Metamorphoses of resonance curves for two coupled oscillators: the case of small nonlinearities in the main mass frame.

Jan Kyzioł
Department of Mechatronics and Mechanical Engineering,
Politechnika Świętokrzyska,
Al. 1000-lecia PP7, 25-314 Kielce, Poland

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Abstract

We study dynamics of two coupled periodically driven oscillators in general case. Periodic steady-state solutions of the system of two equations are determined within the Krylov-Bogoliubov-Mitropolsky approach. The corresponding amplitude profiles, $A(\omega)$, $B(\omega)$, which are given by two implicit equations, $F(A, B, \omega) = 0$, $G(A, B, \omega) = 0$, where ω is frequency of the driving force, are computed. These two equations, each describing a surface, define a $3D$ curve - intersection of these surfaces. In the present paper we carry out preliminary investigation of metamorphoses of this curve, induced by changes of control parameters. The corresponding changes of dynamics near singular points of the curve are studied.

1 Introduction

In this work we study general case of two coupled oscillators, one of which is driven by an external periodic force. Equations governing dynamics of such system are of form:

$$\left.\begin{array}{l} m\ddot{x} - V(\dot{x}) - R(x) + V_e(\dot{y}) + R_e(y) = F(t) \\ m_e(\ddot{x} + \ddot{y}) - V_e(\dot{y}) - R_e(y) = 0 \end{array}\right\} \tag{1}$$

where $x \equiv x_1$ is position of primary mass m, $y \equiv x_2 - x_1$ is relative position of another mass m_e attached to m and R, V and R_e, V_e are nonlinear elastic restoring force and nonlinear force of internal friction for masses m, m_e, respectively (we use convention $\dot{x} \equiv \frac{dx}{dt}$, etc.). Dynamic vibration absorber is a typical mechanical model described by (1) [1, 2] (in this case m is usually much larger than m_e). Dynamics of coupled periodically driven oscillators is very complicated [3, 4, 5, 6, 7, 8]. In our earlier papers we were able, applying simplifying assumptions, $R(x) = -\alpha x$, $V(\dot{x}) = -\nu \dot{x}$, to derive exact 4th-order nonlinear equation for internal motion (i.e. for variable y only) [9, 10].

In the present paper we study more difficult, entirely nonlinear case:

$$\left.\begin{array}{ll} R(x) = -\alpha x - \gamma x^3, & R_e(y) = -\alpha_e y - \gamma_e y^3 \\ V(\dot{x}) = -\nu \dot{x} + \beta \dot{x}^3, & V_e(\dot{y}) = -\nu_e \dot{y} + \beta_e \dot{y}^3 \\ F(t) = f \cos(\omega t) & \end{array}\right\} \tag{2}$$

which cannot be reduced to exact differential equation for internal motion.

We find approximate steady-state solutions (nonlinear resonances) of the system (1), (2) using the Krylov-Bogoliubov-Mitropolsky (KBM) approach. More exactly, we study dependence of amplitudes A, B on the frequency ω, given implicitly by the KBM method as $F(A, B, \omega; a, b, \ldots) = 0$, $G(A, B, \omega; a, b, \ldots) = 0$ where $a, b, \ldots$ are parameters and F, G are polynomial functions. The implicit functions $A(\omega)$, $B(\omega)$ will be referred to as amplitude profiles or resonance surfaces. We have learned recently that idea to use Implicit Function Theorem to study amplitude profiles was put forward in [11]. Metamorphoses of three-dimensional resonance curve, obtained as intersection of two resonance surfaces, $F(A, B, \omega) = 0$, $G(A, B, \omega) = 0$, induced by changes of the control parameters $a, b, \ldots$, are signatures of nonlinear phenomena. In a simpler case of exact 4-th order nonlinear equation for function $y(t)$ only, i.e. when $R(x)$, $V(\dot{x})$ are linear functions of x, $\dot{x}$, respectively (in this case variable $x(t)$ can be separated off), we deal with resonance curve $A(\omega)$, given by one implicit equation $F(A, \omega; a, b, \ldots) = 0$ only [12, 13, 14, 15].

In the present paper we thus study singular points of the three-dimensional resonance curve, obtained within the KBM approach for the general nonlinear case (1), (2).

