Relativistic Approach to Circular Motion and Solution to Sagnac Effect

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Abstract: The Sagnac effect could not have been exactly explained with consistency under the theory of special relativity (TSR). The conundrum of TSR has been completely solved fully in the relativistic context. Special relativity is reformulated without the postulates of the relativity principle and the light velocity constancy, employing a complex Euclidean space (CES), which is an extension of Euclidean space from the real number to the complex number. In the reformulation, the relativity in the representation and the light velocity constancy are obtained as properties that isotropic space-time spaces have. The coordinate systems each have perpendicular axes in CES and the relativistic transformation has the form of rotation. These characteristics of the formulation pave the way for the relativistic approach to circular motion. The relativistic transformation from the inertial frame to the circular frame is shown to have the same representation as that from the circular to the inertial, which implies that circular motions can be described relative to linear motions and vice versa. The difference between the arrival times of two light beams in the Sagnac experiment can be exactly found by the circular approach presented, which shows that the non-relativistic and relativistic analysis results are the same within a first order approximation. The circular approach can also be applied to the analysis of the Hafele–Keating (HK) experiment. The analysis of Hafele and Keating appears to exploit the results of this paper, though circular motions were treated as liner motions. The relativistic approach for circular motion, which is formulated without any postulates, can lead to propound understanding of relativity and true space-time spaces. These issues are examined.

(**Keywords:** Circular motion, Sagnac effect, Special relativity, True space-time spaces, Paradoxes, Complex space)

I. INTRODUCTION

The theory of special relativity (TSR) [1–5] has been formulated based on the two postulates, the principle of relativity and the constancy of light velocity, which result in the Lorentz transformation between inertial frames. The Sagnac effect [6–10], which is a phenomenon of interference encountered on a rotating plate, seems to show the inconsistency of TSR with the reality. The experiment results appear to be in agreement with the non-relativistic analysis, seemingly violating the postulate that the light speed is constant irrespective of the velocity of an emitting source. Though some relativistic explanations on the Sagnac effect [e.g., 7] have been given, they do not present exact analyses with rigorous theoretical derivations. The arguments on the inconsistency of TSR [e.g., 9] still remain.

This paper completely solves the Sagnac effect in the relativistic context, introducing a complex Euclidean space (CES) in which time is represented as an imaginary number. Special relativity is reformulated in CES without any postulates except the isotropy of inertial space-time spaces [11]. In the reformulation, the relativity, which means that relativistic representations between inertial frames have the same form, and the light velocity constancy are obtained as properties that isotropic space-time spaces have. The relativistic transformation can be regarded as a mapping from one time-space coordinate system S to the other S'. In the Minkowski space [2, 3], the time and space axes of one coordinate system, say S', are not perpendicular to each other though those of the other, say S, are so. In CES, each axis of both coordinate systems is perpendicular to the corresponding one and the relativistic transformation is formulated in the form of rotation. These characteristics of the CES formulation shed light on the relativistic approach to circular motion.

In the relativistic approach to circular motion, which will also be called the circular approach, four coordinate systems S, $\tilde{S}$, $\tilde{S}'$, and S' are exploited. The coordinate systems $\tilde{S}$ and $\tilde{S}'$ with a tilde are rotating while the other ones with no tilde is fixed. The Lorentz transformation from the inertial to the circular is made in the tilde coordinate systems so that the unprimed is relativistically converted to the primed. The transformation results are represented in a time-independent coordinate system S'. The relativistic transformation from the circular to the inertial, in which the primed is relativistically converted to the unprimed, has the same form as the reverse one, from the inertial to the circular. In other words, the relativity between the circular and inertial frames holds in terms of the representation. Moreover the linear velocities β and β' have the same magnitude and opposite directions, i.e., $\beta' = -\beta$ where $\beta = r\omega / c$ ($\beta' = r'\omega' / c$ in the primed) with r and ω (r' and ω') denoting a radius and an angular velocity, respectively, in S (S') and c denotes a light speed in vacuum.

In the Sganac experiment, the effect of the difference between the travel times of two light beams is measured. The circular approach allows us to exactly find the time difference seen through the Lorentz lens, which is shown to be, to a first-order approximation, identical with that by the non-relativistic analysis. The circular approach can also be applied to the analysis of the Hafele–Keating (HK) experiment [12, 13]. The analysis of Hafele and Keating has been known to be consistent with the experiment results. Surprisingly, their analysis, except for part of general relativity, appears to exploit the results of the circular approach, though circular motions were dealt with as linear motions.

Inertial frames are associated with linear motions. For convenience, the word 'linear' in place of 'inertial' is often used such as linear frame and linear motion. It is contrasted with 'circular'. The terminology 'linear inertial frame' is also used to distinguish it and 'circular inertial frame', which is addressed in Section IV.

Following this Introduction, Section II presents the reformulation of special relativity in CES. In Section III, based on the reformulation, the relativistic approach for circular motion is dealt with and time differences in the Sagnac experiment are analyzed. The relativistic approaches presented are formulated without any postulates except the isotropy, which may lead to profound understanding of postulates, paradoxes, relativity, and true space-time spaces. Section IV examines these issues, and is followed by Conclusions.

II. SPECIAL RELATIVITY IN COMPLEX EUCLIDEAN SPACE

The TSR was derived under the postulates of the relativity principle and the light velocity constancy [1]. Here, we reformulate special relativity without the postulates. To this end, we introduce a coordinate system S with an imaginary time axis of $\tau = i\kappa t$ where $i = (-1)^{1/2}$ and t denotes time. The constant κ represents the ratio of time to space in S. For a point $p = (\tau, x)$ in S, the scalar quantity $d(\tau, x)$, which represents the distance between the point and the coordinate origin, is defined as

$$d(\tau, x) = d(p) = (\tau^2 + x^2)^{1/2}. \tag{2.1}$$

In the Minkowski time-space which employs the real time axis, the distance is expressed as $d(p) = (x^2 - ct^2)^{1/2}$.

An observer O' is moving along the x-axis with a constant velocity of v with respect to O who is located at $x = 0$. When $\tau = 0$, the two observers O and O' meet. At $\tau = \tau_0$, O' is at $p_0 = (\tau_0, x_0)$ where $x_0 = -iv\tau_0 / \kappa$. As time passes, O goes along the τ-axis while O' follows the τ'-line which is the line crossing the coordinate origin and p_0, as shown in Fig. 1. In fact, the τ-axis is the set of observation points of O, the observation line of O. The τ'-line is the set of observation points of O', the observation line of O'. Therefore the τ'-line is the time axis for O' and can be written as $\tau' = i\kappa' t'$. In Fig. 1, the x-axis is located perpendicular to the τ-axis in a counter clockwise direction. Accordingly, the space axis, x'-axis, for O' is set. Space-time spaces are assumed to be isotropic in inertial frames so that $\kappa' = \kappa$. The coordinate system with the τ'- and x'-axes is denoted by S', the primed one.

The coordinates of an arbitrary point $p = (\tau, x)$ can be expressed in vector form as $\boldsymbol{p} = [\tau, x]^T$ where T stands for the transpose. As seen in Fig. 1, θ and θ_p represent the angles between the τ- and τ'-axes and between the τ-axis and $\boldsymbol{p}$, respectively. Since τ and τ' are complex numbers, the angles also become complex numbers. For a complex number θ, the trigonometric functions $\cos\theta$ and $\sin\theta$ are defined as $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$. It is straightforward to show that $\cos^2\theta + \sin^2\theta = 1$. When θ is given as shown in Fig. 1, the trigonometric functions can be expressed as

$$\cos\theta = \frac{1}{(1-\beta^2)^{1/2}} \quad (= \tau_0 / d(p_0)) \tag{2.2}$$

$$\sin\theta = \frac{\beta}{i(1-\beta^2)^{1/2}} \quad (= x_0 / d(p_0)) \tag{2.3}$$

where $\beta = v/\kappa$. The vector $\boldsymbol{p}$ can be written in polar form as $\boldsymbol{p} = \|\boldsymbol{p}\| [\cos\theta_p, \sin\theta_p]^T$ where $\|\boldsymbol{p}\| = d(p)$. From Fig. 1, the coordinate vector $\boldsymbol{p}$ is represented in S' as

$$\boldsymbol{p}' = \|\boldsymbol{p}\| \begin{bmatrix} \cos(\theta_p - \theta) \\ \sin(\theta_p - \theta) \end{bmatrix}. \tag{2.4}$$

Exploiting the sum and difference identities in trigonometric function [14], (2.4) is expressed as [11]

$$\boldsymbol{p}' = \boldsymbol{T}_{L2}(\theta)\,\boldsymbol{p} \tag{2.5}$$

where

$$\boldsymbol{T}_{L2}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{2.6}$$

Even if τ, τ', x, and x' are changed to x, x', $-\tau$, and $-\tau'$, respectively, (2.5) is the same, which implies that space and time have the relationship of duality. The matrix $\boldsymbol{T}_{L2}(\theta)$ is the 2 x 2 Lorentz transformation matrix in CES. It is straightforward to see that $\boldsymbol{T}_{L2}(-\theta) = \boldsymbol{T}_{L2}^T(\theta)$ and $\boldsymbol{T}_{L2}^T(\theta)\boldsymbol{T}_{L2}(\theta) = \boldsymbol{I}$ where $\boldsymbol{I}$ is an identity matrix. The relationship (2.5) can be written in differential form as $d\boldsymbol{p}' = \boldsymbol{T}_{L2}(\theta)\, d\boldsymbol{p}$ where $d\boldsymbol{p}$ and $d\boldsymbol{p}'$ are differential vectors. It is easy to see that

$$\| d\boldsymbol{p}' \|^2 = \| d\boldsymbol{p} \|^2 \tag{2.7}$$

which implies that the transformation preserves the distance.

