1)$ and $nc_S(u_5) = (1, 1, 0)$. Also $S_2 = \{u_1, u_3, u_4\}$, $S_3 = \{u_1, u_2, u_4\}$, $S_4 = \{u_1, u_3, u_5\}$ are neighborhood resolving sets of G. For this graph, $nr(G) = NR(G) = 3$.

Observation 2.3 The above definition holds good even if G is disconnected.

Theorem 2.4([31]) *Let G be a connected graph of order* $n \geq 3$. *Then G does not have any neighborhood resolving set if and only if there exist two non adjacent vertices u and v in* $V(G)$ *such that* $N(u) = N(v)$.

Definition 2.5([33]) *A subset S of* $V(G)$ *is called an nr-irredundant set of G if for every* $u \in S$, *there exist* $x, y \in V$ *which are privately resolved by u.*

Theorem 2.6([33]) *Every minimal neighborhood resolving set of G is a maximal neighborhood resolving irredundant set of G.*

Definition 2.7([33]) *The minimum cardinality of a maximal neighborhood resolving irredundant set of G is called the neighborhood resolving irredundance number of G and is denoted by* $ir_{nr}(G)$.

The maximum cardinality is called the upper neighborhood resolving irrundance number of G and is denoted by $IR_{nr}(G)$.

Observation 2.8([33]) For any graph G, $ir_{nr}(G) \leq nr(G) \leq NR(G) \leq IR_{nr}(G)$.

Theorem 2.9([34]) *For any graph G, $nr(G) \leq n1$.*

Theorem 2.10([32]) *Let G be a connected graph of order n such that $nr(G) = k$. Then $log_2 n \leq k$.*

Observation 2.11([32]) There exists a graph G in which $n = 2k$ and there exists a neighborhood resolving set of cardinality k such that $nr(G) = k$. Hence all the distinct binary k-vectors appear as codes for the n vertices.

Theorem 2.12([34]) *Let G be a connected graph of order n admitting neighborhood resolving sets of G and let $nr(G) = k$. Then k = 1 if and only if G is either K_2 or K_1.*

Theorem 2.13([34]) *Let G be a connected graph of order n admitting neighborhood resolving sets of G. Then $nr(G)$ = 2 if and only if G is either K_3 or K_3 + a pendant edge or $K_3 \cup K_1$ or $K_2 \cup K_1$.*

Definition 2.14([36]) *Let G = (V, E) be a simple graph. Let $u \in V(G)$. Then u is said to be nr-good if u is contained in a minimum neighborhood resolving set of G. A vertex u is said to be nr-bad if there exists no minimum neighborhood resolving set of G containing u.*

Definition 2.15([36]) *A graph G is said to be nr-excellent if every vertex of G is nr-good.*

Theorem 2.16([36]) *Let G be a non nr-excellent graph. Then G can be embedded in a nr-excellent graph (say) H such that $nr(H) = nr(G)$ + number of nr-bad vertices of G.*

Theorem 2.17([36]) *Let G be a connected non-nr-excellent graph. Let $\{u_1, u_2, \cdots, u_k\}$ be the set of all nr-bad vertices of G. Add vertices v_1, v_2, v_3, v_4 with V(G). Join v_i with v_j, $1 \leq i, j \leq 4$, $i \neq j$. Join u_i with v_1, $1 \leq i \leq k$. Let H be the resulting graph. Suppose there exists no nr-set T of H such that v_1 privately resolves nr-good vertices and nr-bad vertices of G. Then H is nr-excellent, G is an induced subgraph of H and $nr(H) = nr(G) + 3$.*

§3. Just nr-Excellent Graphs

Definition 3.1 *Let G = (V, E) be a simple graph. Let $u \in V(G)$. Then u is said to be nr-good if u is contained in a minimum neighborhood resolving set of G. A vertex u is said to be nr-bad if there exists no minimum neighborhood resolving set of G containing u.*

Definition 3.2 *A graph G is said to be nr-excellent if every vertex of G is nr-good.*

Definition 3.3 *A graph G is said to be just nr-excellent graph if for each $u \in V$, there exists a unique nr-set of G containing u.*

Example 3.4 Let $G = C_5 \Box K_2$.

G :

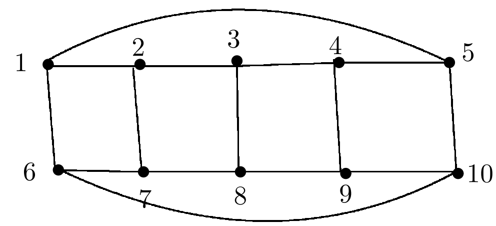

Fig.2

The only nr-sets of $C_5 \Box K_2$ are $\{1,2,3,4,5\}$ and $\{6,7,8,9,10\}$. Therefore, $C_5 \Box K_2$ is just nr-excellent.

Theorem 3.5 *Let G be a just nr-excellent graph. Then $deg(u) \geq \dfrac{n}{nr(G)} - 1$ for every u which does not have 0-code with respect to more than one nr-set S_i of G.*

Proof Let $V = S_1 \cup S_2 \cup \cdots \cup S_m$ be a partition of $V(G)$ into nr-sets of G. Let $x \in V(G)$. Suppose x does not have 0-code with respect to any S_i. Then x is adjacent to at least one vertex in each S_i. Therefore $deg(u) \geq m = \dfrac{n}{nr(G)}$.

Suppose x has 0-code with respect to exactly one nr-set (say) S_i. Then x is adjacent to at least one vertex in each S_j, $j \neq i$. $deg(u) \geq m - 1 = \dfrac{n}{nr(G)} - 1$. □

Note 3.6 These graphs $G1$ to $G72$ referred to the appendix of this paper.

Theorem 3.7 *If G is just nr-excellent, then $nr(G) \geq 4$.*

Proof Let G be just nr-excellent. If $nr(G) = 2$, then G is K_3 or K_3 + a pendant edge or $K_3 \cup K_1$ or $K_2 \cup K_1$. None of them is just nr-excellent.

Let $nr(G) = 3$. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a nr-partition of G. Suppose $k \geq 3$. Then $|V(G)| \geq 9$. But $|V(G)| \leq 2^{nr(G)} = 2^3 = 8$, a contradiction. Therefore $k \leq 2$. Suppose $k = 1$. Then $|V(G)| = 3 = nr(G)$, a contradiction since $nr(G) \leq |V(G)| - 1$. Therefore $k = 2$. Then $|V(G)| = 6$.

Now $\langle S_1 \rangle$, $\langle S_2 \rangle$ are one of graphs P_3 or $K_3 \cup K_1$ or K_3. Clearly $\langle S_1 \rangle$, $\langle S_2 \rangle$ cannot be P_3.

Case 1 $\langle S_1 \rangle = K_3 = \langle S_2 \rangle$.

Let $V(S_1) = \{u_1, u_2, u_3\}$ and $V(S_2) = \{v_1, v_2, v_3\}$. Since v_i has 0-code with respect to S_1, if there exists no edge between S_1 and S_2, there should be at least one edge between S_1 and S_2.

Subcase 1.1 Suppose u_i is adjacent with v_i, $1 \leq i \leq 3$. From $G1$, it is clear that $S = \{u_1, u_2, v_3\}$ is an nr-set of G, a contradiction since G is a just nr-excellent graph.

Subcase 1.2 Suppose u_i is adjacent with v_i for exactly two of the values from $i = 1, 2, 3$. Without loss of generality, let u_2 be adjacent with v_2 and u_3 be adjacent with v_3. Then in $G2$, it is clear that $S = \{u_1, u_2, v_3\}$ is an nr-set of G, a contradiction since G is a just nr-excellent graph. The other cases can be proved by similar reasoning.

Subcase 1.3 Suppose u_i is adjacent with v_i, $1 \le i \le 3$ and one or more u_i, $1 \le i \le 3$ are adjacent with every v_j, $1 \le j \le 3$. Let u_i be adjacent with v_i, $1 \le i \le 3$. If every u_i is adjacent with every v_j, $1 \le i, j \le 3$, then each v_i has the same code with respect to S_1, a contradiction.

Suppose exactly one u_i is adjacent with every v_j, $1 \le i, j \le 3$. Without loss of generality, let u_1 is adjacent with every v_j, $1 \le j \le 3$. Then v_2 and u_3 have the same code with respect to S_1, a contradiction. Suppose u_{i_1} and u_{i_2} are adjacent with every v_j, $1 \le i_1, i_2, j \le 3$, $i_1, \ne i_2$. Without loss of generality, let u_1 and u_2 are adjacent with every v_j, $1 \le j \le 3$, then v_1 and v_2 have the same code with respect to S_1, a contradiction in $G3$.

Subcase 1.4 Suppose u_i is adjacent with v_i for exactly two of the values of i, $1 \le i \le 3$ and for exactly one i, u_i is adjacent with every v_j, $1 \le j \le 3$. Without loss of generality let u_1 and u_2 be adjacent with v_1 and v_2 respectively. If u_1 is adjacent with v_1, v_2, v_3, then $nc_{S_1}(v_2) = nc_{S_1}(u_3)$, a contradiction. If u_2 is adjacent with v_1, v_2, v_3, then $nc_{S_1}(v_1) = nc_{S_1}(u_3)$, a contradiction. If u_3 is adjacent with v_1, v_2, v_3, then $nc_{S_1}(v_1) = nc_{S_1}(u_2)$, a contradiction in $G4$.

Subcase 1.5 Suppose u_i is adjacent with v_i, for every i, $1 \le i \le 3$ and one or more u_i are adjacent with exactly two of the vertices $\{v_1, v_2, v_3\}$. Suppose u_1 is adjacent with v_1, v_2 (u_2 may be adjacent with v_1, v_3 or u_3 may be adjacent with v_1, v_2). Then $nc_{S_1}(v_2) = nc_{S_1}(u_3)$, a contradiction in $G5$. The other cases can be proved similarly.