The paper is organized as follows. In the next Section equations (1), (2) are transformed into nondimensional form. In Section 3 implicit equations for resonance surfaces $A(\omega)$, $B(\omega)$ are derived within the Krylov-Bogoliubov-Mitropolsky approach, where the amplitudes A, B correspond to small and large masses, respectively. The problem is more difficult than before because these two equations are coupled. In Section 4 we review necessary facts from theory of algebraic curves which are used to compute singular points on three-dimensional resonance curve (intersection of resonance surfaces $A(\omega)$, $B(\omega)$). In Section 5 preliminary computational results are presented. Our results are summarized in the last Section.

2 Equations in nondimensional form

In the first step we transform equations (1), (2) into nondimensional form. Apart from obvious advantages of working with nondimensional variables and parameters this procedure conveniently decreases number of control parameters. We thus introduce nondimensional time τ and frequency Ω and rescale variables

x, y:

$$t=\sqrt{\frac{\mu}{\alpha_e}}\tau,\ \omega=\sqrt{\frac{\alpha_e}{\mu}}\Omega,\ x=\sqrt{\frac{\alpha_e}{\gamma_e}}u,\ y=\sqrt{\frac{\alpha_e}{\gamma_e}}z, \tag{3}$$

to get:

$$\left.\begin{array}{rcl}\ddot{u}+\hat{H}\dot{u}-c\dot{u}^3+\hat{a}u+du^3-\hat{\kappa}h\dot{z}+\hat{\kappa}b\dot{z}^3-\hat{\kappa}z-\hat{\kappa}z^3 & = & \lambda\cos\left(\Omega\tau\right)\\ \ddot{z}+h\dot{z}-b\dot{z}^3+z+z^3-\hat{H}\dot{u}+c\dot{u}^3-\hat{a}u-du^3 & = & -\lambda\cos\left(\Omega\tau\right)\end{array}\right\} \tag{4}$$

where new parameters are given by:

$$\left.\begin{array}{c}a=\frac{\mu\alpha}{M\alpha_e},\ b=\frac{\beta_e}{\gamma_e}\left(\frac{\alpha_e}{\mu}\right)^{\frac{3}{2}},\ c=\frac{\beta\left(\alpha_e\right)^{\frac{3}{2}}}{\sqrt{\mu}m\gamma_e},\ d=\frac{\mu\gamma}{m\gamma_e}\\ h=\frac{\nu_e}{\sqrt{\mu\alpha_e}},\ H=\frac{\nu}{M}\sqrt{\frac{\mu}{\alpha_e}},\ G=\frac{1}{\alpha_e}\sqrt{\frac{\gamma_e}{\alpha_e}}f,\ \kappa=\frac{m_e}{m}\\ \lambda=\frac{\kappa}{\kappa+1}G,\ \hat{H}=H\left(1+\kappa\right),\ \hat{a}=a\left(1+\kappa\right),\ \hat{\kappa}=\frac{\kappa}{\kappa+1}\end{array}\right\} \tag{5}$$

where $M=m+m_e$, $\mu=\frac{mm_e}{M}$ and $\dot{u}\equiv\frac{du}{d\tau}$. Note that $\ddot{u}$ was eliminated from the second of Eqs. (4).

3 Nonlinear resonances

System of equations (4) is written in form:

$$\frac{d^2u}{d\tau^2}+\Omega^2u+\varepsilon\left(\sigma u+g\left(u,z,\tau\right)\right)=0, \tag{6a}$$

$$\frac{d^2z}{d\tau^2}+\Omega^2z+\varepsilon\left(\sigma z+k\left(u,z,\tau\right)\right)=0, \tag{6b}$$

where:

$$\left.\begin{array}{l}\varepsilon\sigma=\Theta^2-\Omega^2,\hat{H}=\varepsilon\hat{H}_0,\hat{a}=\varepsilon\hat{a}_0,b=\varepsilon b_0,c=\varepsilon c_0\\ \Theta^2=\varepsilon\Theta_0^2,d=\varepsilon d_0,h=\varepsilon h_0,\varepsilon\delta_0=1,\lambda=\varepsilon\lambda_0\end{array}\right\} \tag{7}$$