Describing the motion of O from the perspective of O', the observer O is moving with a velocity of $-v$ along the x'-axis in S'. In Fig. 1, if the point for $\boldsymbol{p}$ is p, the position of the point p' for $\boldsymbol{p}'$, which is obtained as (2.5), is different from the position of p since θ_p is a complex number. For example, when $p_0 = (\tau_0, \tau_0 \tan\theta)$, it is represented in S' as $p_0' = (\tau_0 / \cos\theta, 0)$, which is located at a position different from p_0, as shown in Fig. 1. Let the point that is located between the x- and τ'-axes in Fig. 1 represent that for $\boldsymbol{p}'$. It is easy to see form Fig. 1 that $\boldsymbol{p}$ is written as

$$\boldsymbol{p} = \boldsymbol{T}_{L2}(-\theta)\boldsymbol{p}'. \tag{2.8}$$

The comparison of (2.5) and (2.8) leads to the relativity that the relativistic representations between inertial frames have the same form. In TSR, the Lorentz transformation has been derived in the real number with the postulate of the relativity principle. In the CES approach, the relativity is a property derived from the space-time isotropy, which means that $\kappa' = \kappa$. When the space-time space is of isotropy, the observation lines of observers in linear motion have the same characteristics, which enables inertial frames to have the relativity. In fact, the relativity is one of properties that isotropic space-time spaces inherently have and the isotropy of space-time spaces is more fundamental than the relativity.

Distances are preserved as in (2.7). It is well known that a moving clock runs at a slower rate than a clock at rest, which is due to the property of distance preservation. The moving clock corresponds to the clock on an observation line. In Fig. 1, the squared distance between $\boldsymbol{p}_2$ and $\boldsymbol{p}_1$ on the O's observation line is written as $ds^2 = \| d\boldsymbol{p} \|^2 = d\tau^2$ where $d\boldsymbol{p} = \boldsymbol{p}_2 - \boldsymbol{p}_1$ and $d\tau = \tau_2 - \tau_1$ with $\boldsymbol{p}_k = [\tau_k, 0]^T$, $k = 1, 2$. In this case the O's clock is the moving clock. The $\boldsymbol{p}_k$ is expressed in S' as $\boldsymbol{p}_k' = [\tau_k \cos\theta, -\tau_k \sin\theta]^T$ and ds'^2 is given by $ds'^2 = \| d\boldsymbol{p}' \|^2 = (d\tau'^2 + dx'^2)$ where $d\boldsymbol{p}' = \boldsymbol{p}_2' - \boldsymbol{p}_1'$. From $ds'^2 = ds^2$, the time interval $|d\tau'|$ is written as $|d\tau'| = |d\tau^2 - dx'^2|^{1/2}$, which can be rewritten as $|d\tau'| = |d\tau| \cos\theta$ so that $|d\tau|$ is smaller than $|d\tau'|$. In case $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$ are on the observation line of O', on the contrary, $|d\tau'|$ becomes smaller than $|d\tau|$. The differential vector on an observation line has zero spatial components and thus the absolute value of its norm corresponds to the time interval. However, the corresponding one in the other coordinate system has nonzero spatial components. As a result of it, the clock on an observation line runs slower than the other.

The time dilation of clocks on observation lines can also be seen directly from (2.5) and (2.8). The observation line of O' is represented in S as $\boldsymbol{p} = (\tau, \tau \tan\theta)$. Substituting this vector into (2.5), the time coordinate of $\boldsymbol{p}'$ is given by

$$\tau' = \frac{\tau}{\cos\theta}. \tag{2.9}$$

The observation line of O is represented in S' as $\boldsymbol{p}' = (\tau', -\tau' \tan\theta)$. Substituting it into (2.8), the time coordinate of $\boldsymbol{p}$ is given by

$$\tau = \frac{\tau'}{\cos\theta}. \tag{2.10}$$

If either $\| d\boldsymbol{p} \| = 0$ or $\| d\boldsymbol{p}' \| = 0$, the other, from (2.7), is also zero. The differential distance $\| d\boldsymbol{p} \|$ becomes zero when dx/dt, which represents a velocity, is equal to κ. In the isotropic space-time spaces, the value of κ corresponds to an invariant speed. It is known (or postulated) that the light speed is invariant in inertial frames, and then $\kappa = c$. The slope of the line crossing the origin and the point p in Fig. 1 is $\tan\theta_p$. The velocity of an object on the line is given by $v_p / c = i \tan\theta_p$, which is represented in S', by the sum and difference identities, as

$$v_p'/c = i \tan(\theta_p - \theta) = \frac{v_p / c - v/c}{1 - v_p v / c^2}. \tag{2.11}$$

It is shown in (2.11) that v_p' becomes c when $v_p = c$. The invariant speed is also one of properties that the isotropic space-time spaces inherently have. The speed that has the same value as κ becomes invariant and it is the light speed.

The formulation in CES provides some profound points in the fundamental concepts, which can be summarized as follows:

1) The special relativity in CES has been formulated without any postulates except the isotropy of space-time spaces. The isotropy means that the scale ratios of time to space in space-time spaces with units are the same. The inherent value of the scale ratio κ appears to be one in unit-less time-space before the manifestation.

2) The relativity is obtained as one of the properties that the isotropic space-time spaces have. It is not identical with the postulation of the relativity principle in TSR. The former means that the relativistic representations between inertial frames have the same form as (2.5) and (2.8), but not the equivalence between them. The light velocity constancy is also obtained as a property. The isotropy of spaces allows them to have an invariant speed. If they are not isotropic so that $\kappa' \neq \kappa$, the speeds are different even though $\| d\boldsymbol{p}' \| = 0$ when $\| d\boldsymbol{p} \| = 0$. The isotropic spaces have an invariant speed of

κ .

3) The coordinate system S' obtained form S according to (2.5) is just for the observer O'. In other words, there is just one observation line, the one for O', and any $x' = x_1'(\neq 0)$ in Fig. 1 are not observation lines. The coordinate system S' shows how the observer sees the world, in connection with the view that O sees.

4) Time and space are in the duality relationship. One is represented in the real number while the other is represented in the imaginary number.

5) The CES formulation can deal with an inertial frame moving faster than c . When $v > c$, the roles of space and time are interchanged.

As a matter of fact, the Lorentz transformation with the imaginary time is older than that with the real number and was already used by Poincare [3]. However, the old formulation may be nothing more than a mathematical manipulation. The CES approach is based on the concept of the observation line and derives the Lorentz transformation without any postulates except the isotropy. It enables us to get insight into the fundamental concepts, as presented above.

III. CIRCULAR MOTION AND RELATIVISTIC APPROACH

Here, we extend the CES to 3-dimension (3-D), including the y- and y'-axes. Unprimed and primed coordinate systems are related by the Lorentz transformation. In addition to S and S', coordinate systems $\tilde{S}$ and $\tilde{S}'$ are introduced to handle circular motions. The observers of S , $\tilde{S}$, $\tilde{S}'$, and S' are denoted by O , $\tilde{O}$, $\tilde{O}'$, and O', respectively. Motions between O and $\tilde{O}'$ are dealt with. The velocity directions of $\tilde{O}$ and $\tilde{O}'$ always correspond to the $\tilde{x}$ -axis of $\tilde{S}$ and $\tilde{x}'$-axis of $\tilde{S}'$, respectively.

A. Framework for Relativistic Approach

The unprimed and primed coordinate systems S , $\tilde{S}$, $\tilde{S}'$, and S' are shown in Fig. 2, where the radius of the sphere, though part of it is shown, is one. For convenience, negative parts of the τ - and τ'-axes are displayed and the $\tilde{x}$ -axis is rotated by φ with respect to the x -axis. The $\tilde{\tau}$ - and $\tilde{x}$ -axes and the $\tilde{\tau}'$- and $\tilde{x}'$-axes lie on the same plane as the axes of S and S' in Fig. 1 are on the same plane.

The observer $\tilde{O}$ is moving with a velocity of $\beta = v/c$ in the direction of $\tilde{x}$ -axis when seen by O . Given a coordinate vector $\tilde{\boldsymbol{p}} = [\tilde{\tau}, \tilde{x}, \tilde{y}]^T$ in $\tilde{S}$, the Lorentz-transformed vector

$\tilde{\boldsymbol{p}}' = [\tilde{\tau}', \tilde{x}', \tilde{y}']^T$ in $\tilde{S}'$ is written as

$$\tilde{\boldsymbol{p}}' = \boldsymbol{T}_L(\theta)\tilde{\boldsymbol{p}} \tag{3.1}$$

where $\boldsymbol{T}_L(\theta)$ is the 3 x 3 Lorentz transformation matrix

$$\boldsymbol{T}_L(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.2}$$

A problem in the representation into $\tilde{S}$ and $\tilde{S}'$ is that if their axes are rotating it is meaningless to represent coordinates in them. Hence, it is necessary to represent coordinates in another coordinate systems which are time-independent.

