On Some Novel Consequences of Clifford Space Relativity Theory

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Abstract

Some of the novel physical consequences of the Extended Relativity Theory in C-spaces (Clifford spaces) are presented. In particular, generalized photon dispersion relations which allow for energy-dependent speeds of propagation while still *retaining* the Lorentz symmetry in ordinary spacetimes, while breaking the *extended* Lorentz symmetry in C-spaces. We analyze in further detail the extended Lorentz transformations in Clifford Space and their physical implications. Based on the notion of "extended events" one finds a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity (DSR) framework. We finalize with a discussion of the modified dispersion relations, rainbow metrics and generalized uncertainty relations in C-spaces which are extensions of the stringy uncertainty relations.

Keywords : Clifford algebras; Extended Relativity in Clifford Spaces; Modified Dispersion Relations ; Rainbow Metrics; Generalized Uncertainty Principle.

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing.

Our theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1]. The polyvector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate).

These polyvector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background. C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes . By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string. When $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions it can be decomposed as

$$\mathbf{X} = s\ \mathbf{1} + x^\mu\ \gamma_\mu + x^{\mu\nu}\ \gamma_\mu \wedge \gamma_\nu + x^{\mu\nu\rho}\ \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho + x^{\mu\nu\rho\tau}\ \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\tau \quad (1)$$

where we have omitted combinatorial numerical factors for convenience in the expansion of eq-(1). To avoid introducing powers of a length parameter L (like the Planck scale L_p), in order to match physical units in the expansion of the polyvector X in eq-(1), we can set it to unity to simplify matters.

The component s is the Clifford scalar component of the polyvector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval

$$(d\Sigma)^2 = (ds)^2 + dx_\mu\ dx^\mu + dx_{\mu\nu}\ dx^{\mu\nu} + \ \ldots \quad (2)$$

that is *invariant* under $Cl(1,3)$ transformations and which are the Clifford-algebraic extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$ is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$. Generalized Lorentz transformations (poly-rotations) in flat C-spaces were discussed in [1]. In this work we shall provide an extensive analysis of the C-space generalized Lorentz transformations and their physical implications.

Let us provide several examples of generalized Lorentz transformations in C-space. For example, given γ_{02} the transformation involving the rotor $R_1 = cosh(\beta/2) - \gamma_{02}\ sinh(\beta/2)$ corresponds to an ordinary Lorentz boost transformation along the X^2 direction and involving the ordinary temporal variable X^0. The ordinary Lorentz boots generators are given by the bivectors $\gamma_{\mu\nu}$, and which in turn are also expressed as the commutators $[\gamma_\mu, \gamma_\nu]$. The physical significance of the latter commutators is that they represent a "rotation" along the $X^\mu - X^\nu$ directions.

However, since one may also write the bivector γ_{02} as the commutator $[\gamma_{12}, \gamma_{01}] = -2\gamma_{02}$, the transformation involving the above rotor R_1 also corresponds to an *areal* boost along the X^{12} direction but involving the areal temporal coordinate X^{01}. Namely, it is a "rotation" along the $X^{12} - X^{01}$ directions. Whereas the ordinary boost is a "rotation" along the $X^2 - X^0$ directions.

After writing

$$(X^B)'\,\Gamma_B \;=\; (\; cosh(\beta/2) - \gamma_{02}\; sinh(\beta/2)\;)\;(\; X^A\,\Gamma_A\;)\;(\; cosh(\beta/2) + \gamma_{02}\; sinh(\beta/2)\;) \quad (3)$$

straightforward algebra yields the transformation of the following bivector coordinates

$$(X^{12})' \;=\; X^{12}\; cosh\beta \;+\; X^{01}\; sinh\beta \quad (4a)$$

$$(X^{01})' \;=\; X^{01}\; cosh\beta \;+\; X^{12}\; sinh\beta \quad (4b)$$

One has a mixing of the spatial and temporal areal bivector coordinates in the new frame of reference.

Furthermore, since $[\gamma_{013}, \gamma_{123}] \sim \gamma_{02}$, the transformation involving the above rotor R_1 also corresponds to a 3-volume boost along the X^{123} direction but involving the 3-volume temporal coordinate X^{013}. Namely, it is a "rotation" along the $X^{123} - X^{013}$ directions giving

$$(X^{123})' \;=\; X^{123}\; cosh\beta \;+\; X^{013}\; sinh\beta \quad (5a)$$

$$(X^{013})' \;=\; X^{013}\; cosh\beta \;+\; X^{123}\; sinh\beta \quad (5b)$$

One has a mixing of the spatial and temporal trivector coordinates in the new frame of reference. The ordinary Lorentz boosts of the vector coordinates give