and

$$g\left(u,z,\tau\right)=g_1\left(u\right)+g_2\left(z,\tau\right), \tag{8a}$$

$$k\left(u,z,\tau\right)=k_1\left(z\right)+k_2\left(u,\tau\right), \tag{8b}$$

$$g_1\left(u\right)=\left(\hat{a}_0+d_0u^2-\Theta_0^2\right)u+\hat{H}_0\dot{u}-c_0\dot{u}^3, \tag{8c}$$

$$g_2\left(z,\tau\right)=-\hat{\kappa}h_0\dot{z}+\hat{\kappa}b_0\dot{z}^3-\hat{\kappa}\delta_0z-\hat{\kappa}\delta_0z^3-\lambda_0\cos\left(\Omega\tau\right), \tag{8d}$$

$$k_1\left(z\right)=\left(\delta_0+\delta_0z^2-\Theta_0^2\right)z+h_0\dot{z}-b_0\dot{z}^3, \tag{8e}$$

$$k_2\left(u,\tau\right)=-\hat{H}_0\dot{u}+c_0\dot{u}^3-\hat{a}_0u-d_0u^3+\lambda_0\cos\left(\Omega\tau\right). \tag{8f}$$

Equations (6a), (6b) have been prepared in such way that for $\varepsilon=0$ the solutions are $u\left(\tau\right)=B\cos\left(\Omega\tau+\psi\right)$, $z=A\cos\left(\Omega\tau+\varphi\right)$.

We shall now look for $1:1$ resonance using the Krylov-Bogoliubov-Mitropolsky (KBM) perturbation approach [17, 18]. For small nonzero ε the solutions of Eqs.(6a), (6b) are assumed in form:

$$
\begin{aligned}
u(\tau) &= B\cos(\Omega\tau+\psi)+\varepsilon u_1(B,\psi,\tau)+\ldots && (9)\\
z(\tau) &= A\cos(\Omega\tau+\varphi)+\varepsilon z_1(A,\varphi,\tau)+\ldots && (10)
\end{aligned}
$$

with slowly varying amplitudes and phases:

$$
\begin{aligned}
\frac{dA}{d\tau} &= \varepsilon M_1(A,\varphi)+\ldots, \quad \frac{dB}{d\tau}=\varepsilon P_1(B,\psi)+\ldots, && (11)\\
\frac{d\varphi}{d\tau} &= \varepsilon N_1(A,\varphi)+\ldots, \quad \frac{d\psi}{d\tau}=\varepsilon Q_1(B,\psi)+\ldots. && (12)
\end{aligned}
$$

Computing now derivatives of z from Eqs.(9), (10), (11), (12) and substituting to Eqs.(6) and eliminating secular terms and demanding $M_1=0$, $N_1=0$, $P_1=0$, $Q_1=0$ we obtain the following equations for the amplitude and phase of steady-states:

$$
\left.
\begin{array}{r}
\hat{\kappa}Aw_1x_1+\hat{\kappa}A\Omega w_2x_2-Bw_3x_3+B\Omega w_4x_4+\lambda y_1=0\\
-\hat{\kappa}A\Omega w_2x_1+\hat{\kappa}Aw_1x_2-B\Omega w_4x_3-Bw_3x_4+\lambda y_2=0\\
A\left(\Omega^2-w_1\right)x_1-A\Omega w_2x_2+B\left(\Omega^2+w_3\right)x_3-B\Omega w_4x_4-\lambda y_1=0\\
A\Omega w_2x_1+A\left(\Omega^2-w_1\right)x_2+B\Omega w_4x_3+B\left(\Omega^2+w_3\right)x_4-\lambda y_2=0
\end{array}
\right\} \qquad (13)
$$

where

$$
\left\{
\begin{array}{l}
w_1=\frac{3}{4}A^2+1,\ w_2=\frac{3}{4}\Omega^2bA^2-h\\
w_3=\frac{3}{4}dB^2+\hat{a}-\Omega^2,\ w_4=-\frac{3}{4}\Omega^2cB^2+\hat{H}\\
x_1=\sin\psi,\ x_2=\cos\psi,\ x_3=\sin\varphi,\ x_4=\cos\varphi\\
y_1=\sin(\varphi+\psi)=x_2x_3+x_1x_4,\ y_2=\cos(\varphi+\psi)=x_2x_4-x_1x_3
\end{array}
\right. \qquad (14)
$$