The coordinates of $\tilde{\boldsymbol{p}}$ can be readily converted into a time-independent coordinate system S. Considering the representation of $\tilde{\boldsymbol{p}}$ into S, (3.1) is rewritten as

$$\tilde{\boldsymbol{p}}' = \boldsymbol{T}_L(\theta)(\boldsymbol{A}(\varphi)\boldsymbol{A}^{-1}(\varphi))\tilde{\boldsymbol{p}} = \boldsymbol{T}_L(\theta)\boldsymbol{A}(\varphi)\boldsymbol{p} \tag{3.3}$$

where $\boldsymbol{A}(\varphi)$ is a rotation matrix, which given by

$$\boldsymbol{A}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \tag{3.4}$$

and $\tilde{\boldsymbol{p}}$ and $\boldsymbol{p}$ are related by

$$\tilde{\boldsymbol{p}} = \boldsymbol{A}(\varphi)\boldsymbol{p}. \tag{3.5}$$

The inverse of $\boldsymbol{A}(\varphi)$ is given by $\boldsymbol{A}(-\varphi)$. Form (3.5), $\boldsymbol{p} = \boldsymbol{A}(-\varphi)\boldsymbol{T}_L(-\theta)\tilde{\boldsymbol{p}}'$, which can be rewritten as

$$\begin{aligned} \boldsymbol{p} &= \boldsymbol{A}(-\varphi)\boldsymbol{T}_L(-\theta)(\boldsymbol{A}'(\varphi')\boldsymbol{A}'^{-1}(\varphi'))\tilde{\boldsymbol{p}}' \\ &= \boldsymbol{A}(-\varphi)\boldsymbol{T}_L(-\theta)\boldsymbol{A}'(\varphi')\boldsymbol{p}' \end{aligned} \tag{3.6}$$

where

$$\boldsymbol{p}' = \boldsymbol{A}'^{-1}(\varphi')\tilde{\boldsymbol{p}}'. \tag{3.7}$$

A time-independent coordinate system in the primed is denoted by S'. In (3.7), $\boldsymbol{A}'^{-1}(\varphi')$ converts $\tilde{S}'$-coordinates into S'-coordinates. It can be assumed without loss of generality that when $\varphi = 0$, $\varphi' = 0$ so that the τ'- and x'-axes of S' can be placed in the same plane as the τ- and x-axes of S, as in Fig. 2 where the y- and y'-axes are overlapped. If $\theta = 0$, S' corresponds to S. The S' is the primed coordinate system corresponding to the unprimed S.

To find $\boldsymbol{A}'(\varphi')$, we use Fig. 2. Recall that the Lorentz transformation has been already done, and the remaining task is to find the difference in the orientation between S' and $\tilde{S}'$ to obtain φ', which can be a single variable or a vector with multiple variables. In Fig. 2, the $\tilde{y}'$-axis lies on the x-y plane and the locus of the $\tilde{x}'$-axis forms a cone as φ increases from zero to 2π. It is obvious that if the $\tilde{x}'$-axis is rotated by $-\varphi_u'$ on the surface of the cone, $\tilde{S}'$ exactly overlaps S'. Fig. 3 clearly shows the orientation of $\tilde{S}'$ with respect to S'. Note that $\tilde{\tau}'=\tau'$. Since the radius of the sphere in Fig. 2 is equal to one, the azimuth angle φ_u' is identical with the arc length between P and $\tilde{P}$. To use Fig. 2 for the calculation of the arc length, we change the τ-axis to a real axis, z-axis. Accordingly, θ becomes a real number θ_z and $\cos\theta_z$ is given by

$$\cos\theta_z = \frac{1}{(1+\beta^2)^{1/2}}. \tag{3.8}$$

In the coordinate system S_z with the z-axis, the coordinates of P_z, which has the same position as the point P in Fig. 2, are written as

$$P_z = (x_1, y_1, z_1) = (\cos\theta_z, 0, \sin\theta_z). \tag{3.9}$$

The arc length is equal to $x_1\varphi$ and thus φ_u' is given by

$$\varphi_u' = \varphi\cos\theta_z. \tag{3.10}$$

Since relative motions between O and $\tilde{O}'$ are dealt with, φ' should be used in accordance with the perspective of $\tilde{O}'$ as φ is used in accordance with the perspective of O in the coordinate conversion of (3.5). Fig. 2 shows the coordinate systems from the perspective of the non-rotation (O and O'). According to the perspective of the rotation ($\tilde{O}'$ and $\tilde{O}$), S and S' are rotated by $-\varphi$ and $-\varphi_u'$ respectively. Thus φ' is obtained as

$$\varphi' = -\varphi_u' = -\varphi\cos\theta_z. \tag{3.11}$$

If $\theta_z = 0$, $\varphi' = -\varphi$. For $\theta_z \neq 0$, $|\varphi'| < |\varphi|$.

As $\boldsymbol{A}'(\varphi')$ is also a rotation matrix, $\boldsymbol{A}'(\varphi') = \boldsymbol{A}(\varphi')$. Using this relationship, (3.3) and (3.7), we have

$$\boldsymbol{p}' = \boldsymbol{T}_{LR}(\theta,\varphi)\boldsymbol{p} \tag{3.12}$$

where

$$\boldsymbol{T}_{LR}(\theta,\varphi) = \boldsymbol{A}(-\varphi')\boldsymbol{T}_L(\theta)\boldsymbol{A}(\varphi). \tag{3.13}$$

B. Circular/Linear Transformation

1) Linear-to-Circular

In the linear-to-circular, motions are described relative to O or O'. We have established a framework for the relativistic transformation of circular motion. Consider that φ varies linearly with time:

$$\varphi = \omega_c \tau \tag{3.14}$$

where $\omega_c = -i\omega / c$ and ω is a constant. The $\tilde{x}$- and $\tilde{y}$-axes in Fig, 2 are rotating around the rotating center C_R with an angular velocity ω. The observer $\tilde{O}$ is located at a radius r in $\tilde{S}$, and the velocity of $\tilde{O}$ is $\beta = ir\omega_c = r\omega / c$ in the $\tilde{x}$-axis direction. We use a notation $\boldsymbol{v}_s$ to denote the 2-D spatial vector of a space-time 3-D vector $\boldsymbol{v}$, excluding the time component. The spatial vectors $\boldsymbol{p}_s = [x, y]^T$ and $\tilde{\boldsymbol{p}}_s = [\tilde{x}, \tilde{y}]^T$ are related by

$$\tilde{\boldsymbol{p}}_s = \boldsymbol{A}_s(\varphi)\boldsymbol{p}_s \tag{3.15}$$

where $\boldsymbol{A}_s(\varphi)$ is a rotation matrix for spatial coordinates,

$$\boldsymbol{A}_s(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}. \tag{3.16}$$

The direct transformation of the unprimed coordinates into the primed ones according to (3.12) can be applied to the trivial case where φ is constant. In that case, with φ' obtained as (3.11), we can represent $\tilde{\boldsymbol{p}}'$ into S'. However, in case φ is varying with time, the relativistic transformation should be handled with differential vectors which can be considered to be constant during an infinitesimal time interval $d\tau$. In other words, the vectors in (3.1) and (3.7) should be replaced with differential vectors in such a way that

$$d\overline{\boldsymbol{p}}'(\tau) = \boldsymbol{T}_L(\theta)\, d\tilde{\boldsymbol{p}}(\tau) \tag{3.17}$$

$$d\boldsymbol{p}'(\tau') = \boldsymbol{A}(-\varphi')\, d\overline{\boldsymbol{p}}'(\tau') \tag{3.18}$$

$$\tilde{\boldsymbol{p}}'(\tau') = \boldsymbol{A}(\varphi')\, \boldsymbol{p}'(\tau')\,. \tag{3.19}$$

where $d\boldsymbol{v}(\tau) = \boldsymbol{v}(\tau + d\tau) - \boldsymbol{v}(\tau)$ for a vector $\boldsymbol{v}(\tau)$. Note that the new notation $d\overline{\boldsymbol{p}}'$, not $d\tilde{\boldsymbol{p}}'$, has been introduced. The $\tilde{\boldsymbol{p}}'(\tau)$ can be obtained as (3.19) only after $\boldsymbol{p}'(\tau)$ has been found. Let us explain the reason, which also explains why the differential vector $d\tilde{\boldsymbol{p}}(\tau)$, instead of $\tilde{\boldsymbol{p}}(\tau)$, should be used for the Lorentz transformation of (3.17). As $\tilde{S}$ rotates, $\tilde{S}'$ also does. For simplicity, suppose that the angular velocity of $\tilde{S}'$ is constant. Fig. 4 illustrates the rotation of the $\tilde{x}'$-axis with respect to x' and the differential angle between the axes is $d\varphi'$, which is an angle shift during $d\tau'$.

Let φ' be zero at $\tau' = 0$. During a time interval $0 < \tau' \le \tau_1'$, where $\tau_1' = N\,d\tau'$, the differential vector $d\bar{\boldsymbol{p}}'$ changes from $d\bar{\boldsymbol{p}}'(d\tau')$ to $d\bar{\boldsymbol{p}}'(\tau_1')$ and the rotation angle φ' increases from $d\varphi'$ to φ_1' where $\varphi_1' = N\,d\varphi'$. Accordingly, for example, the differentials $d\bar{\boldsymbol{p}}'(d\tau')$ and $d\bar{\boldsymbol{p}}'(\tau_1')$ should be converted in such a way that $d\boldsymbol{p}'(d\tau') = \boldsymbol{A}(-d\varphi')\,d\bar{\boldsymbol{p}}'(d\tau')$ and $d\boldsymbol{p}'(\tau_1') = \boldsymbol{A}(-\varphi_1')\,d\bar{\boldsymbol{p}}'(\tau_1')$. If $\boldsymbol{p}'(\tau_1')$ is calculated as $\boldsymbol{p}'(\tau_1') = \boldsymbol{A}(-\varphi_1')\,\bar{\boldsymbol{p}}'(\tau_1')$ after first integrating $d\bar{\boldsymbol{p}}'(\tau')$ to obtain $\bar{\boldsymbol{p}}'(\tau_1')$ or after directly finding $\bar{\boldsymbol{p}}'(\tau_1')$ from $\tilde{\boldsymbol{p}}(\tau_1')$ without using differential vectors, all differentials $d\bar{\boldsymbol{p}}'(k\,d\tau')$ for $k < N$ are incorrectly converted to the S'-coordinates. Therefore, differential vectors should be used and converted into S' as (3.18).