$$(X^{2})' \;=\; X^{2}\; cosh\beta \;+\; X^{0}\; sinh\beta \quad (6a)$$

$$(X^{0})' \;=\; X^{0}\; cosh\beta \;+\; X^{2}\; sinh\beta \quad (6b)$$

while the remaining coordinates remain invariant and such that the quadratic form $X^A X_A = (X^A)'(X_A)'$ remains invariant. Straightforward algebra leads to

$$-\,(X'_0)^2 \;+\; (X'_1)^2 \;-\; L^{-2}\,(X'_{01})^2 \;+\; L^{-2}\,(X'_{12})^2 \;-\; L^{-4}\,(X'_{013})^2 \;+\; L^{-4}\,(X'_{123})^2 \;=$$

$$-\,(X_0)^2 \;+\; (X_1)^2 \;-\; L^{-2}\,(X_{01})^2 \;+\; L^{-2}\,(X_{12})^2 \;-\; L^{-4}\,(X_{013})^2 \;+\; L^{-4}\,(X_{123})^2 \quad (7)$$

The quadratic form is defined as

$$<\; \mathbf{X}^\dagger\,\mathbf{X}\; > \;=\; X_A\,X^A \;=\; s^2 + X_\mu\,X^\mu + X_{\mu_1\mu_2}\,X^{\mu_1\mu_2} + \ldots\ldots\, X_{\mu_1\mu_2\ldots\ldots\mu_D}\,X^{\mu_1\mu_2\ldots\ldots\mu_D} \quad (8)$$

where $\mathbf{X}^\dagger$ denotes the reversal operation obtained by reversing the order of the gamma generators in the wedge products. The symbol $<\; \Gamma_A\;\Gamma_B\; >$ denotes taking the scalar part in the Clifford geometric product of $\Gamma_A\,\Gamma_B$. It is the analog of the trace of a product of matrices. Such scalar part can be obtained from the (anti) commutator relations of the Clifford algebra generators. For example

$$<\; \gamma_\mu\;\gamma^\nu\; > \;=\; \delta^\nu_\mu, \quad <\; \gamma_{\mu_1\mu_2}\;\gamma^{\nu_1\nu_2}\; > \;=\; -\;\delta^{\nu_1\nu_2}_{\mu_1\mu_2}$$

$$<\; \gamma_{\mu_1\mu_2\mu_3}\;\gamma^{\nu_1\nu_2\nu_3}\; > \;=\; -\;\delta^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3}, \quad <\; \gamma_{\mu_1\mu_2\mu_3\mu_4}\;\gamma^{\nu_1\nu_2\nu_3\nu_4}\; > \;=\; \delta^{\nu_1\nu_2\nu_3\nu_4}_{\mu_1\mu_2\mu_3\mu_4}, \quad \ldots\ldots \quad (9)$$

One should note the presence of $\pm$ signs in the right hand side of eqs-(9). They are connected to the even/odd behavior of the reversal operation $(\gamma_C)^\dagger = \pm\gamma_C$.

The quadratic form is invariant under the isometry transformations

$$\mathbf{X}' = \mathbf{R}\,\mathbf{X}\,\mathbf{L}^\dagger, \quad \mathbf{R}^\dagger\,\mathbf{R} = 1, \quad \mathbf{L}^\dagger\,\mathbf{L} = 1 \;\Rightarrow\; <\mathbf{X}'^\dagger\,\mathbf{X}'> \;=\; <\mathbf{X}^\dagger\,\mathbf{X}> \tag{10}$$

due to the cyclic property of the scalar part projection

$$<\mathbf{X}'^\dagger\,\mathbf{X}'> \;=\; <\mathbf{L}\,\mathbf{X}^\dagger\,\mathbf{R}^\dagger\,\mathbf{R}\,\mathbf{X}\,\mathbf{L}^\dagger,> \;=\; <\mathbf{L}\,\mathbf{X}^\dagger\,\mathbf{X}\,\mathbf{L}^\dagger> \;=$$
$$<\mathbf{L}^\dagger\,\mathbf{L}\,\mathbf{X}^\dagger\,\mathbf{X}> \;=\; <\mathbf{X}^\dagger\,\mathbf{X}> \tag{11}$$

where $\mathbf{R}, \mathbf{L}$ are Clifford-valued rotors acting on the right and left respectively.