We solve Eqs. (13) for $x_1,\ x_2,\ x_3,\ x_4$ obtaining:

$$
\left\{
\begin{array}{lcl}
x_1 &=& -\frac{\eta_4\left(\left(\hat{\kappa}\Omega^2+\eta_1\right)^2+\eta_2^2\right)+\hat{\kappa}\Omega^4\eta_2}{(\hat{\kappa}-1)\left((\hat{\kappa}\Omega^2+\eta_1)^2+\eta_2^2\right)\lambda}B\\
x_2 &=& \frac{\left(\eta_3+\hat{\kappa}\Omega^2\right)\left(\left(\hat{\kappa}\Omega^2+\eta_1\right)^2+\eta_2^2\right)-\hat{\kappa}\Omega^4\left(\hat{\kappa}\Omega^2+\eta_1\right)}{(\hat{\kappa}-1)\left((\hat{\kappa}\Omega^2+\eta_1)^2+\eta_2^2\right)\lambda}B\\
x_3 &=& \frac{\hat{\kappa}\Omega^2(\eta_2+\eta_4)+\eta_1\eta_4+\eta_2\eta_3}{(\hat{\kappa}-1)\lambda\Omega^2}A\\
x_4 &=& -\frac{\hat{\kappa}\Omega^2\left((\hat{\kappa}-1)\Omega^2+\eta_1+\eta_3\right)+\eta_1\eta_3-\eta_2\eta_4}{(\hat{\kappa}-1)\lambda\Omega^2}A
\end{array}
\right. \qquad (15)
$$

with

$$
\left\{
\begin{array}{ll}
\eta_1=\left(\frac{3}{4}A^2+1-\Omega^2\right)(\hat{\kappa}-1), & \eta_2=\Omega\left(-\frac{3}{4}bA^2\Omega^2+h\right)(\hat{\kappa}-1)\\
\eta_3=\left(\hat{a}+\frac{3}{4}dB^2-\Omega^2\right)(\hat{\kappa}-1), & \eta_4=\Omega\left(-\frac{3}{4}cB^2\Omega^2+\hat{H}\right)(\hat{\kappa}-1)
\end{array}
\right. \qquad (16)
$$

Identities $x_1^2+x_2^2=1,\ x_3^2+x_4^2=1$ lead to implicit nonlinear equations for $X,\ Y,\ Z$:

$$\left.\begin{array}{c} L_1\left(X,\ Y,\ Z\right)=0 \\ L_2\left(X,\ Y,\ Z\right)=0 \end{array}\right\} \tag{17}$$

where

$$X=\Omega^2,\ Y=A^2,\ Z=B^2, \tag{18}$$

and

$$\left\{\begin{array}{l} L_1=Z\left(\left(\eta_4^2+u_2^2\right)\left(u_1^2+\eta_2^2\right)+\hat{\kappa}X^2u_3\right)-\left(\hat{\kappa}-1\right)^2\left(u_1^2+\eta_2^2\right)\lambda^2 \\ u_1=\hat{\kappa}X+\eta_1,\ u_2=\hat{\kappa}X+\eta_3,\ u_3=\hat{\kappa}X^2+2\eta_2\eta_4-2u_1u_2 \end{array}\right. \tag{19a}$$

$$\left\{\begin{array}{l} L_2=\dfrac{Y}{\left(\hat{\kappa}-1\right)^2}\left(u_4^2+\left(\hat{\kappa}Xu_5+\eta_1\eta_3-\eta_2\eta_4\right)^2\right)-\lambda^2X^2 \\ u_4=\left(\hat{\kappa}X\left(\eta_2+\eta_4\right)+\eta_1\eta_4+\eta_2\eta_3\right),\ u_5=\left(\hat{\kappa}-1\right)X+\eta_1+\eta_3 \end{array}\right. \tag{19b}$$

and substitutions (18) are also made in (16). Phases are easily computed from Eqs. (15):

$$\tan\psi=\frac{x_1}{x_2},\ \tan\varphi=\frac{x_3}{x_4}. \tag{20}$$

If we put $c=d=0$ in Eqs. (19), (16) (or $\beta=\gamma=0$ in Eqs. (2)) then the function L_2 becomes independent on Z. In this case it is possible to separate variables in Eqs. (1), (2) obtaining the fourth-order effective equation for the small mass [15]. The function L_2, defined above, for $c=d=0$ is equal to the function $L\left(X,Y\right)$ defined in Eq. (4.1) in [15].