Hereafter, for simplicity, we will drop the argument indicating time dependency in notations, if not necessary. The motion of $\tilde{O}$ seen in S can be described as

$$\boldsymbol{p}_s = r[\sin\varphi, -\cos\varphi]^T . \tag{3.20}$$

The tilde vector for $\boldsymbol{p}_s$ is obtained by inserting (3.20) into (3.15) as

$$\tilde{\boldsymbol{p}}_s = [0, -r]^T . \tag{3.21}$$

Equation (3.21) indicates that $\tilde{O}$ is at rest on the $\tilde{y}$-axis of $\tilde{S}$. Recall that $\varphi = \omega t$. The velocity vector $\dot{\boldsymbol{p}}_s$ of $\boldsymbol{p}_s$ is given by

$$\dot{\boldsymbol{p}}_s = \frac{d\boldsymbol{p}_s}{dt} = r\omega[\cos\varphi, \sin\varphi]^T . \tag{3.22}$$

Representing $\dot{\boldsymbol{p}}_s$ in $\tilde{S}$, it is expressed by substituting (3.22) into (3.15) as

$$\tilde{\dot{\boldsymbol{p}}}_s = [r\omega, 0]^T . \tag{3.23}$$

The angular velocity of $\tilde{O}$ is in the $\tilde{x}$-axis direction, as seen in (3.23), and $\tilde{O}$ rotates in the $\tilde{x}$-axis direction that is perpendicular to the $\tilde{y}$-axis on which $\tilde{O}$ is located. The Lorentz transformation in (3.17) should be made under the condition that the velocity has the same direction as the $\tilde{x}$-axis. Note that the velocity of $\tilde{O}$ is in the $\tilde{x}$-axis direction, as required.

Given $\boldsymbol{p} = [\tau, \boldsymbol{p}_s^T]^T$ together with the initial condition that $\tau' = 0$ when $\tau = 0$, the elements of $\boldsymbol{p}' = [\tau', \boldsymbol{p}_s'^T]^T$ are obtained as

$$\tau' = \tau\cos\theta \tag{3.24}$$

$$\boldsymbol{p}_s' = r'[\sin\varphi', -\cos\varphi']^T \tag{3.25}$$

and φ' is related to τ' by

$$\varphi' = \omega_c' \tau' \tag{3.26}$$

where

$$r' = \frac{r\cos\theta}{\cos\theta_z} \tag{3.27}$$

$$\omega_c' = -\frac{\omega_c \cos\theta_z}{\cos\theta} \tag{3.28}$$

with $\omega_c' = -i\omega'/c$. In (3.25), the initial condition that the phase of $\boldsymbol{p}_s'$ is $-\pi/2$ at $\tau' = 0$ was used. For derivation, refer to Appendix A. In case $\beta = 0$, any relativistic things must not occur. It is not difficult to see that when $\beta = 0$, $\boldsymbol{p}'$ is reduced to $\boldsymbol{p}$. The coordinate vector $\tilde{\boldsymbol{p}} = [\tau, \tilde{\boldsymbol{p}}_s^T]^T$, where $\tilde{\boldsymbol{p}}_s$ is given by (3.21), represents the time axis of $\tilde{O}$. Equation (3.24) implies that the clock of $\tilde{O}$, which corresponds to a moving clock in the Lorentz transformation of (3.17) since $\tilde{\boldsymbol{p}}$ represents the time axis of $\tilde{O}$, runs slower than that of $\tilde{O}'$.

In the primed, β' is written as $\beta' = i\,r'\omega_c' = r'\omega'/c$. From (3.27) and (3.28),

$$r'\omega_c' = -r\,\omega_c \tag{3.29}$$

which leads to

$$\beta' = -\beta \ . \tag{3.30}$$

The β and β' are linear velocities. Though the angular velocities in S' and S are different, the linear speeds are the same.

The r' and ω_c' are both functions of r and ω. It is easy to see that $f(\beta) > 1$ and $f(\beta)$ is a monotonic increasing function where $f(\beta) = \cos\theta/\cos\theta_z$ and $0 < |\beta| < 1$. Hence $r' > r$ and $|\omega'| < |\omega|$, which indicate the radius increases and the angular speed decreases in S'. The dependency of r' on the angular velocity ω, the direction of which is perpendicular to the radial direction, the $\tilde{y}$-axis direction in $\tilde{S}$, may be pointed out. The radial component does not change in the Lorentz transformation process, but comes to change due to the coordinate conversion of $d\overline{\boldsymbol{p}}'$ form $\tilde{S}'$ to S'. For details, refer to Appendix A.

The spatial vectors $\boldsymbol{p}_s'$ and $\tilde{\boldsymbol{p}}_s'$ are related by $\tilde{\boldsymbol{p}}_s' = \boldsymbol{A}_s(\varphi')\boldsymbol{p}_s'$. The position vector of $\tilde{O}'$ is given by

$$\tilde{\boldsymbol{p}}_s' = [0, -r']^T \ . \tag{3.31}$$

Equation (3.31) indicates that $\tilde{O}'$ is at rest on the $\tilde{y}'$-axis of $\tilde{S}'$. The velocity $\dot{\boldsymbol{p}}_s'$ of $\boldsymbol{p}_s'$ is the same as (3.22) with r, ω and φ changed to primed ones and it is represented in $\tilde{S}'$ as $\dot{\tilde{\boldsymbol{p}}}_s' = [r'\omega', 0]^T$. The velocity is in the $\tilde{x}'$-axis direction. Accordingly $\tilde{O}'$ rotates in the $\tilde{x}'$-axis direction that is perpendicular to the $\tilde{y}'$-axis on which $\tilde{O}'$ is located, which is consistent with the requirement of the $\tilde{x}$-direction motion in the unprimed.

In general, $\boldsymbol{p}_s$ can be expressed as

$$\begin{aligned}\boldsymbol{p}_s &= r[\sin(\varphi+\varphi_0), -\cos(\varphi+\varphi_0)]^T \\ &= r[\sin\omega_c(\tau+\tau_0), -\cos\omega_c(\tau+\tau_0)]^T\end{aligned} \tag{3.32}$$

where $\varphi_0 = \omega_c \tau_0$ and $-\pi < \varphi_0 \le \pi$. The spatial vector (3.32) has a phase of $-\pi/2$ at $\tau = -\tau_0$. Comparing (3.32) and (3.20), the latter is rotated by φ_0 relative to the former. The tilde vector $\tilde{\boldsymbol{p}}_s$ for (3.32) is written as

$$\tilde{\boldsymbol{p}}_s = \boldsymbol{A}_s(\psi)\boldsymbol{p}_s = [0, -r]^T \tag{3.33}$$

where $\psi = \varphi + \varphi_0$. The velocity $\dot{\tilde{\boldsymbol{p}}}_s = \boldsymbol{A}_s(\psi)(d\boldsymbol{p}_s/dt)$ is given as (3.23). The relativistic transformation of $\boldsymbol{p} = [\tau, \boldsymbol{p}_s^T]^T$ yields

$$\tau' = \tau\cos\theta \tag{3.34}$$

$$\boldsymbol{p}_s' = r'[\sin(\varphi'+\varphi_0'), -\cos(\varphi'+\varphi_0')]^T \tag{3.35}$$

where $\varphi_0' = -\varphi_0\cos\theta_z$. The radius r' and the angular velocity ω_c' are equal to (3.27) and (3.28), respectively. In (3.34), the initial condition was used that $\tau' = 0$ when $\tau = 0$, and in (3.35) that the phase of $\boldsymbol{p}_s'$ is $-\pi/2$ when $\tau' = -\tau_0'$ with $\tau_0' = \tau_0\cos\theta$. The derivations are presented in Appendix B. The vector $\boldsymbol{p}_s'$ is represented in $\tilde{S}'$ as

$$\tilde{\boldsymbol{p}}_s' = \boldsymbol{A}_s(\varphi'+\varphi_0')\boldsymbol{p}_s' = [0, -r]^T. \tag{3.36}$$

2) Circular-to-Linear

Motions are described relative to $\tilde{O}'$ or $\tilde{O}$, as shown in Fig. 5 where the linear velocity in S' of O is assumed to be positive. From the point of view of $\tilde{O}'$ who is at rest in $\tilde{S}'$, an observer O' located at a radius r' in S' is rotating with an angular velocity of ω', which leads to $\beta' = \omega' r'/c$. Therefore, S' and S become rotating coordinate systems whereas $\tilde{S}'$ and $\tilde{S}$ are fixed. The Lorentz transformation is performed from S' to S for the motion in the x'-axis direction and the resulting differential vector is converted into $\tilde{S}$. The relativistic transformation process from the primed to the unprimed is as follows:

$$d\bar{\boldsymbol{p}} = \boldsymbol{T}_L(\theta)\, d\boldsymbol{p}' \tag{3.37}$$

$$d\tilde{\boldsymbol{p}} = \boldsymbol{A}(-\varphi)\, d\bar{\boldsymbol{p}} \tag{3.38}$$

$$\boldsymbol{p} = \boldsymbol{A}(\varphi)\, \tilde{\boldsymbol{p}} \tag{3.39}$$

where φ is related to φ' by

$$\varphi = -\varphi' \cos\theta_{z'}\,. \tag{3.40}$$

In (3.40), the z'-axis is a real axis that the imaginary τ'-axis of $\tilde{S}'$ is changed to, and $\cos\theta_{z'}$ is the same as (3.8) with β replace by β'. Fig. 5 is drawn from the point of view of the primed observer $\tilde{O}'$ and φ_p is equal to $-\varphi$.

The motion of O' can be described in $\tilde{S}'$ as

$$\tilde{\boldsymbol{p}}_s' = r'[\sin\varphi', -\cos\varphi']^T \tag{3.41}$$

where $\varphi' = \omega_c' \tau'$. The velocity vector of $\tilde{\boldsymbol{p}}_s'$ is expressed in S' as

$$\dot{\boldsymbol{p}}_s' = \boldsymbol{A}(\varphi') \frac{d\tilde{\boldsymbol{p}}_s'}{dt'} = [r'\omega', 0]^T \tag{3.42}$$

which implies that the angular velocity is in the x'-axis direction.