The second example corresponds to the case when there is a mixing of different grades. It involves the commutator $[\gamma_{0123}, \gamma_3] \sim \gamma_{012}$ and such that the transformation involving the rotor $R_2 = cosh(\beta'/2) - \gamma_{012}\; sinh(\beta'/2)$ corresponds to a boost along the spatial X^3 direction but involving now the *temporal* 4-volume polyvector-valued coordinate X^{0123}. The reason being that γ_{012} can be rewritten as the commutator of γ_{0123} and γ_3, so we have now "rotations" along the $X^3 - X^{0123}$ directions. Straightforward algebra yields now the transformation of the following (poly) vector coordinates

$$(X^3)' \;=\; X^3\; cosh(\beta') \;-\; L^{-3}\; X^{0123}\; sinh(\beta') \tag{12a}$$

$$(X^{0123})' \;=\; X^{0123}\; cosh(\beta') \;-\; L^3\; X^3 sinh(\beta') \tag{12b}$$

In this case one has a *mixing* of polyvector-valued coordinates of $different\ grade$. In the new frame of reference the spatial X^3 coordinate and the temporal 4-volume coordinate X^{0123} are mixed.

Furthermore, since $[\gamma_{03}, \gamma_{123}] \sim \gamma_{012}$, the transformation involving the rotor $R_2 = cosh(\beta'/2) - \gamma_{012}\; sinh(\beta'/2)$ also corresponds to a boost along the spatial *trivector* X^{123} direction but involving now the *temporal* bivector coordinate X^{03}. These transformations are

$$(X^{123})' \;=\; X^{123}\; cosh(\beta') \;-\; L\; X^{03}\; sinh(\beta') \tag{13a}$$

$$(X^{03})' \;=\; X^{03}\; cosh(\beta') \;-\; L^{-1}\; X^{123}\; sinh(\beta') \tag{13b}$$

In the above equations we have used the relations

$$\gamma_{01}^2 = 1, \quad \gamma_{02}^\dagger = -\gamma_{02}, \quad \gamma_{012}^2 = 1, \quad \gamma_{012}^\dagger = -\gamma_{012}$$

$$\{\gamma_{12}, \gamma_{02}\} = 0, \quad [\gamma_{0123}, \gamma_{012}] \;=\; -\,2\;\gamma_3, \quad \{\gamma_{0123}, \gamma_{012}\} \;=\; 0$$

$$\gamma_{02}\;\gamma_{12}\;\gamma_{02} = -\gamma_{12}, \quad [\gamma_{012},\; \gamma_3] \;=\; 2\;\gamma_{0123}, \quad \{\gamma_{012}, \gamma_3\} = 0, ... \tag{14}$$

$$cosh^2(\xi) - sinh^2(\xi) \ = \ 1, \ \ cosh^2(\xi) + sinh^2(\xi) \ = \ cosh(2\xi), \ \ \sinh(2\xi) \ = \ 2\, sinh(\xi)\, cosh(\xi) \tag{15}$$

Given in general a transformation of the form

$$(\ cosh(\beta/2) \ - \ \Gamma_C \ sinh(\beta/2)) \ \ X^A \ \Gamma_A \ \ (\ cosh(\beta/2) \ + \ \Gamma_C \ sinh(\beta/2)) \ \ = \ \ X'^B \ \Gamma_B \tag{16}$$

one learns that

$$X'^B \ = \ X^B \ cosh^2(\beta/2) \ - \ X^A \ sinh^2(\beta/2) \ < \ \Gamma_C \ \Gamma_A \ \Gamma_C \ \Gamma^B \ > \ + \\ X^A \ cosh(\beta/2) \ sinh(\beta/2) \ < \ [\Gamma_A, \ \Gamma_C] \ \Gamma^B \ > \tag{17}$$

The generator Γ_C of generalized Lorentz boosts is of the form $(\gamma_{0\mu_1\mu_2...\mu_{n-1}})$ with the provision that under the reversal operation it changes sign

$$(\gamma_{0\mu_1\mu_2...\mu_{n-1}})^\dagger \ = \ - \ \gamma_{0\mu_1\mu_2...\mu_{n-1}} \tag{18a}$$

so that $\mathbf{R}\mathbf{R}^\dagger = 1$. This condition will *restrict* the values of n to be $n = 2, 3, 6, ...$and obeying

$$(\gamma_{0\mu_1\mu_2...\mu_{n-1}})^2 \ = \ 1 \tag{18b}$$

Generalized *spatial* rotations don't involve the temporal directions and are generated by $\gamma_{\mu_1\mu_2...\mu_m}$ obeying

$$(\gamma_{\mu_1\mu_2...\mu_m})^\dagger \ = \ - \ \gamma_{\mu_1\mu_2...\mu_m} \tag{19}$$

and

$$(\gamma_{\mu_1\mu_2...\mu_m})^2 \ = \ - \ 1 \tag{20}$$

For instance, a generalized rotation in $D > 4$ and generated by $\gamma_{12...6}$ involving the parameter $\alpha^{12....6}$ yields a rotor whose Taylor series expansion becomes

