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The 3D Visualization of E_8 using an H_4 Folding Matrix, math version

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This paper will present various techniques for visualizing a split real even E_8 representation in 2 and 3 dimensions using an E_8 to H_4 folding matrix. This matrix is shown to be useful in providing direct relationships between E_8 and the lower dimensional Dynkin and Coxeter-Dynkin geometries contained within it, geometries that are visualized in the form of real and virtual 3 dimensional objects.

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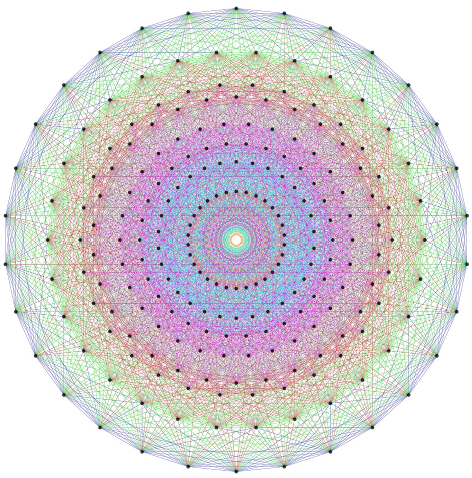

FIG. 1: E_8 Petrie projection

I. INTRODUCTION

Fig. 1 is the Petrie projection of the largest of the exceptional simple Lie algebras, groups and lattices called E_8. It has 240 vertices and 6720 edges of 8 dimensional (8D) length $\sqrt{2}$. Interestingly, in addition to containing the 8D structures of D_8 (aka. the rectified 8-orthoplex) and BC_8 (aka. the 8 demicube or alternated octeract), E_8 has been shown to fold to the 4D Polychora of H_4 (aka. the 120 vertex 600-cell) and a scaled copy H_4/φ, where $\varphi = \frac{1}{2}\left(1+\sqrt{5}\right)$ is the Golden Ratio[4][6]. Fig. 2 shows the folding orientation of E_8 and D6 Dynkin diagrams above the H_4 and H_3 Coxeter-Dynkin diagrams (respectively).

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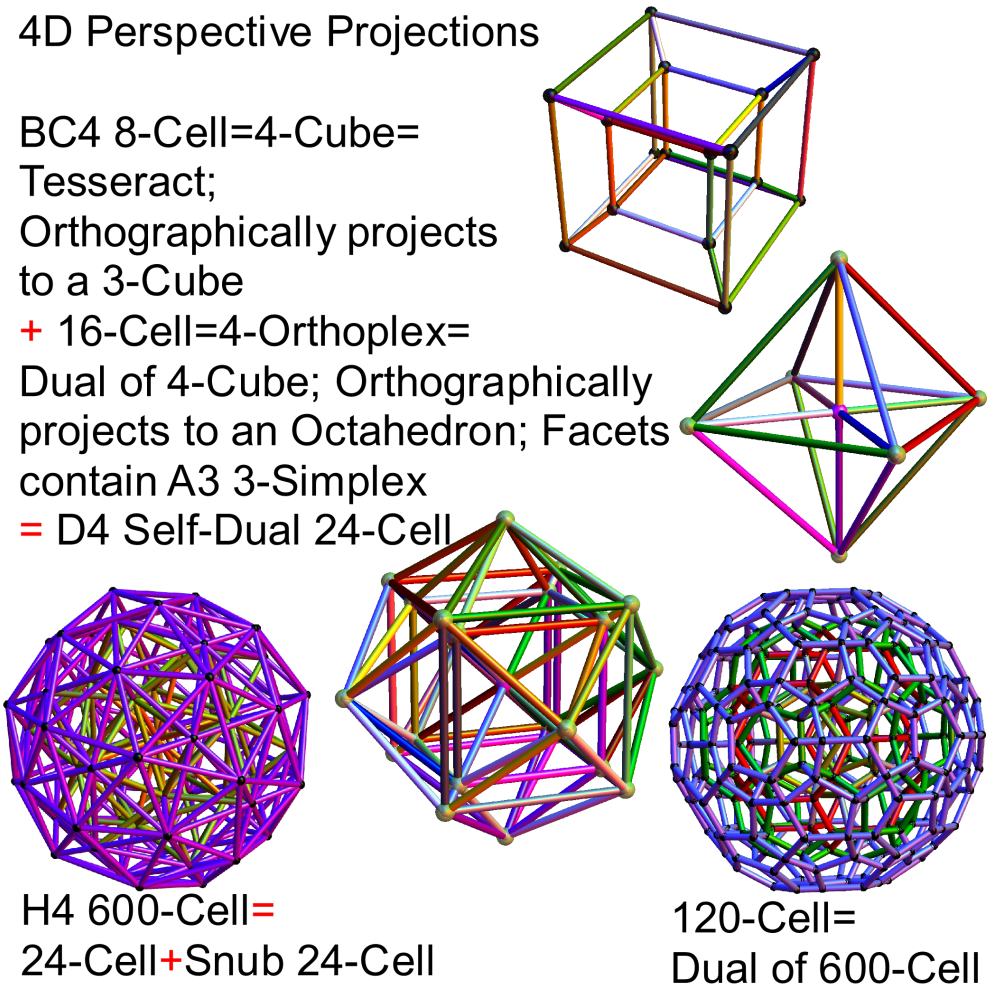