Subcase 1.6 Suppose u_i is adjacent with v_i for exactly two of the values of i, $1 \le i \le 3$, and one of the vertices which is adjacent with some v_i is also adjacent with exactly one v_j, $j \ne i$. If u_1 is adjacent with v_1, v_2; u_2 is adjacent with v_2, but u_3 is not adjacent with v_1, v_2, v_3, then $nc_{S_1}(v_2) = nc_{S_1}(u_3)$, a contradiction in $G6$. The other cases also lead to contradiction.

Subcase 1.7 Suppose exactly one u_i is adjacent with v_i, $1 \le i \le 3$ (say) u_1 is adjacent with v_1. If u_1 is not adjacent with v_2, v_3, then v_2 and v_3 receive 0-code with respect to S_1, a contradiction. If u_1 is adjacent with v_2 and not with v_3, then v_1 and v_2 receive the same code with respect to S_1, a contradiction. If u_1 is adjacent with v_1, v_2 and v_3 then v_1, v_2 and v_3 receive the same code with respect to S_1, a contradiction in $G7$. The other cases can be similarly proved. Since $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ form cycles, any other case of adjacency between S_1 and S_2 will fall in one of the seven cases discussed above. Hence when $k = 2$ and $\langle S_1 \rangle = \langle S_2 \rangle = K_3$, then G is not just nr-excellent.

Case 2 $\langle S_1 \rangle = K_3$ and $\langle S_2 \rangle = K_2 \cup K_1$.

Let $V(S_1) = \{u_1, u_2, u_3\}$ and $V(S_2) = \{v_1, v_2, v_3\}$. Let v_1 and v_2 be adjacent. Since G is connected, v_3 is adjacent with some u_i. Since the argument in Case 1 does not depend on the nature of $\langle S_2 \rangle$, we get that G is not just nr-excellent.

Case 3 $\langle S_1 \rangle = \langle S_2 \rangle = K_2 \cup K_1$.

Let $V(S_1) = \{u_1, u_2, u_3\}$ and $V(S_2) = \{v_1, v_2, v_3\}$. Without loss of generality, let u_1 be adjacent with u_2 and v_1 be adjacent with v_2.

Subcase 3.1 Suppose u_i is adjacent with v_i, $1 \leq i \leq 3$. Then G is disconnected, a contradiction, since G is just nr-excellent.

Subcase 3.2 Suppose u_i is adjacent with v_i for exactly two of the values from $i = 1, 2, 3$. Then G is disconnected, a contradiction since G is just nr-excellent.

Subcase 3.3 Suppose u_i is adjacent with v_i, $1 \leq i \leq 3$ and one or more u_i, $1 \leq i \leq 3$ are adjacent with every v_j, $1 \leq j \leq 3$. If every u_i is adjacent with every v_j, $1 \leq i, j \leq 3$, then each v_j has the same code with respect to S_1, a contradiction, since S_1 is an nr-set of G in $G8$.

If u_1 and u_2 are adjacent with every v_j, $1 \leq j \leq 3$, v_1 and v_2 have the same code with respect to S_1, a contradiction, since S_1 is an nr-set of G ($G9$).

If $u_i (i = 1, 2)$ and u_3 are adjacent with every v_j, $1 \leq j \leq 3$, then v_i and v_3 have the same code with respect to S_1, a contradiction, since S_1 is an nr-set of G ($G10$).

Subcase 3.4 Suppose u_i is adjacent with v_i for exactly two of the values i, $1 \leq i \leq 3$ and for exactly one i, u_i, $1 \leq i \leq 3$ are adjacent with every v_j, $1 \leq j \leq 3$.

Subcase 3.4.1 Suppose u_1 is adjacent with v_1 and u_2 is adjacent with v_2. If u_1 or u_2 is adjacent with every v_j, $1 \leq j \leq 3$, then G is disconnected, a contradiction, since G is just nr-excellent. If u_3 is adjacent with every v_j, $1 \leq j \leq 3$, then $nc_{S_2}(v_1) = nc_{S_2}(u_2)$, a contradiction since S_2 is an nr-set of G in $G11$.

Subcase 3.4.2 Suppose u_1 is adjacent with v_1 and u_3 is adjacent with v_3. If u_1 or u_3 is adjacent with every v_j, $1 \leq j \leq 3$, then $nc_{S_1}(v_1) = nc_{S_1}(u_2)$, a contradiction, S_1 is an nr-set of G ($G1$).

If u_2 is adjacent with every v_j, $1 \leq j \leq 3$, then $nc_{S_1}(v_2) = nc_{S_1}(u_1)$, a contradiction, S_1 is an nr-set of G ($G13$). The other cases can be similarly proved.

Subcase 3.5 Suppose u_i is adjacent with v_i, for every i, $1 \leq i \leq 3$ and one or more u_i are adjacent with exactly two of the vertices $\{v_1, v_2, v_3\}$. Let u_i is adjacent with v_i, for every i, $1 \leq i \leq 3$.

Subcase 3.5.1 Suppose u_1 is adjacent with v_1 and v_2 or u_1 and u_2 are adjacent with v_1 and v_2. Then G is disconnected, a contradiction, G is just nr-excellent.

Subcase 3.5.2 Suppose u_1 is adjacent with v_1 and v_2, u_i, $i = 2, 3$ are adjacent with v_2 and v_3 ($G14$), or u_1 is adjacent with v_1 and v_2, u_2, u_3 are adjacent with v_2 and v_3 ($G15$), or u_1 is adjacent with v_2 and v_3, u_2 is adjacent with v_2 and v_3 ($G16$), or u_1 is adjacent with v_2 and v_3, u_2, u_3 are adjacent with v_2 and v_3 ($G17$), or u_1 is adjacent with v_1, v_3, u_2 is adjacent with v_2, v_3 ($G18$), or u_1 is adjacent with v_1, v_3, u_2, u_3 are adjacent with v_2, v_3 ($G19$). Then $nc_{S_1}(v_1) = nc_{S_1}(u_2)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.5.3 Suppose u_1 is adjacent with v_1, v_2, u_2 is adjacent with v_1, v_3 ($G20$), or u_1, u_2, u_3 are adjacent with v_1, v_2 ($G21$), or u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_1, v_2 ($G22$), or u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_1, v_3 ($G23$), or u_1 is adjacent with

v_2, v_3, u_2, u_3 are adjacent with v_1, v_2 ($G24$). Then $nc_{S_1}(v_1) = nc_{S_1}(v_2)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.5.4 Suppose u_1 is adjacent with v_i, v_3, $i = 1, 2$, u_2 is adjacent with v_1, v_3, u_3 is adjacent with v_1, v_2 ($G25$). Then $nc_{S_1}(v_1) = nc_{S_1}(v_3)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.5.5 Suppose u_1, u_2 is adjacent with v_1, v_2, u_3 is adjacent with v_i, v_3, $i = 1, 2$ ($G26$), or u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_1, v_3, u_3 is adjacent with v_i, v_3, $i = 1, 2$ ($G27$). Then $nc_{S_2}(u_1) = nc_{S_2}(u_2)$, a contradiction since S_2 is an nr-set of G.

Subcase 3.5.6 Suppose u_1, u_2 is adjacent with v_2, v_3; u_3 is adjacent with v_1, v_2 ($G28$). Then $nc_{S_1}(v_2) = nc_{S_1}(v_3)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.5.7 Suppose u_1 is adjacent with v_1, v_3; u_2 is adjacent with v_1, v_2 ($G29$), or u_1, u_2 are adjacent with v_1, v_3 ($G30$), or u_1, u_3 are adjacent with v_1, v_3, u_2 is adjacent with v_i, v_j, $1 \leq i,\ j \leq 3, i \neq j$ ($G31$). Then $nc_{S_1}(v_2) = nc_{S_1}(u_1)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.5.8 Suppose u_1 is adjacent with only v_1 and not with v_2 and v_3 ($G32$). Then $nc_{S_2}(v_2) = nc_{S_2}(u_1)$, a contradiction since S_2 is an nr-set of G.

Subcase 3.5.9 Suppose u_2 is adjacent with only v_2 and not with v_1 and v_2 ($G33$). Then $nc_{S_2}(u_2) = nc_{S_2}(v_1)$, a contradiction since S_2 is an nr-set of G.

Subcase 3.5.10 Suppose u_1 is adjacent with only v_1, v_2, u_2 is adjacent with only v_2, v_3, u_3 is adjacent with only v_1, v_i, $i = 2, 3$ ($G34$), or u_1 is adjacent with only v_1, v_2, u_2 is adjacent with only v_1, v_3, u_3 is adjacent with only v_i, v_j, $1 \leq i, j \leq 3$, $i \neq j$ ($G35$). Then $S = \{u_1, v_1, v_3\}$ is an nr-set of G, a contradiction since G is just nr-excellent.

Subcase 3.5.11 Suppose u_1 is adjacent with only v_2, v_3, u_2 is adjacent with only v_1, v_2, u_3 is adjacent with only v_2, v_3 ($G36$). Then $S = \{u_1, u_3, v_1\}$ is an nr-set of G, a contradiction since G is just nr-excellent.

Subcase 3.5.12 Suppose u_1 is adjacent with only v_2, v_3, u_3 is adjacent with only v_1, v_3, u_2 is adjacent with only $v_2, v_i, i = 1, 3$ ($G37$). Then $S = \{u_1, u_3, v_2\}$ is an nr-set of G, a contradiction since G is just nr-excellent.