$$\mathbf{R} \ = \ e^{\alpha^{12...6} \ \gamma_{12.....6}} \ = \ cos(\alpha^{12....6}) \ + \ \gamma_{012.....6} \ sin(\alpha^{12...6}) \tag{21}$$

due to the condition $(\gamma_{12.....6})^2 \ = \ - \ 1$ which is similar to having the imaginary unit $i^2 = -1$ and the expression $e^{i\theta} = cos(\theta) + i \ sin(\theta)$. For an earlier discussion of generalized rotations within C-space see [6]. Whereas a generalized Lorentz boost is like having a "rotation" with an imaginary "angle" leading to the hyperbolic functions

$$\mathbf{R} \ = \ e^{\beta^{012...5} \ \gamma_{02.....5}} \ = \ cosh(\beta^{012....5}) \ + \ \gamma_{012.....5} \ sinh(\beta^{12...5}) \tag{22}$$

due to the condition $(\gamma_{012.....5})^2 \ = \ 1$.

Eq-(17) only simplifies considerably in the very special case when the values of the polyvector valued indices A, B, C are such that

$$< \ \Gamma_C \ \Gamma_A \ \Gamma_C \ \Gamma^B \ > \ = \ - \ 1, \quad < \ [\Gamma_A, \ \Gamma_C] \ \Gamma^B \ > \ = \ \pm \ 2 \tag{23}$$

and it leads to the type of transformations displayed above. In general, for a given set of values of B, C, one must *sum* over *all* the A indices in eq-(17). For this reason the most general expression for X'^B given by eq-(17) is more complicated than that given by the above equations. Another special case occurs when

$$< \Gamma_C \, \Gamma_A \, \Gamma_C \, \Gamma^B > \; = \; 1, \quad < [\Gamma_A, \; \Gamma_C] \, \Gamma^B > \; = \; 0 \tag{24}$$

leading to $X'^B = X^B$ so that these particular polyvector coordinate components remain invariant.

One should emphasize that the functional form of the most general transformations are even *more complicated* than those described in eq-(17). Let us write the rotor associated with a "rotation" along the $X^A - X^B$ directions in C-space with parameter α^{AB}, after writing the commutation relations $[\Gamma_A, \Gamma_B] = f_{AB}^{\;\;C}\Gamma_C$, as follows

$$\mathbf{R} \; = \; e^{\alpha^{AB} \; [\Gamma_A, \Gamma_B]} \; = \; e^{\alpha^{AB} \; f_{AB}^C \; \Gamma_C} \; = \; e^{\beta^C \Gamma_C}, \;\; \beta^C \; = \; \alpha^{AB} \; f_{AB}^{\;\;C} \tag{25}$$

where $f_{AB}^{\;\;C}$ are the structure constants of the algebra. There is a summation over the C indices (but not over the A, B indices) in eq-(25) and the reversal condition reads

$$[\Gamma_A, \; \Gamma_B]^\dagger \; = \; - \; [\Gamma_A, \; \Gamma_B] \; \Rightarrow \; \mathbf{R} \, \mathbf{R}^\dagger = 1 \tag{26}$$

and which is satisfied in particular when $\Gamma_A^\dagger = -\Gamma_A; \Gamma_B^\dagger = -\Gamma_B$ giving $\Gamma_C^\dagger = -\Gamma_C$. This is a result of the relations $(\Gamma_A \Gamma_B)^\dagger = (\Gamma_B)^\dagger (\Gamma_A)^\dagger = \Gamma_B \Gamma_A$. In the most general case, for arbitrary dimensions, due to the *summation* over the C polyvector indices in eq-(25), the rotor $\mathbf{R}$ cannot be expressed in the form displayed in eq-(16) after performing a Taylor series expansion of the exponentials. For instance

$$e^{\beta^{01} \gamma_{01} \; + \; \beta^{023} \gamma_{023}} \; \neq \; \Big(cosh(\beta^{01}) \; + \; \gamma_{01} \; sinh(\beta^{01}) \Big) \; \Big(cosh(\beta^{023}) \; + \; \gamma_{023} \; sinh(\beta^{023}) \Big) \tag{27}$$

as a result of the Baker-Campbell-Hausdorf formula. Because $[\gamma_{01}, \gamma_{023}] \neq 0$ the left hand side of eq-(27) does not factorize.

We learnt from Special Relativity that the concept of simultaneity is *relative*. The typical example arises when a moving observer inside a train sees the front and back doors of a train opening simultaneously. Due to the spatial separation ($\Delta X^3 \neq 0$) between the two doors, an observer at rest in the platform will see the doors opening at *different* times

$$(\Delta X^0)' \; = \; \Delta X^0 \; cosh(\beta) \; + \; \Delta X^3 \; sinh(\beta) \; \neq \; 0, \tag{28}$$

despite $\Delta X^0 = 0$ due to the fact that $\Delta X^3 \neq 0$.