FIG. 3: 4D Polychora

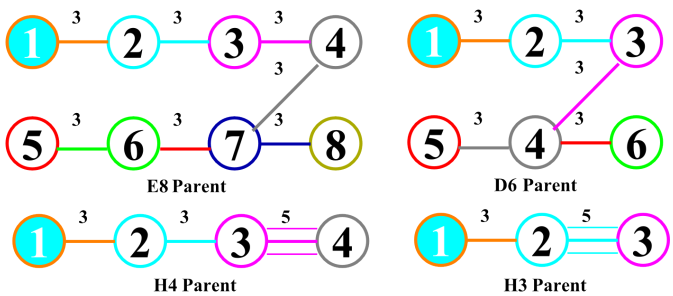

FIG. 2: E_8 and D6 Dynkin diagrams in folding orientation with their associated Coxeter-Dynkin diagrams H_4 and H_3

The 600-cell is constructed from the combination of the 96 vertices of the snub 24-cell and the 24 vertices of the 24-cell shown in Fig. 3. The 24-cell is self-dual and contained within both F_4 and the triality symmetry of the D_4 Dynkin diagram. It is interesting to note that it is constructed from the 16 vertices of the BC_4 tesseract (or 8-cell or 4-cube) and the 8 vertices of it's dual, the 4-orthoplex (or 16-cell). All of these polychora can be found within E_8 with the excluded 8-orthoplex. The snub 24-cell is constructed from even permutations of $\{\varphi, 1, 1/\varphi, 0\}$. Also shown in Fig. 3 is the dual of the 600-cell, namely the 120-cell with 600 vertices and a trirectified H_4 Coxeter-Dynkin diagram (i.e. the filled node is moved to the other end).

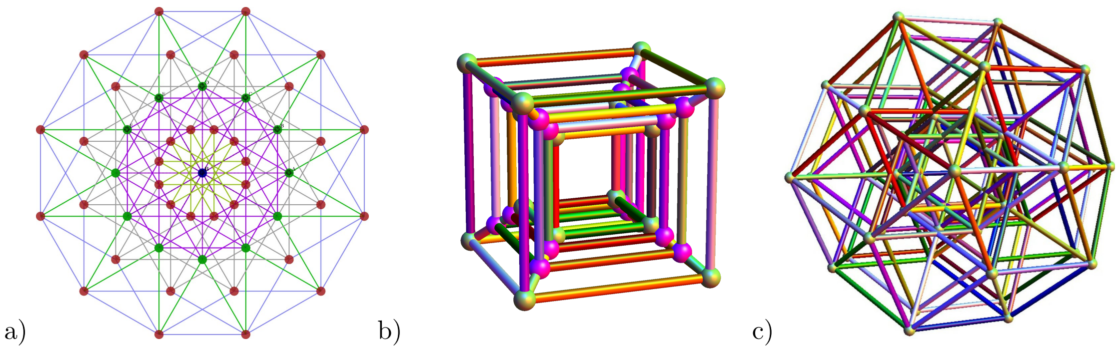

FIG. 4: The 6-cube a) Petrie projection b) 3D perspective c) rhombic triacontahedron