The coordinate vector of O' is given by $\boldsymbol{p}' = [\tau', 0, -r']^T$. With the $\boldsymbol{p}'$, the relativistic transformation from the circular to the linear yields

$$\tau = \tau' \cos\theta \tag{3.43}$$

$$\tilde{\boldsymbol{p}}_s = r\,[\sin\varphi, -\cos\varphi]^T \tag{3.44}$$

$$\varphi = \omega_c \tau \tag{3.45}$$

$$\beta = -\beta' \tag{3.46}$$

where

$$r = \frac{r' \cos\theta}{\cos\theta_{z'}} \tag{3.47}$$

$$\omega_c = -\frac{\omega_c' \cos\theta_{z'}}{\cos\theta}\,. \tag{3.48}$$

Note that the linear speed does not vary. Comparing (3.43) – (3.48) for the circular-to-linear with the corresponding ones for the linear-to-circular, they have the same form. In that sense, the relativity between the circular and linear motions holds.

Now, we have prepared tools for the relativistic approach to circular motion. Let's go to tackle the conundrum, the Sagnac effect.

C. Analysis of Sagnac Effect

In the Sagnac experiment, the coordinate system S represents the one for a laboratory observer O. A circular plate is rotating around its center with an angular velocity Ω and the light detector $\tilde{O}'$ is located at a radius r when seen by O. The motion of $\tilde{O}'$ in S can be described as (3.20) with $\varphi = \Omega t = \Omega_c \tau$ where $\Omega_c = -i\Omega / c$. At $\tau = \tau' = 0$, two light beams leave a light source, which is located at the same place as the detector, and begin to travel on a circumference in different directions. The light signals traveling in the same direction as the plate rotation and in the opposite direction are denoted by p_+ and p_-, respectively. According to the non-relativistic analysis, the light beams p_+ and p_- arrive at the detector at the instants [9]

$$t_\pm = \frac{2\pi r}{c(1 \mp \beta)} \tag{3.49}$$

where t_+ and t_- denote the arrival times of p_+ and p_-, respectively, and $\beta = r\Omega / c$. The difference in the arrival time is calculated as

$$\Delta t_d = (t_+ - t_-) = \frac{4\pi r^2 \Omega}{c^2 (1 - \beta^2)}. \tag{3.50}$$

The time difference (3.50) corresponds to the one seen in S under the assumption that the speeds of the light signals p_+ and p_- are c in S.

Let us find the time difference $\Delta\tau_d{}'$ observed through the relativistic lens. We use the transformation for the circular-to-linear of Subsection III.B.2. In S', the rotated angle for $\Delta\tau_d{}'$ is given by

$$\Delta\varphi' = \Omega_c{}' \Delta\tau_d{}'. \tag{3.51}$$

Then, $\Delta\tau_d$ is expressed, from (3.40), as

$$\Delta\tau_d = -\frac{\Delta\varphi' \cos\theta_{z'}}{\Omega_c}. \tag{3.52}$$

Inserting (3.48) and (3.51) in (3.52), we have

$$\Delta t_d = -\frac{\Delta t_d{}' \, \Omega_c{}' \cos\theta_{z'}}{\Omega_c} = \Delta t_d{}' \cos\theta. \tag{3.53}$$

In a different manner, the time difference can be directly found from (3.43). Replacement τ' in (3.43) by $-i\Delta t_d{}'/c$ results in (3.53). The time difference $\Delta t_d{}'$ is calculated by substituting (3.50) into (3.53) as

$$\Delta t_d' = \frac{4\pi r^2 \Omega}{c^2 (1-\beta^2)^{1/2}} . \qquad (3.54)$$

Comparing (3.50) with (3.54), they are equal within the first-order approximation of β.

IV. BEYOND POSTULATES AND PARADOXES

A. Relativism in Circular Motion

Debates on relativity in circular motions have a long history, including the famous Newton's bucket. The core of the debate in Galileo's relativity is whether or not the motion of $\tilde{O}$, who is on a plate rotating relative to O at rest in S, is an absolute one. In Einstein's relativity, the principle of relativity, together with the light velocity constancy, is applied that 'the laws of physics are the same in all inertial frames of reference', which will be called the conventional principle of relativity (CPR). As the absolute motion (or the ether) could not be found the CPR has led to the equivalence of all inertial frames and to the perception that there are no preferred frames. The relativistic approach for circular motion results in the relativity in the sense that the relativistic representations between the circular and linear motions have the same form. Some relativist may say, "The relativity principle can also be applied to circular motions. It was validated again." Probably, it would not. Instead, the circular approach may lead to deep understanding of the relativity.

Let us tell a story, which is a fiction. Suppose that we don't know any theoretical things of relative motion between circular and linear frames, though some experiment results such as time dilations were known. To explain the results, we made some postulates: 1) The principle of relativity (We cannot tell which one, seemingly circular or seemingly linear motion, is actually rotating. Motions are relative, and the circular and linear frames are equivalent.) and 2) The linear speed constancy, say (3.30). With the postulates, we established the theory of circular relativity, which perfectly and beautifully explained the experiment results. It has been very consistent with almost every experiment. Furthermore, an abundance of astonishing new scientific facts has been discovered from it. Great advances in science and technology have been achieved based on our theory. We, circular relativists, now have big power and big authority. For a few experiment results which seem to contradict our theory, we explain if we can, or neglect if we cannot. Nobody can dare to challenge us.

The relativity world looks magic. We, on a rotating plate, did handshakes with our friends for ten seconds and one minute later, we hugged them for ten seconds. A linear inertial observer *P* says, "You did handshake for ten minutes and one hour later, you hugged for ten minutes." Another person, who is in linear motion relative to *P*, tells another story. Who is right? The postulation-based circular relativity may say that the ten-minute handshake and ten-minute hugging seen by *P* are equivalent to our true action.

In the Sagnac experiment the time difference can be obtained by (3.24) or (3.43). If the postulate that the circular and linear frames are equivalent according to the relativity principle is applied, some problems are caused. Suppose that the time difference $\Delta\tau_{d1}'$ is obtained from $\Delta\tau_{d1}$ in accordance with (3.24). According to the relativity principle, (3.43) also must be able to be equally applied. The $\cos\theta$ is the same irrespective of the use of (3.24) or (3.43) since $|\beta|=|\beta'|$. With $\Delta\tau_{d1}'=\Delta\tau_{d1}\cos\theta$, (3.43) gives $\Delta\tau_{d2}=\Delta\tau_{d1}\cos^2\theta>\Delta\tau_{d1}$. As the calculation repeats continuously, the time difference tends to infinity! On the contrary, in case the time difference $\Delta\tau_{d1}'$ is obtained from $\Delta\tau_{d1}$ in accordance with (3.43), $\Delta\tau_{d1}'=\Delta\tau_{d1}/\cos\theta$. Then, $\Delta\tau_{d2}$ is computed, by substituting the $\Delta\tau_{d1}'$ into (3.24), as $\Delta\tau_{d2}=\Delta\tau_{d1}'/\cos^2\theta<\Delta\tau_{d1}$. As the calculation continues, it goes to zero! Similar problems occur in liner inertial frames as well. Let's look into the twin paradox.

B. Twin Paradoxes

Our twin, O', left us for a space trip when $\tau=\tau'=0$. Suppose that the coordinate systems S and S' are related by (2.5) or (2.8), taking no account of the acceleration. After the space trip, O' comes back to the Earth instantaneously from a point $0'$, $p_0'=(\tau_0',0)$, in Fig. 6. It should be noted in Fig. 6 that the plane of S overlaps the plane of S', and for example, the point 1p' is denoted as $p_{1p'}$ in S and as $p_{1p'}'$ in S'. From the perspective of O', we, O, are moving with a velocity of $-\beta$. Accordingly our coordinates in S' are given by $(\tau_0',-\tau_0'\tan\theta)$, which corresponds to $p_{1n}=(\tau_1,0)$, the coordinates of the point 1n in S, where $\tau_1=\tau_0'/\cos\theta$. In contrast, from our perspective, O' is moving with a velocity of β and O' is currently at $p_{1p'}=(\tau_0,\tau_0\tan\theta)$, the point 1p' in S. Which point, $p_0=(\tau_0,0)$, p_{1n}, or other point, does O' return to? Let us assume that O' moves to p_0, following the perspective of O. The time coordinates τ_0 and τ_0' are then related by $\tau_0=\tau_0'\cos\theta$. According to the CPR, inertial frames are equivalent so that both (2.9) and (2.10) must be able to be equally applied. If an object at p_0 moves instantaneously to S', it must move to the point $p_{1p'}'$ in S'. Otherwise, if it moves to p_0', the equivalence between S and S' is violated, the symmetry being broken. The time coordinate of $p_{1p'}'$ is $\tau_{1p'}'=\tau_0\cos\theta\ (=\tau_0'\cos^2\theta)$. As such instantaneous movement continues, it tends to infinity. On the other hand, in case O' moves to p_{1n}, the continuation of the instantaneous movement makes it approach the coordinate origin. The equivalence between the primed and the unprimed, which prohibits one frame from depending on the other, seems to cause the paradox and to be self-contradictory.

The CPR is inconsistent with the Lorentz transformation if what it does mean is that 'inertial frames are equivalent'. If the frames of O and O' in the twin paradox are equivalent, their clocks must be exactly in the same condition. The same laws of physics must be equally applied to the clock of O as well as that of O'. If the clock of O' indicates τ_0'-time passing after the departure, the clock of O also must indicate τ_0'-time passing so that $\tau_0 = \tau_0'$. Suppose that at $\tau = \tau' = \tau'' = 0$, O and O' also meet another observer O'' who is moving with a different linear velocity β''. According to the equivalence of CPR, $\tau_0 = \tau_0' = \tau_0''$. Their clock rates are independent of motion, which implies Galileo's relativity. Therefore, the CPR must mean that the laws of physics can be represented in the same form in all inertial frames, which, however, are not equivalent.

Twin paradoxes exist also between the circular and linear frames when the relativity principle in the fiction is enforced. Let us introduce another version of twin paradox, nonlinear, circular one. Sadly, our twins A and B have gone to another galaxy, a planet at its edge. Fortunately we could also take a trip close to the galaxy. We admire the magnificent scenery of the giant galaxy beautifully rotating around its center with an angular velocity. We immediately found that the planet is located at a radius *r* from the center. And we saw that as soon as they arrived at the planet, one of our twins, twin A, left with an entity, who is very lovely and very benevolent and is a pilot of an identified flying object (IFO), in the reverse direction of the galaxy rotation. When A returned, we, O, saw our twins again. Who is the youngest?