Something analogous, and more general, occurs in C-space. Let us denote by $\Delta X^3 = X^3_{(2)} - X^3_{(1)}$, $\Delta X^{0123} = X^{0123}_{(2)} - X^{0123}_{(1)}$ the spatial and 4-volume *separation*, respectively, between two events **1** and **2** in a given frame of reference in a *flat* C-space. From eqs-(12) it follows that in the new frame of reference one has

$$(\Delta X^3)' \; = \; \Delta X^3 \; cosh(\beta') \; - \; L^{-3} \; \Delta X^{0123} \; sinh(\beta') \tag{29a}$$

$$(\Delta X^{0123})' \;=\; \Delta X^{0123} cosh(\beta') \;-\; L^3 \;\Delta X^3 sinh(\beta') \qquad (29b)$$

if $\Delta X^{0123} \neq 0$ one has that $(\Delta X^3)' \neq 0$ despite that $\Delta X^3 = 0$. Therefore, because $(\Delta X^3)' \neq 0$ the observer in the new frame of reference does *not* experience events $\mathbf{1}, \mathbf{2}$ at the *same* location.

An "extended" event in C-space described by eqs-(29) can be envisaged as follows. An observer assigns to a physical event the coordinate values X^A where the index A spans 2^D values corresponding to the dimension of a Clifford algebra in D-dim. In particular X^3, X^{0123}. Event **1** can be described in terms of a spherical bubble (a closed 3-brane) moving in spacetime whose 4-volume (swept by the 3-brane at a given time $X^0_{(1)}$) is given by $X^{0123}_{(1)}$. The center of mass of such bubble is given by the $X^\mu_{(1)}$ coordinates, in particular $X^3_{(1)}$ represents the z-component. Whereas event **2** is described in terms of another spherical bubble of *different size* in spacetime whose 4-volume at a given time $X^0_{(2)}$ is given by $X^{0123}_{(2)}$. The center of mass of such bubble is given now by $X^\mu_{(2)}$ coordinates, in particular $X^3_{(2)}$. If the centers of mass of the small and large bubble *coincide* one has that $\Delta X^3 = 0$, while $\Delta X^{0123} \neq 0$ since the bubbles are of *different size*. Consequently one learns from eq-(29a) that $(\Delta X^3)' \neq 0$ in the new frame of reference : namely, the centers of mass of the bubbles in the new frame of reference do *no* longer *coincide*.

Concluding, the concept of spacetime locality is *relative* due to the *mixing* of 4-volume coordinates with spacetime vector coordinates under generalized Lorentz transformations in C-space. In the most general case, there will be mixing of all polyvector valued coordinates. This was the motivation to build a unified theory of all extended objects, p-branes, for all values of p subject to the condition $p+1 = D$. Therefore, the Extended Relativity Theory in C-spaces (Clifford spaces) were provides a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity (DSR) framework [7].

Next we will show how the quadratic Casimir invariant in C-space leads to modified wave equations, dispersion laws and to the generalizations of the stringy-uncertainty principle relations. The on-shell mass condition for a *massless* polyparticle in the 2^4-dimensional C-space corresponding to a Clifford algebra in $D = 4$, can be rewritten in terms of the polyvector valued components of a wave polyvector $\mathbf{K}$, after setting $L = 1, \hbar = c = 1$ for simplicity, as

$$k^2 \;+\; K_\mu K^\mu \;+\; K_{\mu_1\mu_2} K^{\mu_1\mu_2} \;+\; ...... \;+\; K_{\mu_1\mu_2....\mu_4} K^{\mu_1\mu_2...\mu_4} \;=\; \mathcal{M}^2 \;=\; 0 \qquad (30)$$

A particular *slice* through the 2^4-dimensional C-space can be taken by imposing the set of algebraic conditions

$$k^2 \;=\; 0, \;\; K_{\mu_1\mu_2} K^{\mu_1\mu_2} \;=\; \lambda_1 \; (K_\mu K^\mu)^2 \;=\; \lambda_1 \; K^4 \qquad (31a)$$

$$K_{\mu_1\mu_2\mu_3} \, K^{\mu_1\mu_2\mu_3} \;=\; \lambda_2 \, (K_\mu K^\mu)^3 \;=\; \lambda_2 \, K^6, \;\; K_{\mu_1\mu_2\mu_3\mu_4} \, K^{\mu_1\mu_2\mu_3\mu_4} \;=\; \lambda_3 \, (K_\mu K^\mu)^4 \;=\; \lambda_3 \, K^8 \qquad (31b)$$

where the λ's are numerical parameters. Since k is the Clifford scalar part of the wave polyvector it is invariant under C-space transformations. Hence the condition $k^2 = 0$ will not break the C-space symmetry. However the other slice conditions in eqs-(31a, 31b) will *break* the generalized (extended) Lorentz symmetry in C-space because these conditions are *not* preserved under the most general C-space transformations as described earlier. There will be only the residual standard Lorentz symmetry (in ordinary spacetime) remaining which preserves these conditions/constraints in eqs-(31a, 31b).