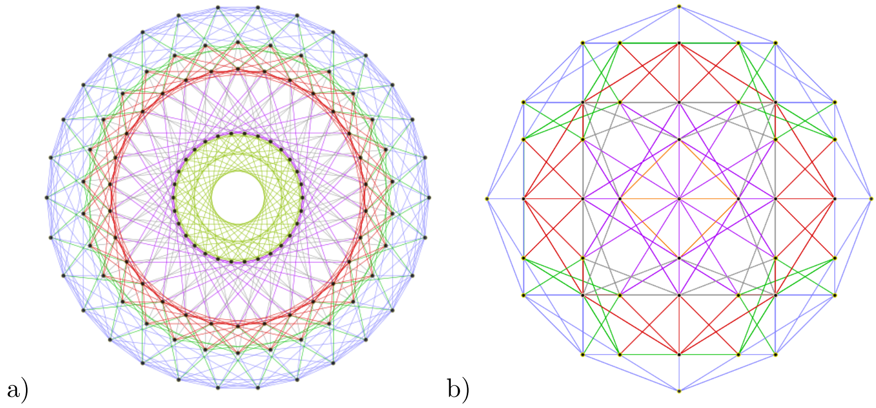

FIG. 5: H_4 600-cell 2D projections: a) Van Oss (or Petrie), b) orthonormal

$$\mathrm{H4_{fold}} = \begin{pmatrix} \varphi^2 & 0 & 0 & 0 & 1/\varphi & 0 & 0 & 0 \\ 0 & 1 & \varphi & 0 & 0 & -1 & \varphi & 0 \\ 0 & \varphi & 0 & 1 & 0 & \varphi & 0 & -1 \\ 0 & 0 & 1 & \varphi & 0 & 0 & -1 & \varphi \\ 1/\varphi & 0 & 0 & 0 & \varphi^2 & 0 & 0 & 0 \\ 0 & -1 & \varphi & 0 & 0 & 1 & \varphi & 0 \\ 0 & \varphi & 0 & -1 & 0 & \varphi & 0 & 1 \\ 0 & 0 & -1 & \varphi & 0 & 0 & 1 & \varphi \end{pmatrix} \tag{1}$$

The specific matrix for performing this folding of E_8 group vertices was shown[5] several years ago to be that of (1). Notice that $\mathrm{H4_{fold}} = \mathrm{H4^T_{fold}}$ such that it is symmetric with a quaternion-octonion Cayley-Dickson like structure. Only the first 4 rows are needed for folding E_8 to H_4, but the 8×8 square matrix is useful in the rotation of 8D vectors by taking its inverse.

E_8 also contains the 6D structures of the 6-cube or hexeract as shown in Fig. 4. It has been shown that using rows 2 through 4 of $\mathrm{H4_{fold}}$ projects the 6-cube[1] down to the 3D Rhombic Triacontahedron[3]. This particular object is interesting in that it contains the Platonic solids including the icosahedron and dodecahedron, and has been used to describe the φ related geometry leading to quasicrystals[2].

$$\begin{array}{llllllllll} x = & \{0 & (1+\sqrt{5})\,\mathrm{Sin}\left[\frac{\pi}{30}\right] & 0 & 1 & 0 & 0 & 0 & 0 & \} \\ y = & \{(1+\sqrt{5})\,\mathrm{Sin}\left[\frac{\pi}{15}\right] & 0 & 2\mathrm{Sin}\left[\frac{2\pi}{15}\right] & 0 & 0 & 0 & 0 & 0 & \} \\ z = & \{0 & 1 & 0 & (1+\sqrt{5})\,\mathrm{Sin}\left[\frac{\pi}{30}\right] & 0 & 0 & 0 & 0 & \} \end{array} \tag{2}$$

$$\begin{array}{lllllllll} X = & \{0 & 0.33821 & 1.2095 & 0.61803 & 0 & -0.33826 & -0.79094 & 0.61803\} \\ Y = & \{1.08863 & 0.50275 & 0 & 0.81347 & -0.25699 & 0.50275 & 0 & -0.81347\} \\ Z = & \{0 & 1 & 0.95629 & 0.20905 & 0 & -1 & 0.27977 & 0.20905\} \end{array} \tag{3}$$