The angular velocities of A and B are $\omega_{a,c} = -i\omega_a / c$ and $\omega_{b,c} = -i\omega_b / c$, respectively. The travel times of A and B from the perspective of O can be calculated as (3.24) with $\tau_\mu = 2\pi r / \omega_{\mu,c}$ and $\cos\theta_\mu = (1-\beta_\mu^2)^{-1/2}$ where $\beta_\mu = r\omega_\mu / c$ and $\mu = a, b$. Then the time ratios are given by $\tau_\mu' / \tau_\mu = \cos\theta_\mu$. However, when the travel times are calculated as (3.43), the ratios are given by $\tau_\mu' / \tau_\mu = \cos^{-1}\theta_\mu$. Which ones are right? Some circular relativist in the fiction may say, "There are no preferred frames, even between seemingly rotating and seemingly linear inertial frames. They are equivalent according to the relativity principle. The time ratios are all right. The confusion results from the ignorance about the relativity of simultaneity. Simultaneous events are valid only in each frame." Then, it becomes another paradox, circular version.

The mechanism causing the paradox between the circular and linear frames is exactly the same as that between the inertial frames. The paradox in the latter is caused by the equivalence of (2.9) and (2.10) and in the former by the equivalence of (3.24) and (3.43). In TSR, elapsed time in the unprimed and in the primed must not be compared since both (2.9) and (2.10) cannot be satisfied at the same time, which led TSR to introduce the so called relativity of simultaneity.

We need more paradoxes, do we? Here is another one, not associated with simultaneity. The observers O and $\tilde{O}'$ met at $\tau = \tau' = 0$. From the perspective of O who thinks that $\tilde{O}'$ is in circular motion, the radius and the angular velocity of $\tilde{O}'$ are given by (3.27) and (3.28), respectively. The $\tilde{O}'$ thinks that O is in circular motion. Applying the relativity principle, the radius and the velocity in S are written, by substituting (3.27) and (3.28) into (3.47) and (3.48) respectively, as $r(\cos\theta/\cos\theta_z)^2$ and $\omega_c(\cos\theta_z/\cos\theta)^2$ respectively, though the measurements of O are r and ω_c. Which ones are right? The postulate seems to create paradoxes and contradictions, doesn't it? The relativistic transformation for circular motion has been formulated without any postulates, which may lead us to resolve the paradoxes.

C. True Space-Time Spaces

In TSR, deep roots of paradoxes and contradictions lie in observation lines or misconceptions of the coordinate system S'. Let us revisit the reformulation of special relativity in CES. Only the τ'-axis in S' is an observation line, on which O' sees events. Any other lines $x' = x_1'(\neq 0)$ are not observation lines. When S is given, S' shows how O' sees the world from S. In TSR, all $x' = x_1'$ seem to be regarded as observation lines. Let every $x' = x_1'$ in Fig.1 be an observation line (according to the view of TSR). At $\tau' = \tau = 0$, if O' meets O, any other objects in S' cannot meet any objects in S and only one object is allowed to meet. This issue is also associated with simultaneity, and it is justified in TSR by the relativity of simultaneity. The situation, however, is different between the circular and linear motions.

Objects that are at rest on a rotating plate belong to the same inertial frame, which is termed a circular inertial frame. Consider N objects located at a radius r' that are at rest on a rotating plate, as in Fig. 7. Let the circular inertial frame $\tilde{S}'$ for the objects be a true local space-time space that consists of observation lines, which will be discussed later. The position vector of the *k*th object $\tilde{O}_k'$ can be written as $\tilde{\boldsymbol{p}}_{k,s}' = r'[\sin\varphi_k', -\cos\varphi_k']^T$ where φ_k' is a constant. The relativistic transformation from the circular to the linear, according to the perspective of $\tilde{O}_k'$, can be performed using (3.37) – (3.39). The spatial vector $\boldsymbol{p}_{k,s}$ is then given by $\boldsymbol{p}_{k,s} = r[\sin\varphi_k, -\cos\varphi_k]^T$, as shown in Appendix C, where $\varphi_k = -\varphi_k'\cos\theta_{z'}$. When $\tilde{O}'$ meets O at $\tau = \tau' = 0$, these objects $\tilde{O}_k'$ in

$\tilde{S}'$, which belong to the same circular inertial frame, can also meet respective objects O_k in the linear inertial frame, the spatial vectors of which are equal to $\boldsymbol{p}_{k,s}$. Recall that the minus sign in the equation '$\varphi_k = -\varphi_k' \cos\theta_{z'}$' results from the fact that O (or O_k) sees $\tilde{S}$ rotating in the reverse direction of the rotation of S' that $\tilde{O}'$ (or $\tilde{O}_k'$) sees. Fig, 7 has been drawn from the perspective of $\tilde{O}'$ so that the phases of $\boldsymbol{p}_{k,s}$ and $\tilde{\boldsymbol{p}}_{k,s}'$ have the same sign in the figure. Though in the circular motion any number of objects in the primed can meet respective ones in the unprimed at the same time, the number has to be only one in TSR. If r' is very large, the objects $\tilde{O}_k'$ are essentially on a line and in uniform linear motion relative to O. They can be moving even very fast as r' is very large, even if the angular velocity is near zero. As explained, these objects can meet respective objects in the same linear inertial frame at the same time. However, the equivalence of CPR does not allow it.

Let us suppose that a linear inertial frame S represents a true local space-time space and that multiple objects in the primed linear inertial frame actually meet respective ones in S at the same time. Though the objects in the primed meet the corresponding ones in S at $\tau = 0$, each sees differently. Let primed objects O' and O_1' located at $x' = 0$ and $x' = x_1'$ meet unprimed ones O and O_1 located at $x = 0$ and $x = x_1$, respectively, as in Fig. 8. Even if S represents a true space-time space, the primed coordinates can be obtained by the Lorentz transformation. The coordinate system S_1' of O_1' is related to the coordinate system S_1 of O_1 by (2.5) or (2.8) with the unprimed coordinate vector $\boldsymbol{p}_{(1)} = [\tau_{(1)}, x_{(1)}]^T$ and the primed one $\boldsymbol{p}_{(1)}' = [\tau_{(1)}', x_{(1)}']^T$ where $\tau_{(1)} = \tau$. The coordinate vector of O is expressed as $\boldsymbol{p}_{(1)} = [\tau_{(1)}, -x_1]^T$ in S_1 and that of O' as $\boldsymbol{p}_{(1)}' = [\tau_{(1)}', -x_1']^T$ in S_1'. The primed vector $\boldsymbol{p}_{(1)}'$ when O' meets O at $\tau_{(1)} = 0$ is calculated as

$$[\tau_{(1)}', -x_1']^T = \boldsymbol{T}_{L2}(\theta)[0, -x_1]^T . \tag{4.1}$$

Equation (4.1) is rewritten as $\tau_{(1)}' = -x_1 \sin\theta$ and $x_1' = x_1 \cos\theta$, which imply that it seems to O_1' that when $\tau_{(1)}' = -x_1 \sin\theta$, O' located at $x_{(1)}' = -x_1 \cos\theta$ met O, as shown in Fig. 8, though they actually met at $\tau = \tau' = 0$. In contrast, O' sees O_1' meet O_1 at $\tau' = x_1 \sin\theta$ though they actually meet at $\tau = \tau_{(1)}' = 0$. Only the τ'-axis of S' is the observation line, and S' shows how O' sees the world from S.

According to Feynman, the principle of relativity is described as [15, 5]: "if a space ship is drifting along at a uniform speed, all experiments performed in the space ship will appear the same as if the ship were not moving, provided, of course, that one does not look outside. This is the meaning of the principle of relativity.” As long as one does not look outside, inertial frames may be equivalent in the restricted sense that the laws of physics are equally applied to every inertial frame. However, returns of twins correspond to ‘looking outside’. If one is looking outside, in other words, if the unprimed and primed observation lines are compared, the equivalence between them does not hold any more. Then, the twin paradox no longer becomes a paradox. In linear motions it may not be necessary to look outside because two observers go far away from each other. However, circular motions may force the observers to look outside because they can see the same outside again. We do not have to worry about losing the elegant Lorentz transformation even if the equivalence between inertial frames does not hold and true space-time spaces are introduced. Of course, the circular approach is also based on the Lorentz transformation in CES. We can use it, but with exact meaning.

What is the true space-time space? An observation line includes a set of events that actually occurred to the corresponding observer. The events on observation lines are true events that actually occur. A true space-time space is a collection of observation lines. One observation line is a true space-time space restricted to the owner of it. Observation lines are like sets of imprints carved on space-time spaces regardless of motions. They cannot be changed by the observation or the representation in other coordinate systems. The representations of true events in a coordinate system disappear as soon as its observer leaves the coordinate system. However, the true events do not disappear. If either $\boldsymbol{p}'$ or $\boldsymbol{p}$ represents a true event, the other can be dependently given by the Lorentz relationship. The word ‘proper’ has often been used such as ‘proper times’, which are found by following so called ‘world lines’. The world line, which probably corresponds to the observation line in basic meaning, can be called ‘the proper line’. We can replace ‘proper’ with ‘true’, can’t we? Though it seems to admit preferred frames and to violate the relativity principle in terms of the equivalence. Are there any other proper lines for a proper one?