Subcase 3.5.13 Suppose u_1 is adjacent with only v_1, v_3, u_2 is adjacent with only v_1, v_2, u_3 is adjacent with only v_2, v_i, $i = 1, 3$ ($G38$). Then $S = \{u_2, v_1, v_3\}$ is an nr-set of G, a contradiction since G is just nr-excellent.

Subcase 3.5.14 Suppose u_1 is adjacent with only v_1, v_3, u_2 is adjacent with only v_2, v_3, u_3 is adjacent with only v_1, v_2 (for fig.39). Then $S = \{u_3, v_1, v_2\}$ is an nr-set of G, a contradiction since G is just nr-excellent.

Subcase 3.5.15 Suppose u_1, u_2 are adjacent with only v_1, v_3; u_3 is adjacent with only v_2, v_3 ($G40$). Then $S = \{u_2, u_3, v_1\}$ is an nr-set of G, a contradiction since G is just nr-excellent.

Subcase 3.6 Suppose u_i is adjacent with v_i for exactly two of the values of i, $1 \leq i \leq 3$.

Subcase 3.6.1 Let u_1 be adjacent with v_1 and u_3 be adjacent with v_3 $(G41-G56)$.

Subcase 3.6.1.1 Suppose u_1, u_3 are adjacent with v_1, v_2 $(G41)$. Then $nc_{S_1}(v_1) = nc_{S_1}(v_2)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.6.1.2 Suppose u_1, u_2 are adjacent with v_1, v_3, u_3 is adjacent with v_2, v_3 $(G42)$, or u_1, u_2 are adjacent with v_1, v_3 $(G43)$. Then $nc_{S_2}(u_1) = nc_{S_2}(u_2)$, a contradiction since S_2 is an nr-set of G.

Subcase 3.6.1.3 Suppose u_1 is adjacent with v_1, v_2, u_3 is adjacent with v_2, v_3 $(G44)$, or u_1 is adjacent with v_i, v_3 $(i = 1, 2)$ (45), or u_1 is adjacent with v_i, v_3 $(i = 1, 2)$, u_3 is adjacent with v_2, v_3 $(G46)$. Then $nc_{S_1}(u_2) = nc_{S_1}(v_1)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.6.1.4 Suppose u_1 is adjacent with only v_1, v_2, u_2 is adjacent with only v_1, v_3, u_3 is adjacent with only v_i, v_2, $i = 1, 3$ $(G47)$, or u_1 is adjacent with only v_2, v_3, u_2 is adjacent with only v_1, v_3, u_3 is adjacent with only v_2, v_3 $(G48)$. Then $S = \{u_1, v_1, v_3\}$ is an nr-set of G, a contradiction since G is just nr-excellent.

Subcase 3.6.1.5 Suppose u_1 is adjacent with v_1, v_2, u_2 is adjacent with v_1, v_3 $(G49)$, or u_1 is adjacent with v_1, v_2, u_2, u_3 are adjacent with v_1, v_3 $(G50)$, or u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_1, v_3 $(G51)$, or u_1 is adjacent with v_2, v_3, u_3 is adjacent with v_1, v_3 $(G52)$. Then $nc_{S_1}(v_2) = nc_{S_1}(u_2)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.6.1.6 Suppose u_1 is adjacent with v_2, v_3, u_3 is adjacent with v_1, v_2 $(G53)$, or u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_1, v_3, u_3 is adjacent with v_1, v_i $(i = 2, 3)$ $(G54)$, or u_1 is adjacent with v_1, v_3, u_3 is adjacent with v_1, v_i $i = 2, 3$ $(G55)$, or u_1, u_2 is adjacent with v_1, v_3, u_3 is adjacent with v_1, v_i $i = 2, 3$ $(G56)$. Then $nc_{S_1}(v_1) = nc_{S_1}(v_3)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.6.2 Let u_1 be adjacent with v_1 and u_2 be adjacent with v_2 $(G57-G64)$.

Subcase 3.6.2.1 Suppose u_1, u_3 are adjacent with v_1, v_2, u_2 is adjacent with v_2, v_3 $(G57)$. Then $nc_{S_2}(u_1) = nc_{S_2}(u_3)$, a contradiction since S_2 is an nr-set of G.

Subcase 3.6.2.2 Suppose u_1 is adjacent with v_1, v_3; u_2, u_3 are adjacent with v_1, v_2 $(G58)$. Then $nc_{S_2}(u_2) = nc_{S_2}(u_3)$, a contradiction since S_2 is an nr-set of G.

Subcase 3.6.2.3 Suppose u_1 is adjacent with only v_i, v_3, $i = 1, 2$, u_2 is adjacent with only v_2, v_3, u_3 is adjacent with only v_1, v_2 $(G59)$. Then $S = \{u_2, u_3, v_1\}$ is an nr-set of G, a contradiction, since G is just nr-excellent.

Subcase 3.6.2.4 Suppose u_1, u_2 are adjacent with only v_1, v_3, u_3 is adjacent with only v_1, v_2 $(G60)$. Then $S = \{u_1, v_2, v_3\}$ is an nr-set of G, a contradiction, since G is just nr-excellent.

Subcase 3.6.2.5 Suppose u_1, u_3 are adjacent with v_1, v_2, u_2 is adjacent with v_1, v_3 $(G61)$, or u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_1, v_i $(i = 2, 3)$, u_3 is adjacent with v_1, v_2 $(G62)$. Then $nc_{S_1}(v_1) = nc_{S_1}(v_2)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.6.2.6 Suppose u_1 is adjacent with v_1, v_2, u_3 is adjacent with v_1, v_3 $(G63)$, or u_1 is adjacent with v_i, v_3, $i = 1, 2$, u_3 is adjacent with v_1, v_2 $(G64)$. Then $nc_{S_2}(u_2) = nc_{S_2}(v_1)$, a contradiction since S_2 is an nr-set of G. The other instances can be similarly argued.

Subcase 3.6.3 Suppose exactly one u_i is adjacent with v_i, $1 \leq i \leq 3$.

Subcase 3.6.3.1 If u_1 is adjacent with v_1, u_2 is adjacent with v_3, u_3 is adjacent with v_i, $i = 1, 2$, or u_3 is adjacent with v_1, v_2, or u_1 is adjacent with v_1, u_2 is adjacent with v_1, v_3, u_3 is adjacent with v_i, $i = 1, 2$, or u_3 is adjacent with v_1, v_2, or u_1 is adjacent with v_1, v_2, u_2 is adjacent with v_3, u_3 is adjacent with v_i, $i = 1, 2$, or u_3 is adjacent with v_1, v_2, or u_1 is adjacent with v_1, v_2, u_2 is adjacent with v_1, v_3, u_3 is adjacent with v_i, $i = 1, 2$, or u_3 is adjacent with v_1, v_2 ($G65$). Then $nc_{S_1}(v_3) = nc_{S_1}(u_1)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.6.3.2 If u_1 is adjacent with v_1, v_3, u_2 is adjacent with v_3, or u_2 is adjacent with v_1, v_3, u_3 is adjacent with v_2, or If u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_3, or u_2 is adjacent with v_1, v_3), u_3 is adjacent with v_2 ($G66$). Then $nc_{S_2}(u_3) = nc_{S_2}(v_1)$, a contradiction since S_2 is an nr-set of G.

Subcase 3.6.3.3 If u_1 is adjacent with v_1, v_3, u_3 is adjacent with v_i, $i = 1, 2$, or u_1 is adjacent with v_1, v_3, u_2 is adjacent with v_1, u_3 is adjacent with v_i, $i = 1, 2$, or u_1 is adjacent with v_1, v_3, u_3 is adjacent with v_1, v_2, or u_1 is adjacent with v_1, v_3, u_2 is adjacent with v_1, u_3 is adjacent with v_1, v_2, or u_1 is adjacent with v_2, v_3, u_3 is adjacent with v_i, $i = 1, 2$, or u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_1 u_3 is adjacent with v_i, $i = 1, 2$, or u_1 is adjacent with v_2, v_3, u_3 is adjacent with v_1, v_2, or u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_1, u_3 is adjacent with v_1, v_2 ($G67$). Then $nc_{S_1}(v_3) = nc_{S_1}(u_2)$, a contradiction, since S_1 is an nr-set of G.

Subcase 3.6.3.4 If u_1 is adjacent with v_1, v_3, u_2 is adjacent with v_3, or u_2 is adjacent with v_1, v_3, u_3 is adjacent with v_1, or if u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_3, or)u_2 is adjacent with v_1, v_3), u_3 is adjacent with v_1 ($G68$). Then $nc_{S_2}(u_3) = nc_{S_2}(v_2)$, a contradiction since S_2 is an nr-set of G.

Subcase 3.6.3.5 If u_1, u_2 are adjacent with v_1, v_3, u_3 is adjacent with v_1, v_2 ($G69$). Then $nc_{S_2}(u_1) = nc_{S_2}(u_2)$, a contradiction since S_2 is an nr-set of G.

Subcase 3.6.3.6 If u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_3, u_3 is adjacent with v_1, v_2 ($G70$). Then $nc_{S_1}(v_1) = nc_{S_1}(v_2)$, a contradiction since S_1 is an nr-set of G.

Subcase 3.6.3.7 If u_1 is adjacent with v_1, v_3, u_2 is adjacent with v_3, u_3 is adjacent with v_1, v_2 ($G71$). Then $S = \{u_1, u_2, v_2\}$ is an nr-set of G, a contradiction since G is just nr-excellent.