4 General properties of functions $A\left(\Omega\right),\ B\left(\Omega\right)$

Functions $A\left(\Omega\right),\ B\left(\Omega\right)$ are defined implicitly by Eqs. (17), (18), (19). More exactly, Eqs. (17) define a three dimensional curve. Singular points of this curve are computed from the following equations [19]:

$$\left.\begin{array}{r} L_1\left(X,\ Y,\ Z\right)=0 \\ L_2\left(X,\ Y,\ Z\right)=0 \\ \det\begin{pmatrix} \frac{\partial L_1}{\partial X} & \frac{\partial L_2}{\partial X} \\ \frac{\partial L_1}{\partial Y} & \frac{\partial L_2}{\partial Y} \end{pmatrix}=0 \\ \det\begin{pmatrix} \frac{\partial L_1}{\partial Y} & \frac{\partial L_2}{\partial Y} \\ \frac{\partial L_1}{\partial Z} & \frac{\partial L_2}{\partial Z} \end{pmatrix}=0 \\ \det\begin{pmatrix} \frac{\partial L_1}{\partial Z} & \frac{\partial L_2}{\partial Z} \\ \frac{\partial L_1}{\partial X} & \frac{\partial L_2}{\partial X} \end{pmatrix}=0 \end{array}\right\} \tag{21}$$

This means that in a singular point solutions $Y\left(X\right),\ Z\left(X\right)$ or $X\left(Y\right),\ Z\left(Y\right)$ or $X\left(Z\right),\ Y\left(Z\right)$ do not exist.

Let us consider a special case when L_2 does not depend on Z. In this case we have $\frac{\partial L_2}{\partial Z} = 0$ and Eqs. (21) reduce to either of two cases below:

$$\left.\begin{aligned} L_1\left(X,\ Y,\ Z\right) = 0 \\ L_2\left(X,\ Y\right) = 0 \\ \frac{\partial L_1}{\partial Z} = 0 \\ \det\begin{pmatrix} \frac{\partial L_1}{\partial X} & \frac{\partial L_2}{\partial X} \\ \frac{\partial L_1}{\partial Y} & \frac{\partial L_2}{\partial Y} \end{pmatrix} = 0 \end{aligned}\right\} \tag{22}$$

or

$$\left.\begin{aligned} & L_1\left(X,\ Y,\ Z\right) = 0 \\ & L_2\left(X,\ Y\right) = 0,\ \frac{\partial L_2}{\partial X} = 0,\ \frac{\partial L_2}{\partial Y} = 0 \end{aligned}\right\} \tag{23}$$

The latter case, which arises for $c = d = 0$, was studied in our previous paper [15].

5 Computational results

In this Section singular points of amplitude profiles are studied. In the neighbourhood of a singular point the form of amplitude profile changes qualitatively (we call it metamorphosis) and this means that the form of the corresponding steady-state solution changes as well.

In what follows we assume that parameters c, d in Eq. (4) are small, i.e. we study small perturbation of the simpler system of coupled oscillators with $c = d = 0$ (or $\beta = \gamma = 0$, cf. Eq. (2)), described in [15] (let us recall that in this case internal motion can be separated off, leading to simpler equation for the corresponding amplitude profile). The results obtained before for $c = d = 0$ can be thus used to elucidate more complicated system with $c \neq 0$, $d \neq 0$ considered in the present work.

We thus start with the case $c = d = 0$, other parameters assumed as in [15]: $a = 5$, $b = -0.001$, $h = 0.5$, $H = 0.4$, $\kappa = 0.05$. Then for $\lambda = 0.981505$ there is isolated point at $X = 4.153001$, $Y = 4.680111$ and self-intersection for $\lambda = 1.018055$ at $X = 4.835083$, $Y = 4.192055$, see [15].