Experiments of twin paradox have already been done on rotating plates, the Hafele–Keating experiment. The analysis of Hafele and Keating has been known to be in good agreement with the experiment results [12, 13]. The analysis, except for part of general relativity, appears to exploit the results of this paper, though circular motions were treated as linear ones. In the HK experiment, three time measurements were obtained, one on the surface of Earth and the others from the planes traveling in the Earth rotation direction and in the opposite. We use the results of Subsection III.B.2 for the analysis of the HK experiment results. Since the flight altitudes are negligible as compared with the Earth radius, the three observers, $\tilde{O}_0'$, $\tilde{O}_1'$, and $\tilde{O}_2'$, can be considered to be rotating at an equal radius r' with different velocities where $\tilde{O}_0'$ denotes the Earth surface observer, and $\tilde{O}_1'$, and $\tilde{O}_2'$ the plane observers in the same and the opposite directions, respectively. The subscript number will be used to indicate the observer with the same number so that for example, ω_k', $k = 0, \cdots, 2$, denotes the angular velocity of $\tilde{O}_k'$. In the HK analysis, the unprimed time is employed for comparison. The unprimed frame S is a rest frame with respect to the Earth rotation. From (3.43), the time differences for the travels are written as

$$\Delta t_l' = t_l' - t_{0,l}' = (\cos^{-1}\theta_l - \cos^{-1}\theta_0)\, t_l, \quad l = 1, 2 \tag{4.2}$$

where t_l', $t_{0,l}'$ and t_l denote the times measured, respectively, by $\tilde{O}_l'$, $\tilde{O}_0'$ and O, an observer of S, during the travel of $\tilde{O}_l'$. In (4.2), $\cos\theta_k = (1 - (\beta_k')^2)^{-1/2}$ where $\beta_k' = r'\omega_k'/c$, $k = 0, \cdots, 2$. The unprimed times were not measured, and t_l seems to have been calculated as $t_l = 2\pi r'/v_l'$ where v_l' is the ground speed of $\tilde{O}_l'$.

Four observers appear in the HK analysis. Thus, there are twelve relationships of relativity between them. The experiment results clearly show that there are no equivalences between them. The HK analysis for the time dilations only due to the circular motions with angular velocities indicates that $\Delta t_1' < 0$ and $\Delta t_2' > 0$ [12], which are also seen from (4.2) by the circular approach. If (3.24), instead of (3.43), for the linear-to-circular is used, their signs change, i.e., $\Delta t_1' > 0$ and $\Delta t_2' < 0$. The observed times in (4.2) are true ones, recorded on the respective observation lines, which can be represented in other coordinate systems. The true events on the time axes of the coordinate systems $\tilde{S}_k'$, $k = 0, \cdots, 2$, can be observed in S, and t_l in (4.2) indicates the travel time of $\tilde{O}_l'$ seen by O. As far as the travel times are concerned, t_l depends on t_l'. The experiment shows that (3.43) only is valid and so t_l' cannot be obtained from t_l by applying (3.24).

As mentioned above, if either $\boldsymbol{p}'$ or $\boldsymbol{p}$ is a true event, the other can be dependently given. What if both $\boldsymbol{p}'$ and $\boldsymbol{p}$ represent true events? In twin paradoxes, both of them represent true events. These cases can be resolved by discovering whose clock is the moving clock, which can be known through experiments. Time dilations, which imply that moving clocks run slow, have been well validated through quite a number of related experiments including the HK one [16]. However, the time dilation is incompatible with the equivalence of inertial frames since one clock is in motion relative to the other. The incompatibility causes the twin paradox, which is about whose clock is the actually moving clock with respect to the other. It is the clock that shows a time dilation relative to the other when elapsed time intervals are compared.

The HK experiment clearly shows whose clocks are the actually moving clocks relative to others. As mentioned in Section II, a moving clock is the one on an observations line. The coordinates on an observation line have zero spatial components, or its differential spatial vector is zero so that the relativistic transformation for the differential vector is written in 2-D as $d\boldsymbol{p}'=\boldsymbol{T}_{L2}(\theta)\,[d\tau,0]^T$ or $d\boldsymbol{p}=\boldsymbol{T}_{L2}(\theta)\,[d\tau',0]^T$, which leads to $d\tau'=d\tau\cos\theta$ or $d\tau=d\tau'\cos\theta$. In the circular approach also, differential times are given by the same equations as in linear inertial frames. In the HK experiment, the travel times when only considering the effect of special relativity can be obtained from the same equations as well. Hence, equivalently the observation lines of $\tilde{O}_0'$, $\tilde{O}_1'$, and $\tilde{O}_2'$ can be depicted as in Fig. 9. The $\tau_{(k)}'$-axis, $k=0,\cdots,2$, is the time axis for $\tilde{O}_k'$. The plane observer $\tilde{O}_l'$, $l=1,2$, who is at the point $0_{(l)}$, instantaneously moves to $\tilde{O}_0'$. If their relative motions are equivalent, the elapsed time intervals must be all equal. Even though the travels of $\tilde{O}_1'$ and $\tilde{O}_2'$ are symmetric with respect to $\tilde{O}_0'$, τ_1' and τ_2' are different. The problem of the twin paradox is where, the point 1 or 2 in Fig. 9, $\tilde{O}_0'$ is on the observation line at the instant that each $\tilde{O}_l'$ does instantaneous movement. The movement to the point 1 (point 2) implies that the clock of $\tilde{O}_l'$ ran at a faster (slower) rate than the clock of $\tilde{O}_0'$. According to the HK experiment, $\tilde{O}_1'$ moves to the point 2 while $\tilde{O}_2'$ moves to the point 1. If an object on the $\tau_{(0)}'$-axis instantaneously moves to the $\tau_{(l)}'$-axis, it does to the point $0_{(l)}$. There is no paradox.

Inertial frames are equivalent, provided that they do not look outside. However, when looking outside, for example, comparing time passing, the inertial frames are not equivalent as experiments show different elapsed time intervals. In general relativity, the acceleration is described as ‘with respect to something’, say acceleration with respect to a tangent plane. In special relativity also, relative motions should be described as ‘with respect to something’. If O'’s time runs slower than O’s time, it implies the motion of O' with respect to O. As a matter of fact, the formulation of special relativity in Section II, which uses Fig.1 drawn from the perspective of O, implicitly implies the motion of O' with respect to O, though (2.5) has the same form as (2.8) so that their motions appear to be equivalent. As the acceleration in an accelerated motion tends to zero, it reduces to a uniform linear motion. When applying ‘with respect to something’ to special relativity, the inconsistency in describing the accelerated and the uniform motions disappears. Relative motions can be represented in the same form, even for circular motions, but they are not equivalent because their elapsed time intervals are different. On the contrary, if they are the same, the special relativity reduces to Galileo’s relativity.

A true space-time space is a collection of observation lines. An observation line is a set of true events. Precisely speaking, a true space-time space is a collection of sets of events which occurred, occur, and will occur on time axes of observers. We can imagine the true space of a linear inertial frame S as follows. An observer O located at x is at rest in S. Let an event on the observation line of O at time τ be $e(\tau, x)$. The event $e(\tau, x)$ is the event at (τ, x) in the true space S. We can also imagine a true space in circular motion. Observers who are at rest on a rotating plate belong to the same circular inertial frame $\tilde{S}'$. An observer $\tilde{O}'$ is located at a spatial coordinate (r', ϕ') in $\tilde{S}'$ where ϕ' is an azimuth angle. If an event on the observation line of $\tilde{O}'$ at time τ' is $e(\tau', r', \phi')$, it is the event at (τ', r', ϕ') in the true space $\tilde{S}'$. If all spatial points of $\tilde{S}'$ belong to a set Π, say $\Pi = \{(r', \phi') \,|\, r_1' \le r' \le r_2', \phi_1' \le \phi' \le \phi_2'\}$, $\tilde{S}'$ is a true space over Π. A true space can have just one entry. In the HK experiment, the observation lines of $\tilde{O}_0'$, $\tilde{O}_1'$, and $\tilde{O}_2'$ are all respective true spaces. Their relative motions can also be described as 'with respect to', according to the HK experiment results, which indicate the motions of $\tilde{O}_0'$, $\tilde{O}_1'$ and $\tilde{O}_2'$ with respect to O, the motion of $\tilde{O}_1'$ with respect to $\tilde{O}_0'$ and so on.

The person *P* says, "You did handshake for ten minutes and one hour later, you hugged for ten minutes." We express our warm respect for *P*'s opinion because *P* is just saying what was seen, it's true, and because we are one. The things 'seen' are *P*'s true events, belong to *P*'s observation line, and the 'actions' of the handshake and hugging are our true events, not *P*'s, which belong to our observation lines and which cannot be changed by others' observation. We each are walking on each one's own observation line on the Earth, engraving each life on each observation line. The Earth plane, which is a collection of these observation lines, is a true local space-time space irrespective of the motion of our Earth, Solar System, and Milky Way. We live in the true space.

V. CONCLUSIONS

The celebrated Sagnac experiment may have hidden precious treasures that can pave the way for relativistic approaches to circular motion and profound understandings of relativity. The presented formulations for circular motion, which have been obtained without any postulates, can allow us to deal with relativity problems without any restrictions. Relativity between circular and linear frames as well as between linear frames holds, but as a property derived, not a postulate. Motions are relative and they can be represented in the same form, even for circular motions. Relativity holds in terms of the representation. However, relative motions are not equivalent since their elapsed time intervals are different, as shown in the HK experiment.

The reformulation of special relativity in CES is more fundamental than in the real number. Besides the points presented in Section II, it has shed light on the relativistic approach to circular motion. Now it's time to be free from, to go beyond the postulates and to look into them as properties. In the reformulation, the relativity between linear inertial frames is obtained as a property derived from the isotropy of the space-time spaces, i.e., the same κ, the ratio of time to space in a manifested universe. When a speed is the same as the value of κ, it becomes invariant from frame to frame. The relativity and the light velocity constancy are properties that the isotropic space-time spaces have. If they are postulates, they must be unconditionally accepted. As a result, paradoxes and contradictions may have been created in TSR. The application of TSR may be limited under the postulates, maybe with paradoxes, since it has been derived from them. However, the CES formulation can be applied without the limitation.