Subcase 3.6.3.8 If u_1 is adjacent with v_2, v_3, u_2 is adjacent with v_1, v_3, u_3 is adjacent with v_1, v_2 ($G72$). Then $S = \{u_2, u_3, v_2\}$ is an nr-set of G, a contradiction since G is just nr-excellent. The other instances can be similarly argued. Hence, if G is just nr-excellent, then $nr(G) \geq 4$. □

Theorem 3.8 *Every just nr-excellent graph G is connected.*

Proof If G is not connected, all the connected components of G contains more than one vertex (since $G \cup K_1$ is not a nr-excellent graph). Let G_1 be one of the component of G. As G_1 is also just nr-excellent, and $nr(G_1) \leq \dfrac{|G_1|}{2}$, G_1 has more than one nr-set. Select two nr-sets say S_1 and S_2 of G_1. Fix one nr-set D for $G - G_1$. Then both $D \cup S_1$ and $D \cup S_2$ are nr-sets of G, which is a contradiction, since G is just nr-excellent. Hence every just nr-excellent graph is connected. □

Theorem 3.9 *The graph G of order n is just nr-excellent if and only if*

(1) *$nr(G)$ divides n;*
(2) *$d_{nr}(G) = \dfrac{n}{nr(G)}$;*
(3) *G has exactly $\dfrac{n}{nr(G)}$ distinct nr-sets.*

Proof Let G be just nr-excellent. Let $S_1, S_2, \cdots, S_m$ be the collection of distinct nr-sets of G. Since G is just nr-excellent these sets are pairwise disjoint and their union is $V(G)$. Therefore $V = S_1 \cup S_2 \cup \cdots \cup S_m$ is a partition of V into nr-sets of G.

Since $|S_i| = nr(G)$, for every $i = 1, 2, \cdots, m$ we have neighborhood resolving partition number of $G = d_{nr}(G) = m$ and $nr(G)m = n$.

Therefore both $nr(G)$ and $d_{nr}(G)$ are divisors of n and $d_{nr}(G) = \frac{n}{nr(G)}$. Also G has exactly $m = \dfrac{n}{nr(G)}$ distinct nr-sets.

Conversely, assume G to be a graph satisfying the hypothesis of the theorem. Let $m = \dfrac{n}{nr(G)}$. Let $V = S_1 \cup S_2 \cup \cdots \cup S_m$ be a decomposition of neighborhood resolving sets of G. Now as $nr(G)m = n = \sum\limits_{i=1}^{m} |S_i| \geq m.nr(G)$, for each i, S_i is an nr-set of G. Since it is given that G has exactly m distinct nr-sets, $S_1, S_2, \cdots, S_m$ are the distinct nr-sets of G.

$V = S_1 \cup S_2 \cup \cdots \cup S_m$ is a partition and hence each vertex of V belongs to exactly one S_i. Hence G is just nr-excellent. □

Theorem 3.10 *Let G be a just nr-excellent graph. Then $\delta(G) \geq 2$.*

Proof Suppose there exists a vertex $u \in V(G)$ such that $deg(u) = 1$. Let v be the support vertex of u. Let $S_1, S_2, \cdots, S_m$ be the nr-partition of G.

Case 1 Let $u \in S_1$ and $v \notin S_1$. Suppose u resolves u and v only. Then $(S_1 - \{u\}) \cup \{v\}$ is an nr-set of G, a contradiction. Suppose u resolves privately and uniquely v and y for some $y \in V(G)$.

Subcase 1.1 v and y are non-adjacent.

Since $v \in S_i$, $i \neq 1$ and S_i is an nr-set of G, there exists some $z \in S_i$ such that z resolves v and y. Further $x_1, x_2 \in V(G)$ where $x_1, x_2 \neq u$, are resolved by the vertices of $S_1 - \{u\}$. Therefore $(S_1 - \{u\}) \cup \{z\}$ is a neighborhood resolving set of G. Since $|(S_1 - \{u\}) \cup \{z\}| = |S_1|$, $(S_1 - \{u\}) \cup \{z\}$ is an nr-set of G, a contradiction to G is just nr-excellent.

Subcase 1.2 v and y are adjacent.

Then $(S_1 - \{u\}) \cup \{v\}$ is an nr-set of G, a contradiction.

Case 2 Suppose $u, v \in S_i$ for some S_i, $1 \leq i \leq m$. Without loss of generality, let $u, v \in S_1$.

Subcase 2.1 Suppose u resolves u and v only. Let $S_1^1 = S_1 - \{u\}$. Suppose there exists a vertex w in S_1 such that w and v have 0-code with respect to $S_1 - \{u\}$. Then u resolves v and

w in S_1, a contradiction, since u resolves u and v only. So v does not have 0-code with respect to S_1^1. Therefore S_1^1 is a neighborhood resolving set, a contradiction.

Subcase 2.2 Suppose u resolves privately and uniquely v and y for some $y \in V(G)$. If v and y are adjacent, then v resolves v and y, a contradiction, since u resolves privately v and y. Therefore v and y are non-adjacent.

Since S_i, $i \neq 1$, is an nr-set of G, there exists a vertex $z \in S_i$, such that z resolves v and y. Consider $S_1^{11} = (S_1 - \{u\}) \cup \{z\}$. Suppose there exists a vertex w in S_1 whose code is zero with respect to $S_1 - \{u\}$ and v also has 0-code with respect to $S_1 - \{u\}$. If $y \neq w$, then u resolves v and w in S_1, a contradiction, since u resolves v and y uniquely. Therefore $y = w$. That is y receives 0-code with respect to $S_1 - \{u\}$.

Since z resolves v and y with respect to S_1, z is either adjacent to v or adjacent to y. If z is adjacent to y, then v receives 0-code with respect to S_1^{11}. $x, y \in V(G)$ where $x, y \neq u$, are resolved by the vertices of $S_1 - \{u\}$. Therefore, S_1^{11} is a neighborhood resolving set of G. Since $|S_1^{11}| = |S_1|$, S_1^{11} is an nr-set of G, a contradiction, since G is just nr-excellent. If z is adjacent to v, then z is not adjacent to y. Then y receives 0-code with respect to S_1^{11}. Arguing as before we get a contradiction. Consequently, $\delta(G) \geq 2$. □

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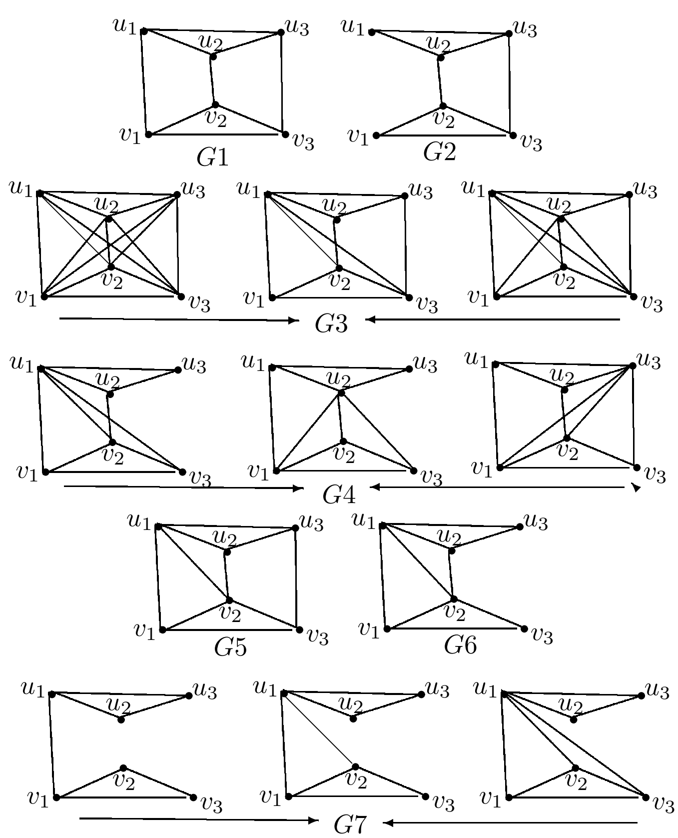
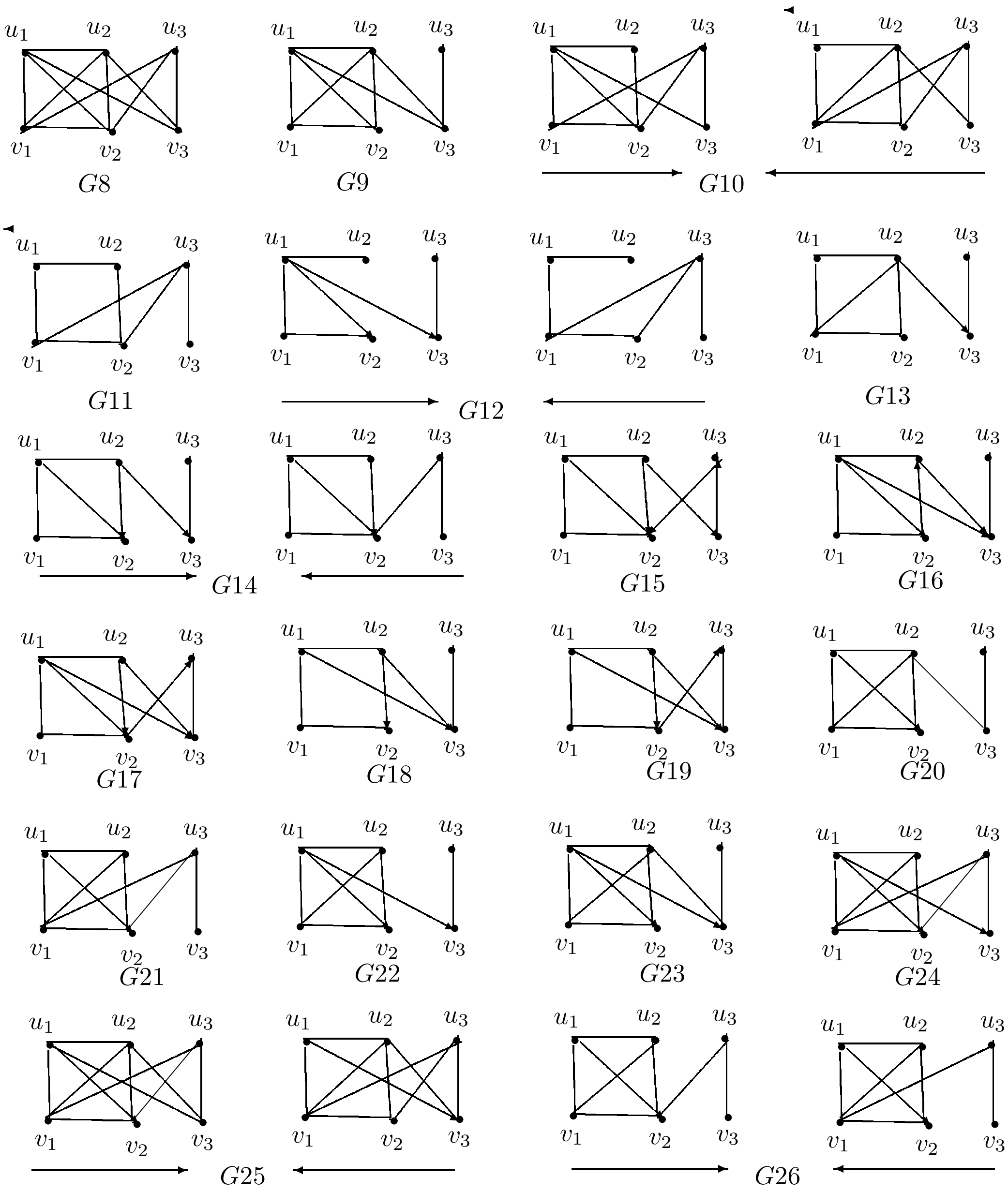
Appendix: Graphs $G1 - G72$