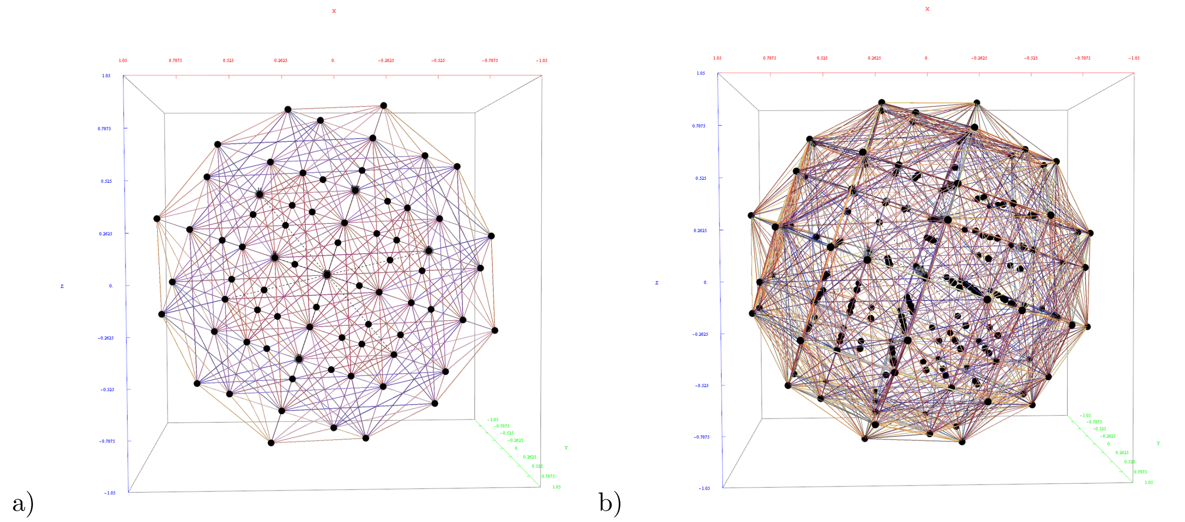

FIG. 6: E_8 projection showing H_4 and H_4/φ orthonormal face orientation in 2D and 3D perspective. Only 1220 of 6720 edges are shown in order prevent occlusion of vertices in 3D.

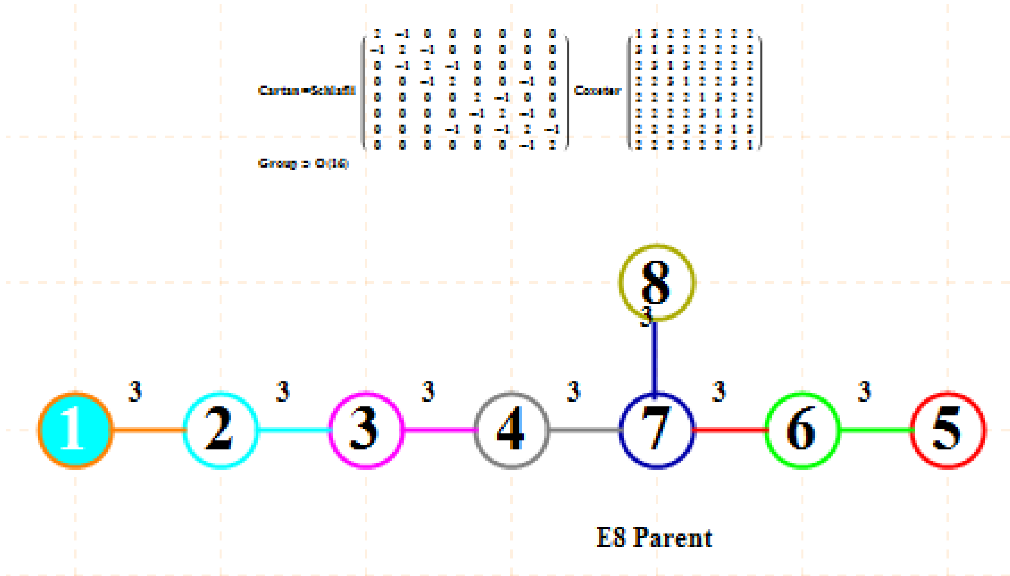

FIG. 7: E_8 Dynkin diagram with Cartan, Schlafli, and Coxeter matrices

Projection E_8 to 2D (or 3D) requires 2 (or 3) basis vectors {X, Y, Z}. We start with (2), which are simply the two 2D Petrie projection basis vectors of the 600-cell (aka. the Van Oss projection) as shown in Fig. 5 a), with a 3rd z basis vector added for the 3D projection. Notice the 8D basis vectors with zero in the last 4 columns (or dimensions).