Next, we have solved equations (21) numerically for small values of parameters c, d, namely for $c = -0.001$, $d = 0.02$. And indeed, there are two singular points: isolated point at $X = 4.154856$, $Y = 4.678062$, $Z = 0.275652$, $\lambda_1 = 0.983240$ and self-intersection at $X = 4.859019$, $Y = 4.171601$, $Z = 0.358503$, $\lambda_2 = 1.020886$, which are clearly equivalents of singular points computed for $c = d = 0$.

Figures 1 show how the singular point (isolated point) at $\Omega = 2.038346$, $A = 2.162883$, $B = 0.525026$ (values of Ω, A, B have been computed from X, Y, Z given above according to Eq. (18)) is created when λ is increased from $\lambda_a = 0.981214 < \lambda_1$ to $\lambda_b = 0.985214 > \lambda_1$. The isolated point is created exactly when green and yellow spire pierces the resonance surface $L_1\left(X, Y, Z\right) = 0$ (red and blue).

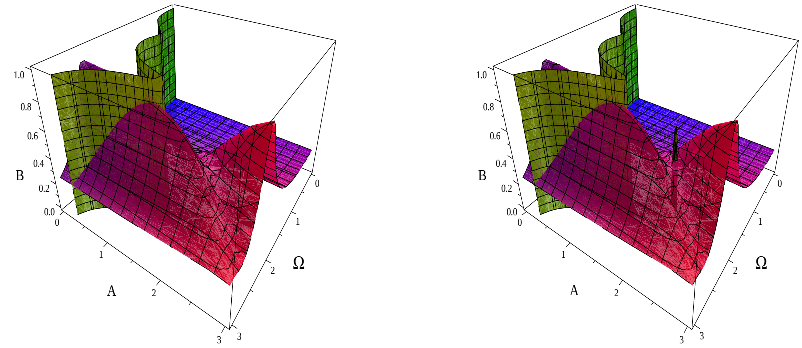

Figure 1: Amplitude profiles: $L_1\,(X,Y,Z) = 0$ (red and blue) and $L_2\,(X,Y,Z) = 0$ (green and yellow); $\lambda = \lambda_a = 0.981214$, left Figure, $\lambda = \lambda_b = 0.985214$, right Figure. Critical value of λ is $\lambda_1 = 0.983240$.

We have computed bifurcation diagrams, shown in Fig. 2, to document change of dynamics occuring in the neighbourhood of the isolated point.

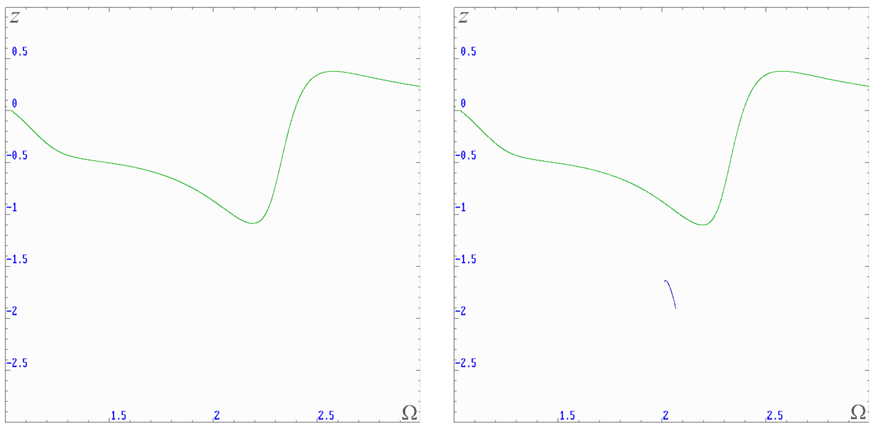

Figure 2: Bifurcation diagrams: $\lambda = \lambda_a = 0.981214$, left Figure, $\lambda = \lambda_b = 0.985214$, right Figure.

Indeed, when λ is increased from λ_a to λ_b another branch of steady-state has been created at $\Omega \cong 2$. This metamorphosis is consistent with change of the amplitude profile shown in Fig. 1.

Similar change of dynamics has been observed near another singular point. In Figs. 3 we see formation of self-intersection at $\Omega = 2.204318$, $A = 2.042450$, $B = 0.598751$ when λ is increased from $\lambda_c = 1.018860$ to $\lambda_d = 1.022860$.