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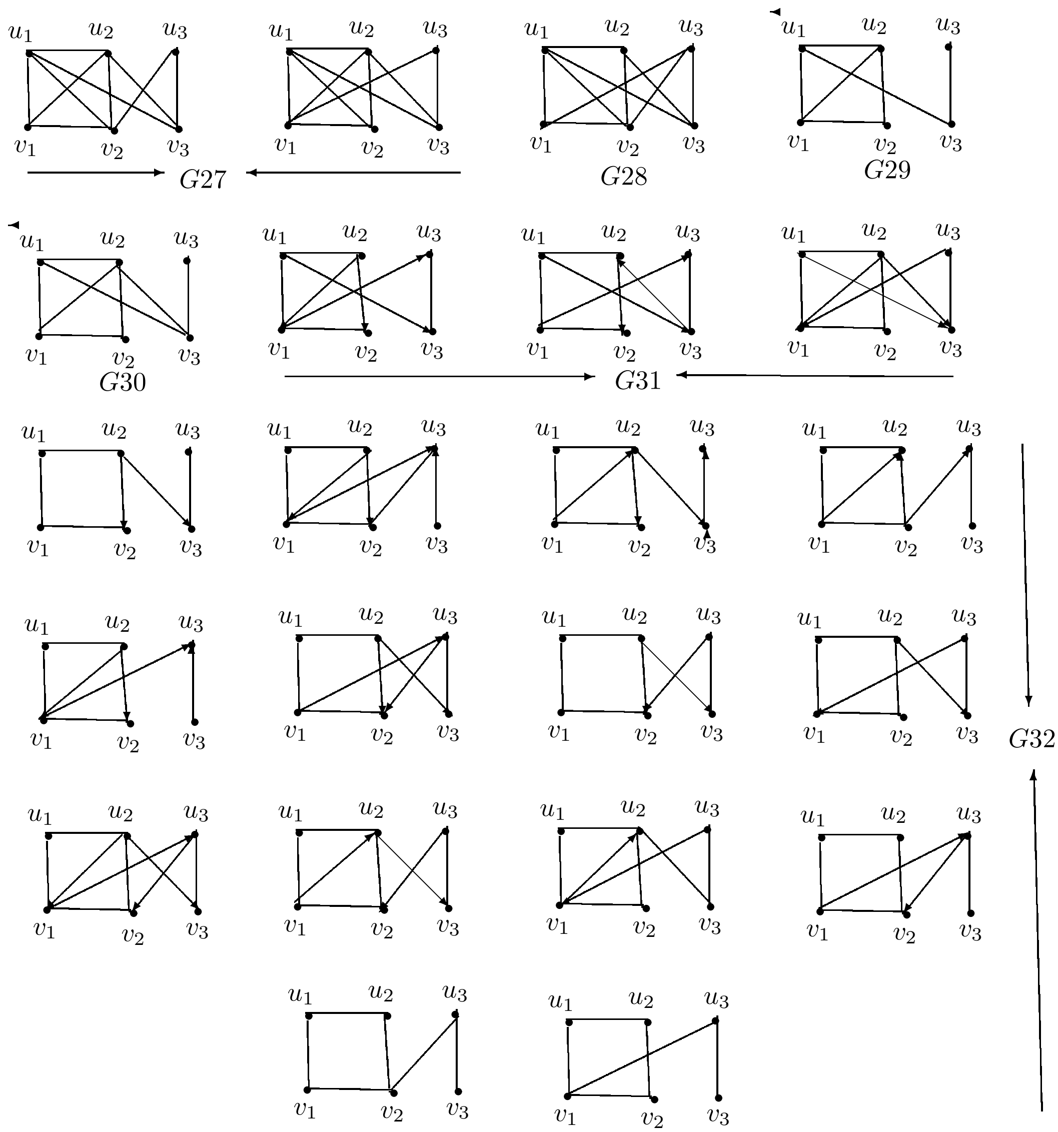
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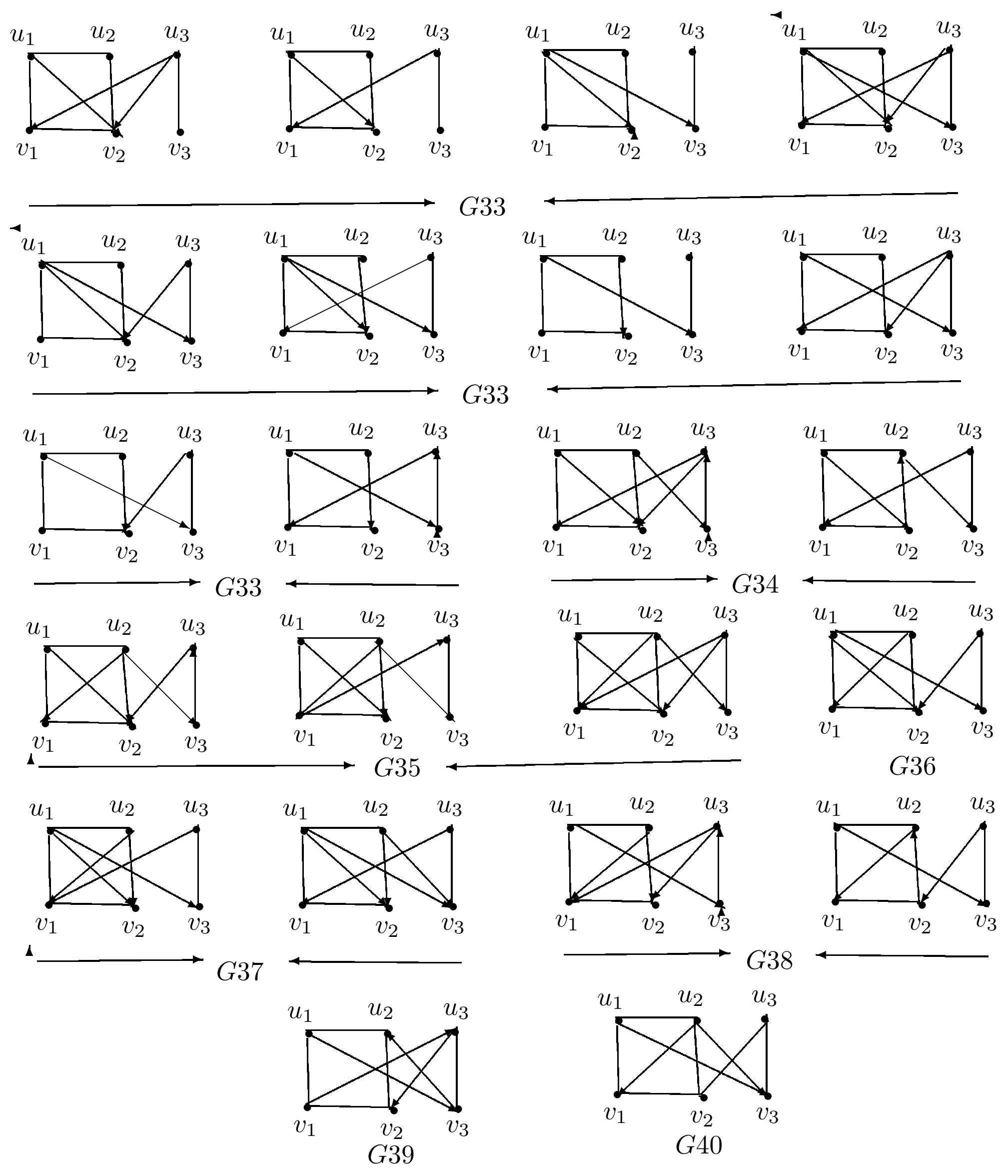
u_1 u_2 u_3
v_1 v_2 v_3
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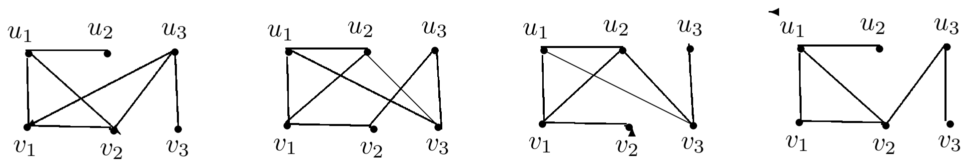
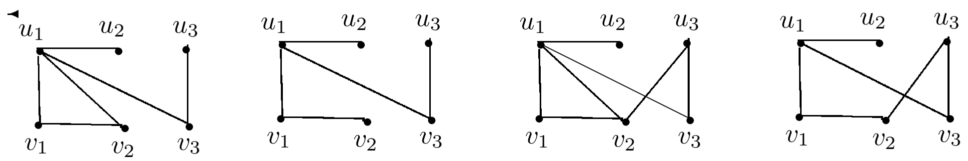
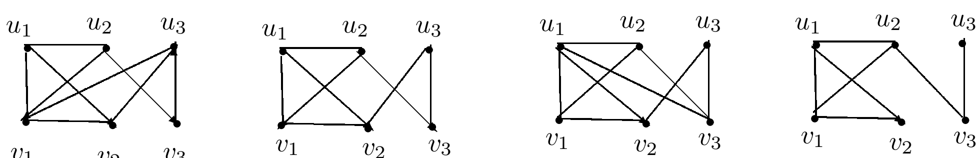
$G35$
$G36$
$G37$
$G38$
$G39$
$G40$