The E_8 projection basis (3) is obtained by $\{X, Y, Z\} = 4 * \mathrm{H4}_{\mathrm{fold}}^{-1}.\{x, y, z\}$. On one face (or 2 of 6 cubic faces, which are the same), they project E_8 to its 2D Petrie projection shown in Fig. 1. On another face of this particular 3D projection is what would be found on all 6 faces of an orthonormal projection to 3D of the H_4 600-cell combined with a scaled H_4/φ, shown in 2D on Fig. 5 b) and in 3D in Fig. 6.

$$\text{E8}_{\text{srm}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \quad (4)$$

There are several choices for the form of E_8, whether it be complex or split real (even or odd). For the purposes of this work, the form selected is split real even (SRE). While the basic topology of the E_8 Dynkin diagram is unique, it has 8!=40320 permutations of node ordering. The node order used here is given in Fig. 7. The 240 specific E_8 group vertex values are determined from the simple roots matrix E8_{srm} shown in (4). The resulting Cartan matrix and generated algebraic roots are directly dependent on these as inputs.

$$\text{E8}_{\text{Cartan}} = \text{E8}_{\text{srm}}.\text{E8}_{\text{srm}}^T \quad (5)$$

$$\text{E8}_{\text{SREvertex}} = \text{E8}_{\text{srm}}^T.\text{E8}_{\text{root}} \quad (6)$$

The Dynkin diagram was constructed as user input with the Mathematica “VisibLie” notebook. Fig. 7 was generated and exported from the referenced tool, as are all of the figures in this paper. It has the same node ordering as the E_8 Dynkin used in Fig. 2. The Cartan matrix can be generated directly by the structure of the Dynkin diagram or from its relationship to the simple roots matrix (5). The positive E_8 algebra roots are generated by the Mathematica “SuperLie” package and listed in Fig. 9 along with its Hasse diagram in Fig. 10 of Appendix A. The 120 positive and 120 negative algebra roots are then used to generate the SRE E_8 vertices using (6).

II. CONCLUSION

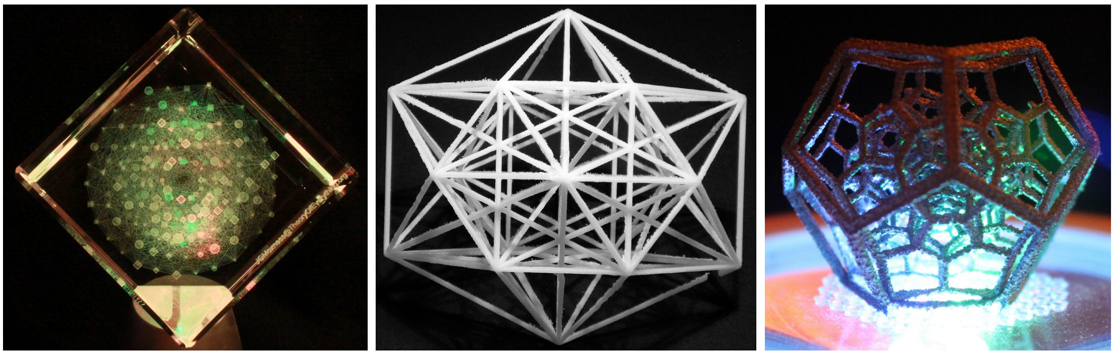

FIG. 8: The E_8 to H_4 3D projection model used to laser etch optical crystal

In terms of mathematical symmetry representing the beauty of Nature, E_8 is one of the most beautiful. It contains a wealth of symmetries, including those of 2D projections, 3D polyhedrons, 4D polychora, and those up to 8D. An SRE E_8 to H_4 folding matrix was determined and used to fold E_8 to the 120 4D vertices of the H_4 600-cell and 120 vertices of H_4/φ.