Inserting the conditions of eqs-(31) into eq-(30), after setting $k^2 = 0$, yields the modified dispersion law

$$K^2 \ (\ 1 \ + \lambda_1 \ K^2 \ + \lambda_2 \ K^4 \ + \lambda_3 \ K^6 \ ) \ = \ \mathcal{M}^2 - k^2 \ = \ 0 \tag{32}$$

Upon writing explicitly

$$K^2 \ = \ K_\mu \ K^\mu \ = \ |\vec{K}|^2 \ - \ (K_0)^2 \ = \ |\vec{K}|^2 \ - \ (\omega)^2 \tag{33}$$

and solving the algebraic equation for ω in terms of $|\vec{K}|$ obtained from eq-(32) leads to $\omega = \omega(|\vec{K}|)$. Finally, the group velocity (after reinstating c) is given by

$$c(|\vec{K}|) \ = \ \frac{\partial \omega(|\vec{K}|)}{\partial |\vec{K}|} \ = \ c \ + \ ... \tag{34}$$

The group velocity might be greater, smaller or equal to c. From eq-(32) one can deduce immediately that one solution is $K^2 = |\vec{K}|^2 \ - \ (\omega)^2 = 0 \Rightarrow \omega = |\vec{K}| \Rightarrow \frac{\partial \omega(|\vec{K}|)}{\partial |\vec{K}|} = 1$ (in $c = 1$ units) and as expected massless particles move at the speed of light. However, there are *other* solutions to eq-(32) besides the trivial one leading to *energy* dependent speed of propagation. Setting $K^2 = Z$ leads to a cubic equation inside the parenthesis of eq-(32)

$$1 \ + \lambda_1 \ Z \ + \lambda_2 \ Z^2 \ + \lambda_3 \ Z^3 \ = \ 0 \tag{35}$$

that can be solved exactly in terms of the λ's parameters giving 3 roots $z_i(\lambda_1, \lambda_2, \lambda_3), \ i = 1, 2, 3$. The roots can be all real, or one real and a pair of complex conjugate roots. In the former case we have (after reinstating c and adjusting the proper units for z_i) the particular solutions are

$$K^2 \ = \ c^2 \ |\vec{K}|^2 \ - \ (\omega)^2 \ = \ z_i(\lambda_1, \lambda_2, \lambda_3), \quad \Rightarrow \ \omega \ = \ \sqrt{c^2 |\vec{K}|^2 \ - z_i} \ \Rightarrow$$

$$c(|\vec{K}|) = \frac{\partial \omega(|\vec{K}|)}{\partial |\vec{K}|} \ = \ c \ \frac{c \ |\vec{K}|}{\sqrt{c^2 \ |\vec{K}|^2 \ - z_i}} \ = \ c \ \frac{\sqrt{(\omega)^2 + z_i}}{\omega} \quad i = 1, 2, 3 \tag{36}$$

Therefore, from eq-(36) one has an *energy* dependent speed of propagation that can be superluminal if $z_i > 0$, or subluminal if $z_i < 0$, in the case one has 3 real roots to the cubic equation (35). One should add that after differentiating $c^2 \ |\vec{K}|^2 \ - \ (\omega)^2 \ = \ z_i$ in eq-(36) gives

$$2 \ c^2 \ |\vec{K}| \ d|\vec{K}| \ = \ 2 \ \omega \ d\omega \ \Rightarrow \ c^2 \ = \ \frac{\omega}{|\vec{K}|} \ \frac{d\omega}{d|\vec{K}|} \tag{37}$$

leading always to the standard relation $v_{group}\ v_{phase} = c^2$ between group and phase velocities for all the possible solutions. The above results were all obtained by setting the Clifford scalar part k of the wave polyvector to zero. The calculations in the simplest $D = 2$ case when $k^2 \neq 0$ can be found in [5] leading also to the possibility of superluminal propagation.

Thus the key *novel* results one obtains from our analysis of wave propagation in C-space when $k^2 = 0$ are :

1. Irrespective of the solutions found in eqs-(35, 36) the standard dispersion relation $K^2 = c^2|\vec{K}|^2 - (\omega)^2 = 0$ is *always* a solution to eq-(32) giving a constant speed of photon propagation. This is a valid solution to choose whether or not an energy-dependent photon speed is found.