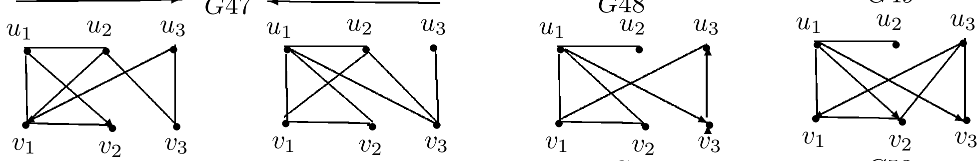
$G41$ $G42$ $G43$ $G44$



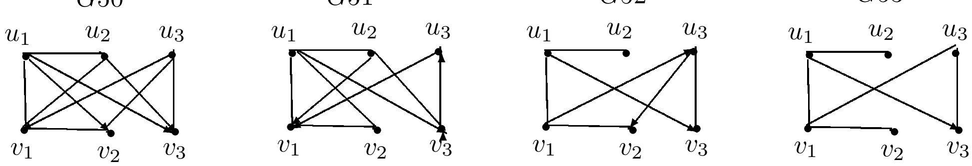
$\longrightarrow G45 \longleftarrow$ $\longrightarrow G46 \longleftarrow$



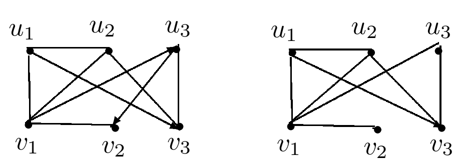
$\longrightarrow G47 \longleftarrow$ $G48$ $G49$



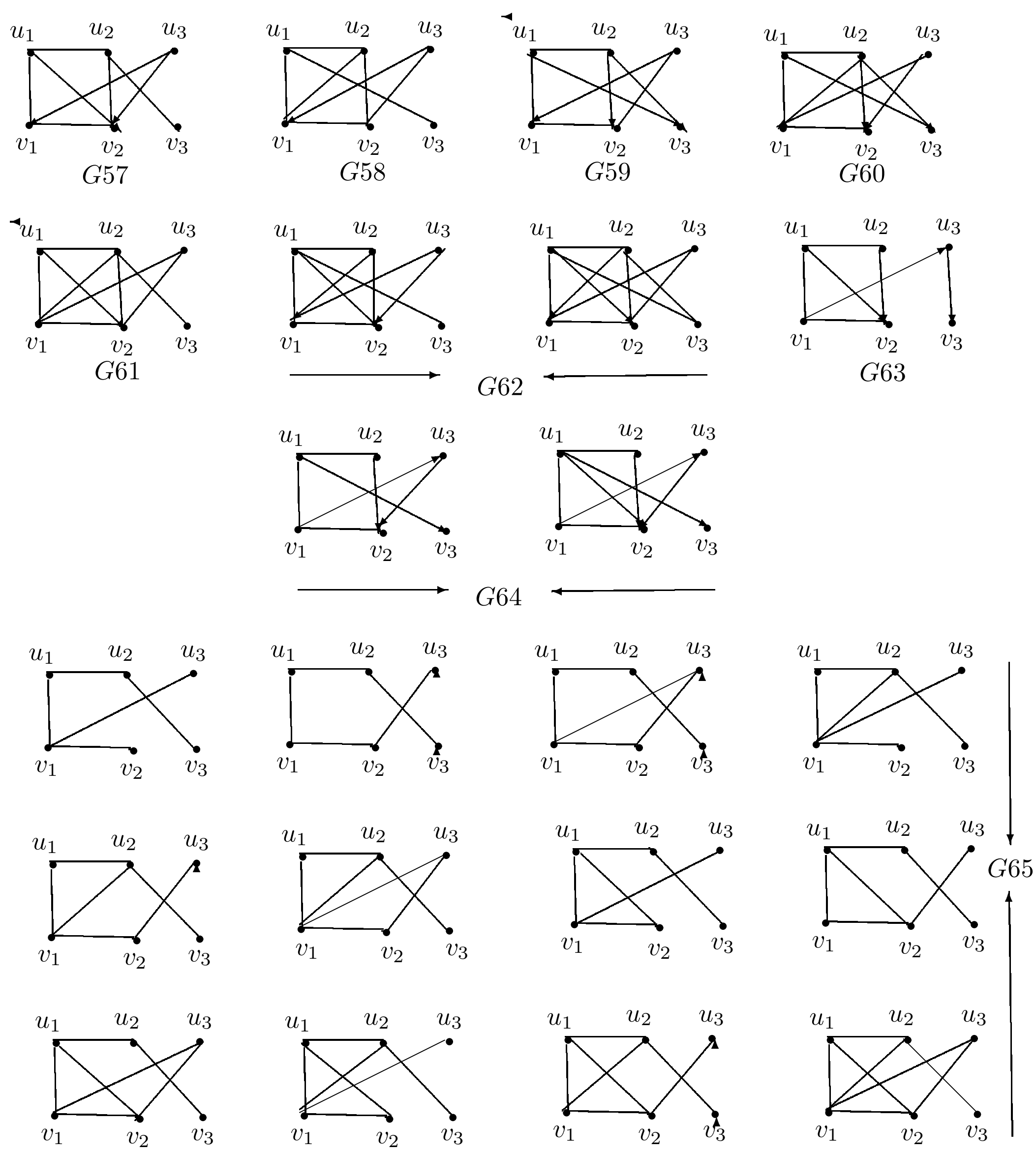
$G50$ $G51$ $G52$ $G53$



$\longrightarrow G54 \longleftarrow$ $\longrightarrow G55 \longleftarrow$



$\longrightarrow G56 \longleftarrow$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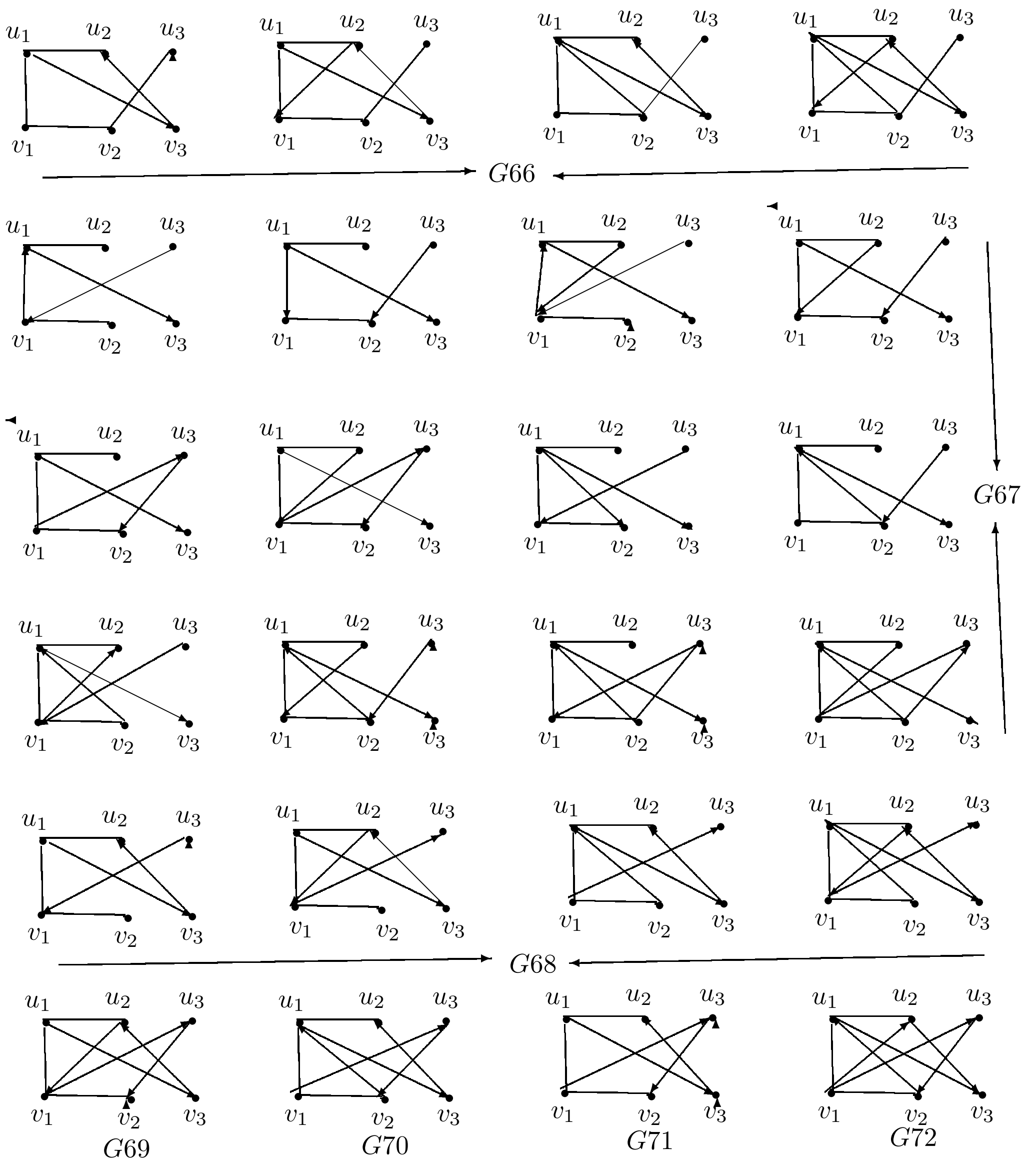
G65