The traditional 2D Petrie projections of high dimensional geometry were extended by adding a carefully chosen third basis vector and generating 3D objects in either orthogonal or perspective views. The folding matrix was shown to generate these basis vectors used in projecting the E_8 vertices. These projected 3D objects can be realized as 3D models, which allow for their realization as animated rotations, models laser etched in optical crystal, and in some cases 3D printed in plastic or even metal as in Fig. 8.

Acknowledgments

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Bibliography

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III. APPENDIX A

The positive E_8 algebra roots generated by the Mathematica "SuperLie" package and its Hasse diagram. The full source version of the "VisibLie" notebook at http://theoryofeverything.org/MyToE may be made available by request in order to generate algebra roots and their Hasse diagrams.

Dimension=248 Rank=8 DetCM=1 # of Positive Roots=120 Coxeter #=30

2	-1	0	0	0	0	0	0
-1	2	-1	0	0	0	0	0
0	-1	2	-1	0	0	0	0
0	0	-1	2	0	0	-1	0
0	0	0	0	2	-1	0	0
0	0	0	0	-1	2	-1	0
0	0	0	-1	0	-1	2	-1
0	0	0	0	0	0	-1	2

1	2	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
2	-1	2	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	1
3	0	-1	2	-1	0	0	0	0	0	0	1	0	0	0	0	0	1
4	0	0	-1	2	0	0	-1	0	0	0	0	1	0	0	0	0	1
5	0	0	0	0	2	-1	0	0	0	0	0	0	1	0	0	0	1
6	0	0	0	0	-1	2	-1	0	0	0	0	0	0	1	0	0	1
7	0	0	0	-1	0	-1	2	-1	0	0	0	0	0	0	1	0	1
8	0	0	0	0	0	0	-1	2	0	0	0	0	0	0	0	1	1
9	1	1	-1	0	0	0	0	0	1	1	0	0	0	0	0	0	2
10	-1	1	1	-1	0	0	0	0	0	1	1	0	0	0	0	0	2
11	0	-1	1	1	0	0	-1	0	0	0	1	1	0	0	0	0	2
12	0	0	-1	1	0	-1	1	-1	0	0	0	1	0	0	1	0	2
13	0	0	0	0	1	1	-1	0	0	0	0	0	1	1	0	0	2
14	0	0	0	-1	-1	1	1	-1	0	0	0	0	0	1	1	0	2
15	0	0	0	-1	0	-1	1	1	0	0	0	0	0	0	1	1	2