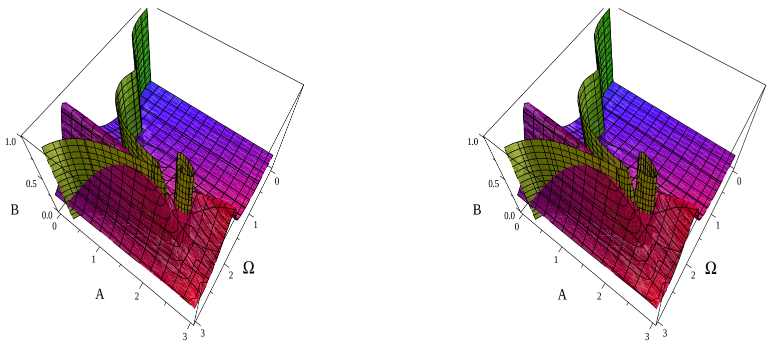

Figure 3: Amplitude profiles: $L_1(X, Y, Z) = 0$ (red and blue) and $L_2(X, Y, Z) = 0$ (green and yellow); $\lambda = \lambda_a = 1.018860$, left Figure, $\lambda = \lambda_b = 1.022860$, right Figure. Critical value of λ is $\lambda_2 = 1.020886$.

Bifurcation diagram shows change of dynamics in the neighbourhood of singular point. Indeed, we see in Figs. 4 that upper branch of the steady-state is disrupted, while lower branch joins another attractor.

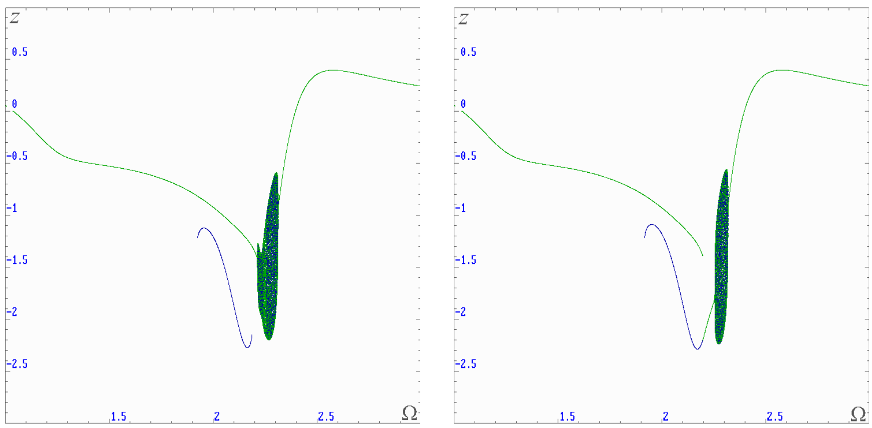

Figure 4: Bifurcation diagrams: $\lambda = \lambda_a = 1.018860$, left Figure, $\lambda = \lambda_b = 1.022860$, right Figure.

6 Summary and discussion

The present work is a preliminary study of general case of dynamics of two coupled periodically driven oscillators, cf. Eq. (1). In this model $R(x)$, $V(\dot{x})$, $R_e(y)$, $V_e(\dot{y})$ are nonlinear functions of the corresponding variables. For the sake of example we have assumed these functions in form (2). For $\beta = \gamma = 0$ (or $c = d = 0$ with c, d defined in (5)) it was possible to separate off motion of small mass to get 4th-order differential equation [15]. Analysis of resonance curve for this equation was relatively simple. On the other hand, the case $c \neq 0$, $d \neq 0$, considered in the present work is much harder since the amplitude profiles are given by system of two coupled implicit equations (17). The amplitude profiles are thus resonance surfaces rather than curves, which arise in the case of a single oscillator.

We have thus studied singular points of the three-dimensional resonance curve (which is obtained as intersection of two resonance surfaces) for small c, d, because we could use our computations, carried out in [15], for $c = d = 0$. We have thus solved equations (21) numerically for $c = -0.001$, $d = 0.02$, calculating two singular points: isolated point and self-intersection. It turns out that qualitative changes of dynamics (metamorphoses) occur in the neighbourhood of singular points of this three-dimensional resonance curve.

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