u_1 u_2 u_3
v_1 v_2 v_3
$G66$
$G67$
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$G69$
$G70$
$G71$
$G72$

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Total Dominator Colorings in Caterpillars

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Abstract: Let G be a graph without isolated vertices. A total dominator coloring of a graph G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we determine the total dominator chromatic number in caterpillars.

Key Words: Total domination number, chromatic number and total dominator chromatic number, Smarandachely k-dominator coloring, Smarandachely k-dominator chromatic number.

AMS(2010): 05C15, 05C69

§1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [4].

Let $G = (V, E)$ be a graph of order n with minimum degree at least one. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\bigcup\limits_{v \in S} N(v)$, and the closed neighborhood of S is $N[S] = N(S) \cup S$.

A subset S of V is called a total dominating set if every vertex in V is adjacent to some vertex in S. A total dominating set is minimal total dominating set if no proper subset of S is a total dominating set of G. The total domination number γ_t is the minimum cardinality taken over all minimal total dominating sets of G. A γ_t-set is any minimal total dominating set with cardinality γ_t.

A proper coloring of G is an assignment of colors to the vertices of G, such that adjacent vertices have different colors.

The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$. Let $V = \{u_1, u_2, u_3, \cdots, u_p\}$ and $C = \{C_1, C_2, C_3, \cdots, C_n\}$, $n \leqslant p$ be a collection of subsets $C_i \subset V$. A color represented in a vertex u is called a non-repeated color if there exists one color class $C_i \in C$ such that $C_i = \{u\}$.

A vertex v of degree 1 is called an end vertex or a pendant vertex of G and any vertex

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which is adjacent to a pendant vertex is called a support.

A caterpillar is a tree with the additional property that the removal of all pendant vertices leaves a path. This path is called the spine of the caterpillar, and the vertices of the spine are called vertebrae. A vertebra which is not a support is called a zero string. In a caterpillar, consider the consecutive i zero string, called zero string of length i. A caterpillar which has no zero string of length at least 2 is said to be of class 1 and all other caterpillars are of class 2.

Let G be a graph without isolated vertices. For an integer $k \geqslant 1$, a Smarandachely k-dominator coloring of G is a proper coloring of G with the extra property that every vertex in G properly dominates a k-color classes and the smallest number of colors for which there exists a Smarandachely k-dominator coloring of G is called the Smarandachely k-dominator chromatic number of G and is denoted by $\chi_{td}^S(G)$. Let G be a graph without isolated vertices. A total dominator coloring of a graph G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we determine total dominator chromatic number in caterpillars.

Throughout this paper, we use the following notations.

Notation 1.1. Usually, the vertices of P_n are denoted by $u_1, u_2, \cdots, u_n$ in order. For $i < j$, we use the notation $\langle [i,j] \rangle$ for the sub path induced by $\langle u_i, u_{i+1}, \cdots, u_j \rangle$. For a given coloring C of P_n , $C/\langle [i,j] \rangle$ refers to the coloring C restricted to $\langle [i,j] \rangle$.

We have the following theorem from [1].

Theorem 1.2([1]) *Let G be any graph with $\delta(G) \geqslant 1$. Then* $\max\{\chi(G), \gamma_t(G)\} \leqslant \chi_{td}(G) \leqslant \chi(G) + \gamma_t(G)$.

From Theorem 1.2, $\chi_{td}(P_n) \in \{\gamma_t(P_n), \gamma_t(P_n)+1, \gamma_t(P_n)+2\}$. We call the integer n, good (respectively bad, very bad) if $\chi_{td}(P_n) = \gamma_t(P_n)+2$ (if respectively $\chi_{td}(P_n) = \gamma_t(P_n)+1, \chi_{td}(P_n) = \gamma_t(P_n)$). First, we prove a result which shows that for large values of n, the behavior of $\chi_{td}(P_n)$ depends only on the residue class of n mod 4 [More precisely, if n is good, $m > n$ and $m \equiv n (\text{mod } 4)$ then m is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their respective residue classes. This therefore classifies the good integers.

Fact 1.3 Let $1 < i < n$ and let C be a td-coloring of P_n. Then, if either u_i has a repeated color or u_{i+2} has a non-repeated color, $C/\langle [i+1, n] \rangle$ is also a td-coloring.

Theorem 1.4([2]) *Let n be a good integer. Then, there exists a minimum td-coloring for P_n with two n-d color classes.*

§2. Total Dominator Colorings in Caterpillars

After the classes of stars and paths, caterpillars are perhaps the simplest class of trees. For this reason, for any newly introduced parameter, we try to obtain the value for this class. In this paper, we give an upper bound for $\chi_{td}(T)$, where T is a caterpillar (with some restriction). First, we prove a theorem for a very simple type which however illustrates the ideas to be used in the general case.

Theorem 2.1 *Let G be a caterpillar such that*

(i) No two vertices of degree two are adjacent;
(ii) The end vertebrae have degree at least 3;
(iii) No vertex of degree 2 is a support vertex.

Then $\chi_{td}(G) \leqslant \left\lceil \frac{3r+2}{2} \right\rceil$.

Proof Let C be the spine of G. Let $u_1, u_2, \cdots, u_r$ be the support vertices and $u_{r+1}, u_{r+2}, \cdots, u_{2r-1}$ be the vertices of degree 2 in C. In a td-coloring of G, all support vertices receive a non-repeated color, say 1 to r and all pendant vertices receive the same repeated color say $r+1$ and the vertices u_{r+1} and u_{2r-1} receive a non-repeated color say $r+2$ and $r+3$ respectively. Consider the vertices $\{u_{r+2}, u_{r+3}, \cdots, u_{2r-2}\}$. We consider the following two cases.

Case 1 r is even.

In this case the vertices $u_{r+3}, u_{r+5}, \cdots, u_{r+\left(\frac{r}{2}-2\right)}, u_{r+\frac{r}{2}}, u_{r+\left(\frac{r}{2}+2\right)}, \cdots, u_{2r-3}$ receive the non-repeated colors say $r+4$ to $r + \left(\frac{r}{2}+1\right) = \frac{3r+2}{2}$ and the remaining vertices $u_{r+2}, u_{r+4}, \cdots, u_{2r-2}$ receive the already used repeated color $r+1$ respectively. Thus $\chi_{td}(G) \leqslant \frac{3r+2}{2}$.

Case 2 r is odd.

In this case the vertices $u_{r+3}, u_{r+5}, \cdots, u_{r+\left(\frac{r}{2}-2\right)}, u_{r+\frac{r}{2}}, u_{r+\left(\frac{r}{2}+2\right)}, \cdots, u_{2r-4}, u_{2r-2}$ receive the non-repeated colors say $r+4$ to $r + \left(\frac{r+3}{2}\right) = \frac{3r+3}{2}$ and the remaining vertices $u_{r+2}, u_{r+4}, \cdots, u_{2r-3}$ receive the already used repeated color $r+1$ respectively. Thus

$$\chi_{td}(G) \leqslant \frac{3r+3}{2} = \left\lceil \frac{3r+2}{2} \right\rceil. \qquad \square$$

Illustration 2.2 In Figures 1 and 2, we present 2 caterpillars holding with the upper bound of $\chi_{td}(G)$ in Theorem 2.1.

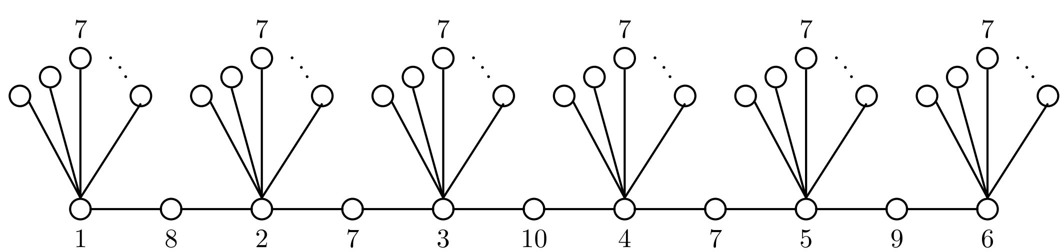

Figure 1

Clearly, $\chi_{td}(G) = 10 = \frac{3r+2}{2}$.