16	1	0	1	-1	0	0	0	0	1	1	1	0	0	0	0	0	3
17	-1	1	0	1	0	0	-1	0	0	1	1	1	0	0	0	0	3
18	0	0	-1	1	-1	1	0	-1	0	0	0	1	0	1	1	0	3
19	0	-1	1	0	0	-1	1	-1	0	0	1	1	0	0	1	0	3
20	0	0	0	-1	1	0	1	-1	0	0	0	0	1	1	1	0	3
21	0	0	-1	1	0	-1	0	1	0	0	0	1	0	0	1	1	3
22	0	0	0	-1	-1	1	0	1	0	0	0	0	0	1	1	1	3
23	0	0	-1	1	-1	1	-1	1	0	0	0	1	0	1	1	1	4
24	1	0	0	1	0	0	-1	0	1	1	1	1	0	0	0	0	4
25	-1	1	0	0	0	-1	1	-1	0	1	1	1	0	0	1	0	4
26	0	-1	1	0	-1	1	0	-1	0	0	1	1	0	1	1	0	4
27	0	-1	1	0	0	-1	0	1	0	0	1	1	0	0	1	1	4
28	0	0	-1	1	1	0	0	-1	0	0	0	1	1	1	1	0	4
29	0	0	0	-1	1	0	0	1	0	0	0	0	1	1	1	1	4
30	0	-1	1	0	-1	1	-1	1	0	0	1	1	0	1	1	1	5
31	1	0	0	0	0	-1	1	-1	1	1	1	1	0	0	1	0	5
32	0	-1	1	0	1	0	0	-1	0	0	1	1	1	1	1	0	5
33	0	0	-1	1	1	0	-1	1	0	0	0	1	1	1	1	1	5
34	-1	1	0	0	-1	1	0	-1	0	1	1	1	0	1	1	0	5
35	-1	1	0	0	0	-1	0	1	0	1	1	1	0	0	1	1	5
36	0	0	-1	0	-1	0	1	0	0	0	0	1	0	1	2	1	5
37	0	-1	1	0	1	0	-1	1	0	0	1	1	1	1	1	1	6
38	1	0	0	0	-1	1	0	-1	1	1	1	1	0	1	1	0	6
39	1	0	0	0	0	-1	0	1	1	1	1	1	0	0	1	1	6
40	-1	1	0	0	1	0	0	-1	0	1	1	1	1	1	1	0	6
41	-1	1	0	0	-1	1	-1	1	0	1	1	1	0	1	1	1	6
42	0	-1	1	-1	-1	0	1	0	0	0	1	1	0	1	2	1	6
43	0	0	-1	0	1	-1	1	0	0	0	0	1	1	1	2	1	6
44	-1	1	0	0	1	0	-1	1	0	1	1	1	1	1	1	1	7
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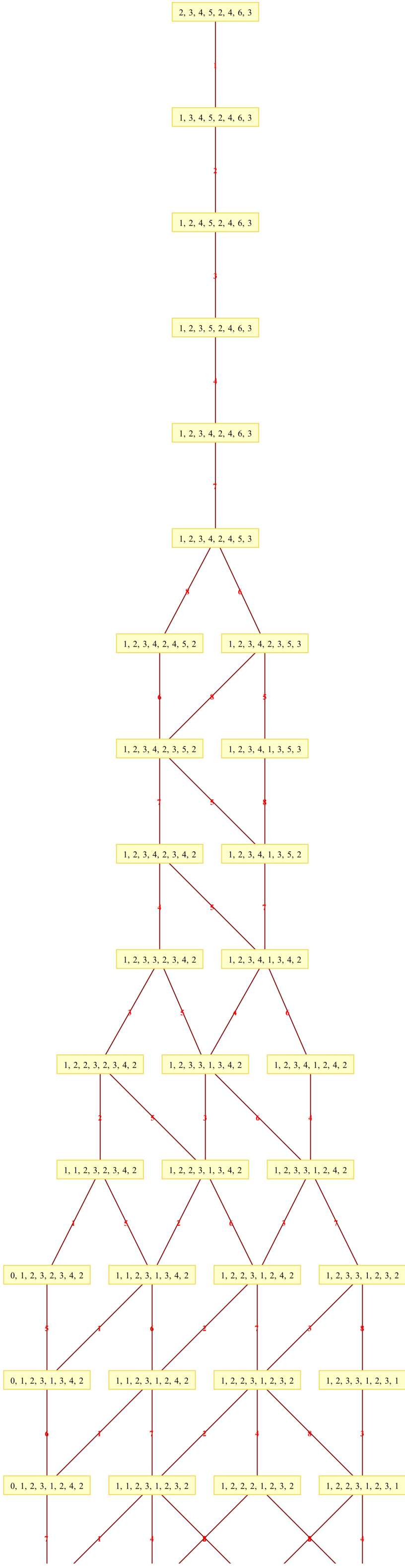
107	0	0	1	-1	1	0	0	0	1	2	3	3	2	3	4	2	20
108	0	0	0	1	-1	1	-1	0	1	2	3	4	1	3	4	2	20
109	0	0	0	0	-1	0	1	-1	1	2	3	4	1	3	5	2	21
110	0	0	0	1	1	0	-1	0	1	2	3	4	2	3	4	2	21
111	0	0	0	0	1	-1	1	-1	1	2	3	4	2	3	5	2	22
112	0	0	0	0	-1	0	0	1	1	2	3	4	1	3	5	3	22
113	0	0	0	0	1	-1	0	1	1	2	3	4	2	3	5	3	23
114	0	0	0	0	0	1	0	-1	1	2	3	4	2	4	5	2	23
115	0	0	0	0	0	1	-1	1	1	2	3	4	2	4	5	3	24
116	0	0	0	-1	0	0	1	0	1	2	3	4	2	4	6	3	25
117	0	0	-1	1	0	0	0	0	1	2	3	5	2	4	6	3	26
118	0	-1	1	0	0	0	0	0	1	2	4	5	2	4	6	3	27
119	-1	1	0	0	0	0	0	0	1	3	4	5	2	4	6	3	28
120	1	0	0	0	0	0	0	0	2	3	4	5	2	4	6	3	29