2 . Because the *modified* dispersion relation in eq-(32) *is Lorentz invariant* since the proper norm $K^2 = c^2|\vec{K}|^2 - (\omega)^2$ is Lorentz invariant, one is able to arrive at the energy-dependent speed of propagation $c(|\vec{K}|)$ in eqs-(36) while still *retaining* the Lorentz symmetry. This does *not* occur in DSR nor in other approaches.

The on-shell mass condition for a massive polyparticle moving in the 2^4-dimensional flat C-space, corresponding to a Clifford algebra in $D = 4$, can be written in terms of the polymomentum (polyvector-valued) components, in natural units $L = L_P = 1, \hbar = c = 1$, as

$$\pi^2 + p_\mu\ p^\mu + p_{\mu_1\mu_2}\ p^{\mu_1\mu_2} + p_{\mu_1\mu_2\mu_3}\ p^{\mu_1\mu_2\mu_3} + p_{\mu_1\mu_2....\mu_4} p^{\mu_1\mu_2...\mu_4} = -\ \mathcal{M}^2 \tag{38}$$

Let us *break* the ordinary Lorentz invariance by imposing the non-Lorentz invariant conditions on the poly-momenta in C-space

$$p_{ij}\ p^{ij} = \beta_1\ |\vec{p}|^4,\quad p_{ijk}\ p^{ijk} = \beta_2\ |\vec{p}|^6$$

$$p_{0i}\ p^{0i} = \alpha_1\ (p_0)^2\ |\vec{p}|^2,\quad p_{0ij}\ p^{0ij} = \alpha_2\ (p_0)^2\ |\vec{p}|^4,\quad p_{0ijk}\ p^{0ijk} = \alpha_3\ (p_0)^2\ |\vec{p}|^6 \tag{39}$$

where the α's and β's are numerical parameters. The mass-shell condition in C-space $P_A P^A = -\mathcal{M}^2$ becomes after inserting the conditions (39) and taking into account the chosen signature $(-, +, +, +)$

$$|\vec{p}|^2 \left(\frac{\pi^2}{|\vec{p}|^2} + 1 + \beta_1\ |\vec{p}|^2 + \beta_2\ |\vec{p}|^4 \right) - (p_0)^2 \left(1 + \alpha_1\ |\vec{p}|^2 + \alpha_2\ |\vec{p}|^4 + \alpha_3\ |\vec{p}|^6 \right) = -\ \mathcal{M}^2 \tag{40}$$

One may notice that the terms inside the parenthesis in eq-(40) behave as if one had a *rainbow* metric as follows

$$g^{ij}(\pi^2,\ |\vec{p}|^2)\ p_i\ p_j + g^{00}(|\vec{p}|^2)\ p_0\ p_0 = g^2(\pi^2,\ |\vec{p}|^2)\ |\vec{p}|^2 - f^2(|\vec{p}|^2)\ E^2 = -\ \mathcal{M}^2 \tag{41}$$

A rainbow metric [8] is a one-parameter family of metrics which depends on the energy (momentum) of the test particles moving in a given spacetime background, and forming a rainbow of metrics (rainbow geometry). Setting $\pi^2 = 0$ in eq-(41) one has then that the squared rainbow functions are given by

$$g^2(\pi^2 = 0, \ |\vec{p}|^2) \ \equiv \ 1 \ + \beta_1 \ |\vec{p}|^2 \ + \beta_2 \ |\vec{p}|^4, \ \ \beta_1, \beta_2 > 0 \qquad (42a)$$

$$f^2(|\vec{p}|^2) \ \equiv \ 1 \ + \alpha_1 \ |\vec{p}|^2 \ + \alpha_2 \ |\vec{p}|^4 \ + \ \alpha_3 \ |\vec{p}|^6, \ \ \alpha_1, \alpha_2, \alpha_3 > 0 \qquad (42b)$$

Given

$$g^{ij} \ = \ g^2(\pi^2 = 0, \ |\vec{p}|^2) \ \delta^{ij} \ = \ \Big(1 \ + \beta_1 \ |\vec{p}|^2 \ + \beta_2 \ |\vec{p}|^4 \Big) \ \delta^{ij} \qquad (43a)$$

$$g^{00} \ = \ - \ f^2(|\vec{p}|^2) \ \delta^{00} \ = \ - \ \Big(1 \ + \alpha_1 \ |\vec{p}|^2 \ + \alpha_2 \ |\vec{p}|^4 \ + \ \alpha_3 \ |\vec{p}|^6 \Big) \qquad (43b)$$

the *rainbow* metric is then *defined* as

$$ds^2 \ = \ g_{\mu\nu} \ dx^\mu \ dx^\nu \ =$$

$$- \ \Big(1 \ + \alpha_1 \ |\vec{p}|^2 \ + \alpha_2 \ |\vec{p}|^4 \ + \ \alpha_3 \ |\vec{p}|^6 \Big)^{-1} \ (dt)^2 \ + \ \Big(1 \ + \beta_1 \ |\vec{p}|^2 \ + \beta_2 \ |\vec{p}|^4 \Big)^{-1} \ (dx^i)^2 \qquad (44)$$

