A Relation between Radiation and Temperature

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It is obtained a relation between the radiated energy density and the absolute temperature.

Key words: radiation, temperature.

For an ideal gas

$$PV = NkT \tag{1}$$

where P is the pressure, V the volume, N the number of particles (atoms or molecules), k the Boltzmann's constant and T the Kelvin's temperature (or absolute temperature). On the other hand, the energy of a single particle ($N = 1$) would be

$$E = \frac{3}{2}kT \tag{2}$$

But also

$$E = \frac{1}{2}mv^2 \tag{3}$$

m and v being the mass and the speed of the particle, respectively. From (1), (2) and (3)

$$PV = NkT = N\frac{2}{3}E = N\frac{2}{3}\frac{1}{2}mv^2 = N\frac{1}{3}mv^2$$

$$P = \frac{1}{3}\frac{Nm}{V}v^2 \tag{4}$$

For the radiation, considered as a gas of photons

$$P = \frac{1}{3}\frac{Nhf}{Vc^2}c^2 = \frac{u}{3} \tag{5}$$

since $v = c$, with an "effective mass" of the photon hf/c^2 and an energy density $u = Nhf/V$, where c is the light speed in vacuum, h the Planck's constant and f the frequency. Note that for a photon, it is $E = mc^2 = (hf/c^2)c^2 = hf$, and from (2) and (3), we have that $3kT = mv^2$, then for $E = mc^2 = hf$ it would be $E = 3kT$, and

$PV = NkT = NE/3 = Nhf/3$ and $P = Nhf/3V = u/3$, which is (5). The total internal energy U of the radiation contained in the volume V would be

$$U = uV = 3PV \tag{6}$$

From the first principle of the thermodynamics $dU = dQ - dW$, where Q is the heat and W the work, with $dW = Fdr = PAdr = PdV$, where F is the force, r the distance and A the area, and from the second principle of the thermodynamics $dQ = TdS$, where S is the entropy, we have that

$$dU = TdS - PdV \tag{7}$$

From (7) and (6), we have

$$dS = \frac{1}{T}dU + \frac{P}{T}dV = 3\frac{V}{T}dP + 4\frac{P}{T}dV$$

$$dS = \left(\frac{\partial S}{\partial P}\right)_V dP + \left(\frac{\partial S}{\partial V}\right)_P dV$$

$$\left(\frac{\partial S}{\partial P}\right)_V = 3\frac{V}{T}$$

$$\left(\frac{\partial S}{\partial V}\right)_P = 4\frac{P}{T}$$

Since

$$\left(\frac{\partial^2 S}{\partial P \partial V}\right)_{V,P} = \left(\frac{\partial^2 S}{\partial V \partial P}\right)_{P,V}$$

then

$$3\left(\frac{\partial (V/T)}{\partial V}\right)_P = 4\left(\frac{\partial (P/T)}{\partial P}\right)_V$$

$$3\left(\frac{\partial (V/T)}{\partial V}\right)_P = \frac{3}{T}\left(\frac{\partial V}{\partial V}\right)_P = \frac{3}{T}$$

$$4\left(\frac{\partial (P/T)}{\partial P}\right)_V = 4\left(\frac{(T\partial P - P\partial T)/T^2}{\partial P}\right)_V = 4\left(\frac{1}{T} - \frac{P}{T^2}\left(\frac{\partial T}{\partial P}\right)_V\right)$$

$$\frac{3}{T} = 4\left(\frac{1}{T} - \frac{P}{T^2}\left(\frac{\partial T}{\partial P}\right)_V\right)$$

$$\left(\frac{\partial P}{P}\right)_V = 4\left(\frac{\partial T}{T}\right)_V$$

$$\int\left(\frac{\partial P}{P}\right)_V = 4\int\left(\frac{\partial T}{T}\right)_V$$

$$\ln P = 4\ln T + \ln b = \ln bT^4$$

$$P = bT^4$$

b being an integration constant. From this last equation and from (5)

$$u = 3bT^4 \tag{8}$$

and the energy density of the radiation is proportional to the fourth power of the absolute temperature.

On the other hand

$$de = \frac{Jds\cos\theta ds'\cos\theta' dt}{\ell^2} \tag{9}$$

where de is the energy of the radiation emitted by the surface ds across the surface ds' in a time dt, J the specific intensity of the radiation, ℓ the distance between the surfaces, and θ and θ' the angles between ℓ and the perpendiculars to ds and ds', respectively. As the solid angle is $d\Omega = ds'\cos\theta'/\ell^2 = ds'_0/\ell^2$ and in spherical coordinates is

$$d\Omega = \sin\theta d\theta d\phi \tag{10}$$

where θ and ϕ are the angular coordinates, then

$$de = Jds\cos\theta d\Omega dt = Jds\cos\theta\sin\theta d\theta d\phi dt$$

$$de = dsdt\int_0^{2\pi} d\phi\int_0^{\pi/2} J\cos\theta\sin\theta d\theta$$

and, in general, J depends on θ and ϕ, but in an isotropic and homogeneous medium, J is a constant, therefore as:

sin 2a = sin(a + a) = sin a cos a + sin a cos a = 2 sin a cos a

sin a cos a = (1/2) sin 2a

$\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta = (1/2)\int_0^{\pi/2} \sin 2\theta\, d\theta = -(1/4)\int_0^{\pi/2} -2\sin 2\theta\, d\theta = -(1/4)\,[\cos 2\theta]_0^{\pi/2}$
$= -(1/4)\,[\cos\pi - \cos 0] = -(1/4)\,(-1 -1) = 1/2$

$\int_0^{2\pi} d\phi = 2\pi$

then $de = J\pi ds dt$ and

$$j = \frac{de}{dsdt} = J\pi \tag{11}$$

where j is the integral radiation and represents the emitted radiation per unit of surface and per unit of time. From (9), $de = Jdsds'dt/\ell^2$, for $\theta = \theta' = 0$, with $\ell = R - r$, where r and R are the radii of ds and ds' respectively, and the volume is $dscdt$, then $du = de/dscdt = Jds'/\ell^2 c$. Since $R >> r$, $\ell \cong R$ and $ds'/\ell^2 \cong ds'/R^2 = d\Omega$. Hence, $du = Jd\Omega/c$, and integrating and using (10)

$$u = \frac{J}{c}\int_0^{4\pi} d\Omega = \frac{J}{c}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta = \frac{J4\pi}{c} \tag{12}$$

And from (11), (12) and (8)

$$j = \frac{cu}{4} = \frac{c3bT^4}{4} = \sigma T^4 \tag{13}$$

where (13) and $\sigma = c3b/4$ are the law and the constant of Stefan-Boltzmann, respectively. For last, integrating the energy density of the Planck radiation formula of a black body for all the frequencies, we have that

$$u = \int_0^{\infty} du(f) = \int_0^{\infty} \frac{8\pi h}{c^3} \frac{f^3}{e^{hf/kT} - 1} df = \frac{8\pi^5 k^4}{15c^3 h^3} T^4 \tag{14}$$

and from (8)

$$\sigma = \frac{c3b}{4} = \frac{2\pi^5 k^4}{15c^2 h^3} \tag{15}$$

and its value is $\sigma = 5.67 \times 10^{-8}\, watt/{}^{\circ}K^4 m^2$. And from (12), (11) and (13): $u = J4\pi/c = (4/c)j = (4\sigma/c)T^4$. That is

$$u = \frac{4\sigma}{c} T^4 \tag{16}$$

which is the relation between the radiated energy density and the absolute temperature, using the Stefan-Boltzmann's constant.