FIG. 9: 120 positive algebraic roots of E_8



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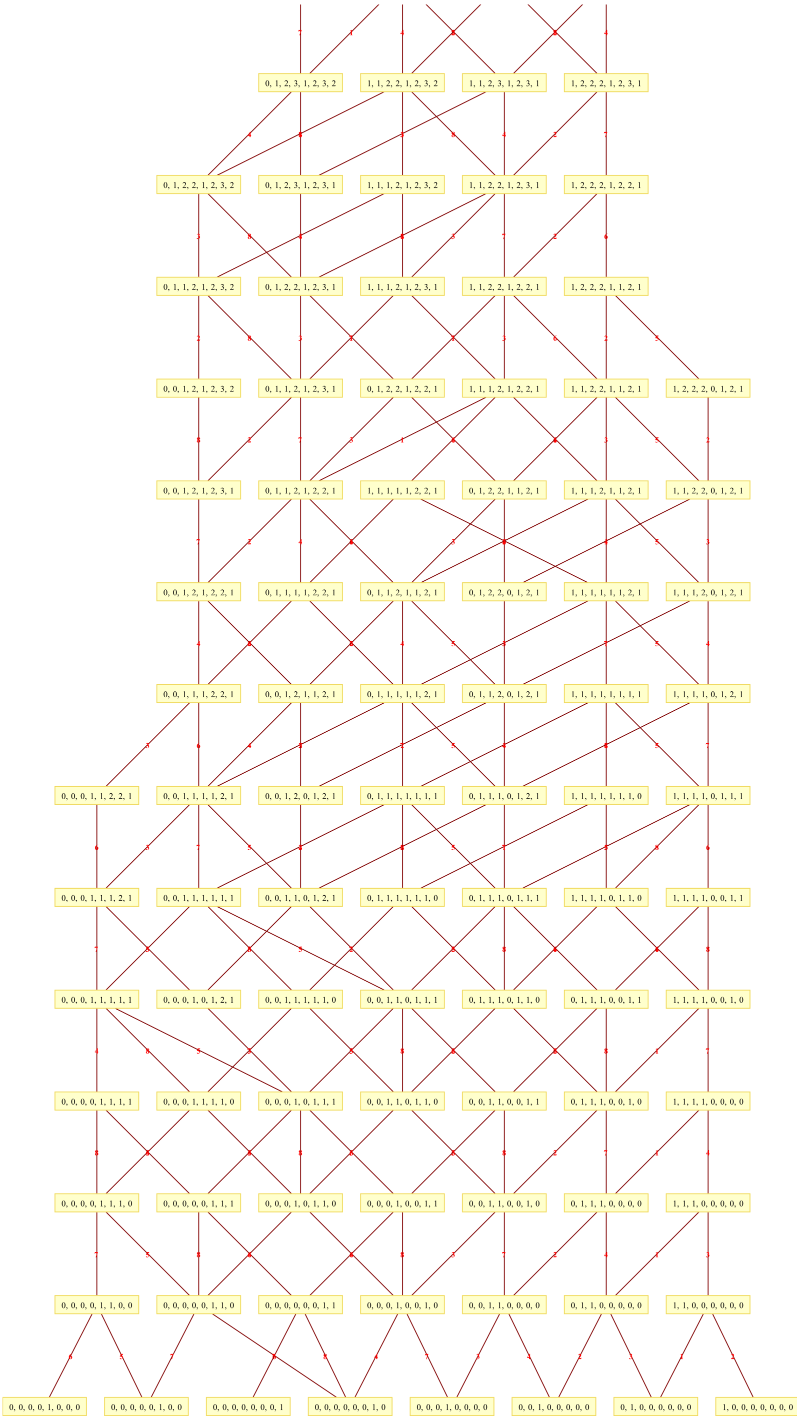

FIG. 10: E_8 Hasse diagram representation