Another physical consequence is that the rainbow metric (44) when $\alpha_3 = 0; \alpha_1 = \beta_1; \alpha_2 = \beta_2$ yields *modifications* of the Weyl-Heisenberg algebra

$$[x^\mu, \ p^\nu] \ = \ i \ \hbar \ g^{\mu\nu}(|\vec{p}|^2) \qquad (45)$$

resulting from the momentum-dependent metric (44), and which in turn leads to the following uncertainty relations

$$\Delta x^\mu \ \Delta p^\nu \ \geq \ \frac{\hbar}{2} \ | \ < \Big(1 \ + \alpha_1 \ |\vec{p}|^2 \ + \alpha_2 \ |\vec{p}|^4 \Big) > \ \eta^{\mu\nu} \ | \qquad (46)$$

where $< .... >$ denote the QM expectation values $< \Psi|......|\Psi >$. See [9] for rigorous mathematical details.

From (46) one arrives at the minimal length stringy uncertainty relations [10]

$$\Delta x \ \Delta p_x \ \geq \ \frac{\hbar}{2} \ \Big(1 \ + \alpha_1 \ (\Delta p_x)^2 \Big) \ \Rightarrow \ \Delta x \ \geq \ \frac{\hbar}{2 \Delta p_x} \ + \ (\frac{\hbar \alpha_1}{2}) \ \Delta p_x \qquad (47)$$

Minimizing the expression in (47) and inserting the Planck scale L_P which was set to unity one has for the minimum position uncertainty a quantity of the order of the Planck scale

$$(\Delta x)_{min} \ = \ L_P \ \sqrt{\alpha_1}, \ \ \alpha_1 > 0 \qquad (48)$$

Higher order corrections to the stringy uncertainty relations in eq-(47) stem from the higher grade polymomentum variables in C-space appearing in eq-(46) and correspond, physically, to the membrane contributions to the modified uncertainty relations. Hence, the stringy and membrane corrections to the uncertainty relations in $D = 4$ are of the form (similar equations follow for the other spatial coordinates)

$$\Delta x \; \Delta p_x \; \geq \; \frac{\hbar}{2} \; [\; 1 \; + \alpha_1 \; (\Delta p_x)^2 \; + \; \alpha_2 \; (\Delta p_x)^4 \;] \tag{49}$$

leading to

$$\Delta x \; \geq \; \frac{\hbar}{2} \; [\; \frac{1}{\Delta p_x} \; + \alpha_1 \; (\Delta p_x) \; + \; \alpha_2 \; (\Delta p_x)^3 \;] \tag{50}$$

the extremization problem of (50) is more complicated but there is a local minimum when $\alpha_1 > 0, \alpha_2 > 0$. The value of Δp_x which yields the local minimum for Δx is

$$(\Delta p_x)_o \; = \; \left(\frac{- \; \alpha_1 \; + \; \sqrt{(\alpha_1)^2 + 12\alpha_2}}{6\alpha_2} \right)^{\frac{1}{2}}, \quad \alpha_1 > 0, \; \alpha_2 > 0 \tag{51}$$

If one sets the above value of $(\Delta p_x)_o$ and minimal length uncertainty to coincide with the Planck momentum and Planck scale, respectively, one can fix the numerical values of α_1, α_2. In higher dimensions than $D = 4$ one will capture the p-brane contributions beyond the membrane case due to the contributions of the higher grade polymomenta components. The dimensions (units) of the parameters in eqs-(49-51) are $[\beta_1] = (L/\hbar)^2, \; [\beta_2] = (L/\hbar)^4$.

Related to the minimal length uncertainty in eq-(48) one should mention that the theory of Scale Relativity proposed by Nottale [11] is based on a minimal observational length-scale, the Planck scale, as there is in Special Relativity a maximum speed, the speed of light, and deserves to be looked within the Clifford algebraic perspective. In future work we shall address the fractal nature of quantum spacetime [11] within the framework of quantum Clifford algebras and Scale Relativity. In the quantization program of gravity a key role must be played by quantum Clifford-Hopf algebras since the latter q-Clifford algebras naturally contain the κ-deformed Poincare algebras [12], [13], which are essential ingredients in the formulation of DSR within the context of Noncommutative spaces. The Minkowski spacetime quantum Clifford algebra structure associated with the conformal group and the Clifford-Hopf alternative κ-deformed quantum Poincare algebra was investigated [14].

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