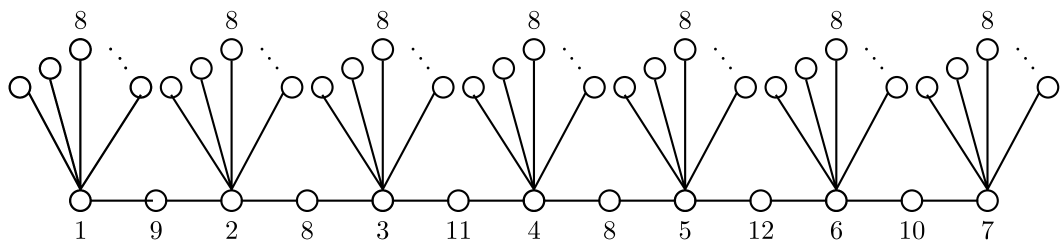

Figure 2

Clearly, $\chi_{td}(G) = 12 = \left\lceil \dfrac{3r+2}{2} \right\rceil$.

Remark 2.3 Let C be a minimal td-coloring of G. We call a color class in C, a non-dominated color class ($n-d$ color class) if it is not dominated by any vertex of G. These color classes are useful because we can add vertices to those color classes without affecting td-coloring.

Theorem 2.4 *Let G be a caterpillar of class 2 having exactly r vertices of degree at least 3 and r_i zero strings of length $i, 2 \leqslant i \leqslant m, m$ = maximum length of a zero string in G. Further suppose that $r_n \neq 0$ for some n, where $n-2$ is a good number and that end vertebrae are of degree at least 3. Then*

$$\chi_{td}(G) \leqslant 2(r+1) + \sum_{\substack{i=3 \\ i \equiv 1,2,3 (mod 4)}}^{m} r_i \left\lceil \frac{i-2}{2} \right\rceil + \sum_{\substack{i=4 \\ i \equiv 0 (mod 4)}}^{m} r_i \left(\left\lceil \frac{i-2}{2} \right\rceil + 1 \right).$$

Proof Let S be the spine of the caterpillar G and let $V(S) = \{u_1, u_2, \cdots, u_r\}$. We give the coloring of G as follows:

Vertices in S receive non-repeated colors, say from 1 to r. The set $N(u_j)$ is given the color $r+j, 1 \leqslant j \leqslant r$ (u_j is not adjacent to an end vertex of zero string of length 3 and if a vertex is adjacent to two supports, it is given one of the two possible colors). This coloring takes care of any zero string of length 1 or 2. Now, we have assumed $r_n \neq 0$ for some n, where $n-2$ is a good number. Hence there is a zero string of length n in G.

By Theorem 1.4, there is a minimum td-coloring of this path in which there are two $n-d$ colors. We give the sub path of length n this coloring with $n-d$ colors being denoted by $2r+1, 2r+2$. The idea is to use these two colors whenever $n-d$ colors occur in the coloring of zero strings. Next, consider a zero string of length 3, say

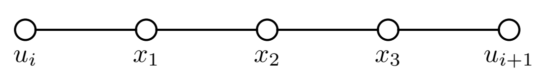

Figure 3

where u_i and u_{i+1} are vertices of degree at least 3 and we have denoted the vertices of the string of length 3 by x_1, x_2, x_3 for simplicity. Then, we give x_1 or x_3, say x_1 with a non-repeated color;

we give x_2 and x_3 the colors $2r+1$ and $2r+2$ respectively. Thus each zero string of length 3 introduces a new color and $\left\lceil \frac{3-2}{2} \right\rceil = 1$. Similarly, each zero string of length i introduces $\left\lceil \frac{i-2}{2} \right\rceil$ new colors when $i \equiv 1, 2, 3 (\text{mod } 4)$. However, the proof in cases when $i > 3$ is different from case $i = 3$ (but are similar in all such cases in that we find a *td*-coloring involving two $n-d$ colors). e.g. a zero string of length 11.

We use the same notation as in case $i = 3$ with a slight difference:

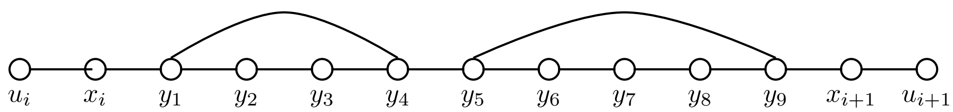

Figure 4

u_i and u_{i+1} being support vertices receive colors i and $i+1$. x_i and x_{i+1} receive $r+i$ and $r+i+1$ respectively. For the coloring of P_9 , we use the color classes $\{y_1, y_4\}, \{y_2\}, \{y_3\}, \{y_5, y_9\}, \{y_6\}$, $\{y_7\}, \{y_8\}$. We note that this is *not* a minimal *td*-coloring which usually has no $n-d$ color classes. This coloring has the advantage of having two $n-d$ color classes which can be given the class $2r+1$ and $2r+2$ and the remaining vertices being given non-repeated colors. In cases where i is a good integer, P_{i-2} requires $\left\lceil \frac{i-2}{2} \right\rceil + 2$ colors. However there will be two $n-d$ color classes for which $2r+1$ and $2r+2$ can be used. Thus each such zero string will require only $\left\lceil \frac{i-2}{2} \right\rceil$ new colors (except for the path containing the vertices we originally colored with $2r+1$ and $2r+2$). However, if $i \equiv 0(\text{mod}4), i-2 \equiv 2(\text{mod}4)$, and we will require $\left\lceil \frac{i-2}{2} \right\rceil + 1$ new colors. It is easily seen this coloring is a *td*-coloring. Hence the result. □

Illustration 2.5 In Figures $5-7$, we present 3 caterpillars with minimum td-coloring.

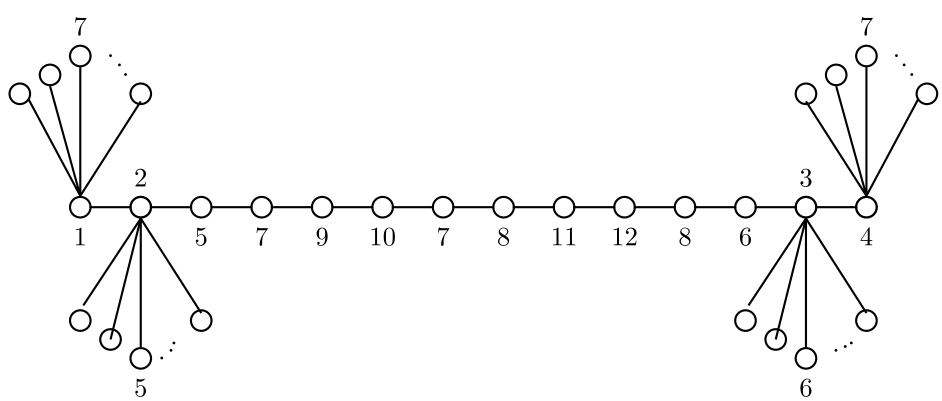

Figure 5

Then, $\chi_{td}(T) = 12 < 2(r+1) + r_{10} \left\lceil \frac{10-2}{2} \right\rceil$.

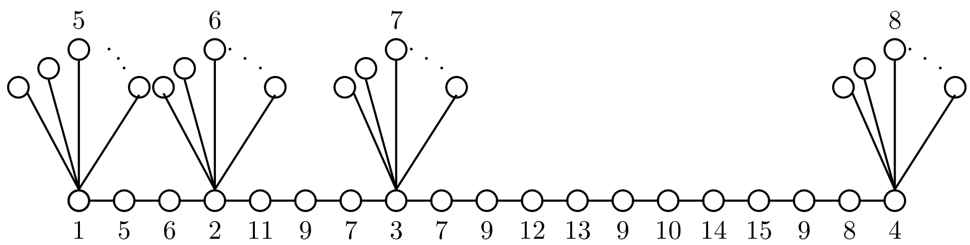

Figure 6

Then, $\chi_{td}(T_2) = 15 = 2(r+1) + r_3\left\lceil\frac{3-2}{2}\right\rceil + r_{10}\left\lceil\frac{10-2}{2}\right\rceil$

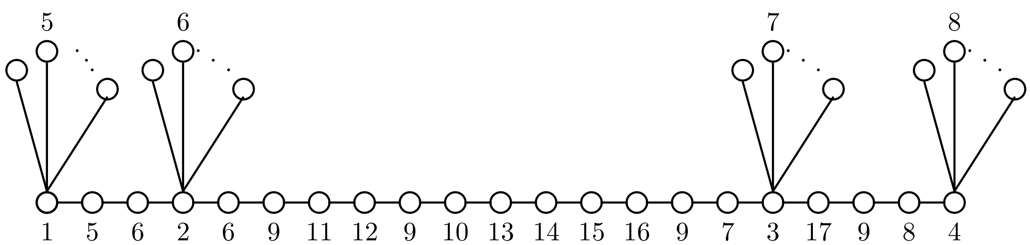

Figure 7

Then, $\chi_{td}(T_3) = 17 = 2(r+1) + r_3 + r_{12}\left(\left\lceil\frac{12-2}{2}\right\rceil + 1\right)$.

Remark 2.7 (1) The condition that end vertebrae are of degree at least 3 is adopted for the sake of simplicity. Otherwise the caterpillar 'begins' or 'ends' (or both) with a segment of a path and we have to add the χ_{td} -values for this (these) path(s).

(2) If in Theorem 2.1, we assume that all the vertices of degree at least 3 are adjacent (instead of (ii)), we get $\chi_{td}(G) = r + 1$.

(3) The bound in Theorem 2.4 does not appear to be tight. We feel that the correct bound will have $2r + 1$ on the right instead of $2r + 2$. There are graphs which attain this bound.

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We know nothing of what will happen in future, but by the analogy of past experience.

By Abraham Lincoln, an American president.

Author Information

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[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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