A true space-time space is a collection of observations lines, which are sets of true events. True events cannot be changed by the observation or the representation of other frames though they are differently seen. In circular motion, a true space consists of observation lines of objects at rest on a rotating plate. The basic motion in universes, macroscopic and microscopic, is circular. The relativistic approach for circular motions may provide a very important base for deep understanding of them. It can allow us to see the Solar System and galaxies with the eyes of an observer at rest with respect to their motions. Though we are also in various circular motions, we live in the true space-time space, maybe coming into a circular age from the linear age.

APPENDIX A

With $\boldsymbol{p}_s = r[\sin\varphi, -\cos\varphi]^T$ given, the tilde vector $\tilde{\boldsymbol{p}}_s$ is written as $\tilde{\boldsymbol{p}}_s = [0, -r]^T$. Then $d\tilde{\boldsymbol{p}}_s$ becomes a zero vector so that

$$d\tilde{\boldsymbol{p}} = [d\tau, 0, 0]^T . \tag{A.1}$$

Recall that $\tilde{\tau}' = \tau'$. Substituting (A.1) into (3.17), it follows that

$$d\tau' = d\tau\cos\theta \tag{A.2}$$

$$d\overline{\boldsymbol{p}}_s' = \begin{bmatrix} -d\tau\sin\theta \\ 0 \end{bmatrix} . \tag{A.3}$$

Under the initial condition that $\tau' = 0$ when $\tau = 0$, the integration of (A.2) leads to (3.24).

Note that the $\tilde{y}'$-component of $d\overline{\boldsymbol{p}}_s'$ is zero, which implies that the Lorentz transformation does not change the radial component of $d\tilde{\boldsymbol{p}}_s$. From $\varphi = \omega_c\tau$ and (3.11), we have

$$d\tau = -\frac{d\varphi'}{\omega_c\cos\theta_z} . \tag{A.4}$$

The differential vector $d\boldsymbol{p}_s'$ is obtained, by using (A.3), (A.4) and (3.18), as

$$d\boldsymbol{p}_s' = \frac{\sin\theta}{\omega_c \cos\theta_z}\begin{bmatrix} d\varphi'\cos\varphi' \\ d\varphi'\sin\varphi' \end{bmatrix} = r'\begin{bmatrix} d\varphi'\cos\varphi' \\ d\varphi'\sin\varphi' \end{bmatrix} \tag{A.5}$$

where r' is given as (3.27), with the use of $\sin\theta/\omega_c = -i\beta\cos\theta/(-i\beta/r) = r\cos\theta$. Integrating (A.5) with respect to φ' and using the initial condition that the phase of $\boldsymbol{p}_s'$ is $-\pi/2$ at $\tau'=0$, we have (3.25). Though the Lorentz transformation does not change the radial component, the coordinate conversion from $\tilde{S}'$ to S' results in the change of it.

Using (A.2) and (A.4), we have

$$d\tau' = -\frac{d\varphi'\cos\theta}{\omega_c \cos\theta_z}. \tag{A.6}$$

Then

$$\omega_c' = \frac{d\varphi'}{d\tau'} = -\frac{\omega_c \cos\theta_z}{\cos\theta}. \tag{A.7}$$

APPENDIX B

Let ς be $\varsigma = \tau + \tau_0$. The variable ψ $(= \varphi + \varphi_0)$ can be expressed as

$$\psi = \omega_c \varsigma. \tag{B.1}$$

The vector $\boldsymbol{p}_s$ is written as $\boldsymbol{p}_s = r[\sin\psi, -\cos\psi]^T$ and the tilde vector $\tilde{\boldsymbol{p}}_s$ is given by (3.33). Then $d\tilde{\boldsymbol{p}}_s$ becomes a zero vector. It is obvious that $d\tau = d\varsigma$. The differential vector can be written as

$$d\tilde{\boldsymbol{p}} = [d\varsigma, 0, 0]^T. \tag{B.2}$$

Substituting (B.2) in (3.17), it follows that

$$d\tau' = d\varsigma\cos\theta \tag{B.3}$$

$$d\overline{\boldsymbol{p}}_s' = \begin{bmatrix} -d\varsigma\sin\theta \\ 0 \end{bmatrix}. \tag{B.4}$$

The integration of $d\tau'$ under the initial condition that $\tau'=0$ when $\tau=0$ leads to (3.34).

We introduce a variable ψ' defined as $\psi' = \varphi' + \varphi_0'$ where $\varphi_0' = -\varphi_0\cos\theta_z$. It is clear that $d\psi = d\varphi$ and $d\psi' = d\varphi'$. The differential $d\psi'$ can be written from (3.11) as

$$d\psi' = -d\psi\cos\theta_z. \tag{B.5}$$

From (B.1) and (B.5)

$$d\varsigma = -\frac{d\psi'}{\omega_c \cos\theta_z}. \tag{B.6}$$

Inserting (B.6) into (B.3), we have

$$d\tau' = -\frac{d\psi' \cos\theta}{\omega_c \cos\theta_z}. \tag{B.7}$$

Then

$$\omega_c' = \frac{d\varphi'}{d\tau'} = \frac{d\varphi'}{d\psi'}\frac{d\psi'}{d\tau'} = -\frac{\omega_c \cos\theta_z}{\cos\theta}. \tag{B.8}$$

Finally, let us derive (3.35). As $\tilde{\boldsymbol{p}}_s$ is obtained as $\tilde{\boldsymbol{p}}_s = \boldsymbol{A}_s(\psi)\boldsymbol{p}_s$, the differentials $d\overline{\boldsymbol{p}}_s'$ and $d\boldsymbol{p}_s'$ are related by

$$d\boldsymbol{p}_s = \boldsymbol{A}_s(-\psi')\, d\overline{\boldsymbol{p}}_s'. \tag{B.9}$$

Using (B.4), (B.6) and (B.9), we have

$$d\boldsymbol{p}_s' = r' \begin{bmatrix} d\psi' \cos\psi' \\ d\psi' \sin\psi' \end{bmatrix} \tag{B.10}$$

where r' is given as (3.28), with the use of $\sin\theta / \omega_c = -i\beta \cos\theta / (-i\beta / r) = r\cos\theta$. The initial condition is that the phase of $\boldsymbol{p}_s'$ is $-\pi/2$ at $\tau' = -\tau_0'$ where $\tau_0' = \tau_0 \cos\theta$. Inserting $\tau_0 = \varphi_0 / \omega_c$ and $\varphi_0 = -\varphi_0' / \cos\theta_z$ into $\tau_0' = \tau_0 \cos\theta$ and using (B.8), it is written as $\tau_0' = -\varphi_0' \cos\theta / (\omega_c \cos\theta_z) = \varphi_0' / \omega_c'$. Integrating (B.10) with respect to ψ' and applying the initial condition at $\tau' = -\varphi_0' / \omega_c'$, it yields (3.35).

APPENDIX C

Let us introduce a coordinate system S_k' the coordinate vector of which is represented as $\boldsymbol{p}_{(k)}' = [\tau_{(k)}', x', y']^T$ where $\tau_{(k)}' = \tau' + \tau_k'$ with $\tau_k' = \varphi_k' / \omega_c'$. A variable $\varphi_{(k)}' = \varphi' + \varphi_k'$ is defined as

$$\varphi_{(k)}' = \omega_c' \tau_{(k)}'. \tag{C.1}$$

From the point of view of $\tilde{O}_k'$, an observer O_k' located at a radius r' in S_k' is rotating with an angular velocity of ω_c'. The motion of O_k' can be described in $\tilde{S}'$ as

$$\tilde{\boldsymbol{p}}_{O_k',s}' = r' [\sin\varphi_{(k)}', -\cos\varphi_{(k)}']^T. \tag{C.2}$$

Note that when $\tau' = 0$, $\tilde{\boldsymbol{p}}_{O_k',s}' = \tilde{\boldsymbol{p}}_{k,s}'$ so that $\tilde{O}_k'$ meets O_k'. The vector $\tilde{\boldsymbol{p}}_{O_k',s}'$ is represented in

S_k' as

$$\boldsymbol{p}_{O_k',s}' = \boldsymbol{A}_s(\varphi_{(k)}')\tilde{\boldsymbol{p}}_{O_k',s}' = [0, -r']^T . \quad \text{(C.3)}$$

Then $d\boldsymbol{p}_{O_k',s}'$ becomes a zero vector so that

$$d\boldsymbol{p}_{O_k'}' = [d\tau_{(k)}', 0, 0]^T . \quad \text{(C.4)}$$

The differential vector $d\boldsymbol{p}_{O_k'}'$ is Lorentz-transformed into S_k where S_k is the unprimed coordinate system corresponding to S_k'. The coordinate vector of S_k is denoted as $\boldsymbol{p}_{(k)} = [\tau_{(k)}, x, y]^T$ where $\tau_{(k)} = \tau + \tau_k$ with $\tau_k = \tau_k' \cos\theta$. The transformed differential vector is converted into $\tilde{S}$. Using (3.37) and (3.38), and similarly following the related derivation in Appendix A with the appropriate change of symbols, we have

$$\tilde{\boldsymbol{p}}_{O_k',s} = r\,[\sin\varphi_{(k)}, -\cos\varphi_{(k)}]^T \quad \text{(C.5)}$$

where $\varphi_{(k)} = \omega_c \tau_{(k)} = \varphi + \varphi_k$ with $\varphi_k = -\varphi_k' \cos\theta_{z'}$. The initial condition was used in (C.5) that the phase of $\tilde{\boldsymbol{p}}_{O_k',s}$ is $-\pi/2$ when $\tau_{(k)} = 0$. The vector $\tilde{\boldsymbol{p}}_{O_k',s}$ is represented in S', not S_k', as

$$\boldsymbol{p}_{k,s} = \boldsymbol{A}_s(\varphi)\tilde{\boldsymbol{p}}_{O_k',s} = r\,[\sin\varphi_k, -\cos\varphi_k]^T . \quad \text{(C.6)}$$

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FIGURE CAPTIONS

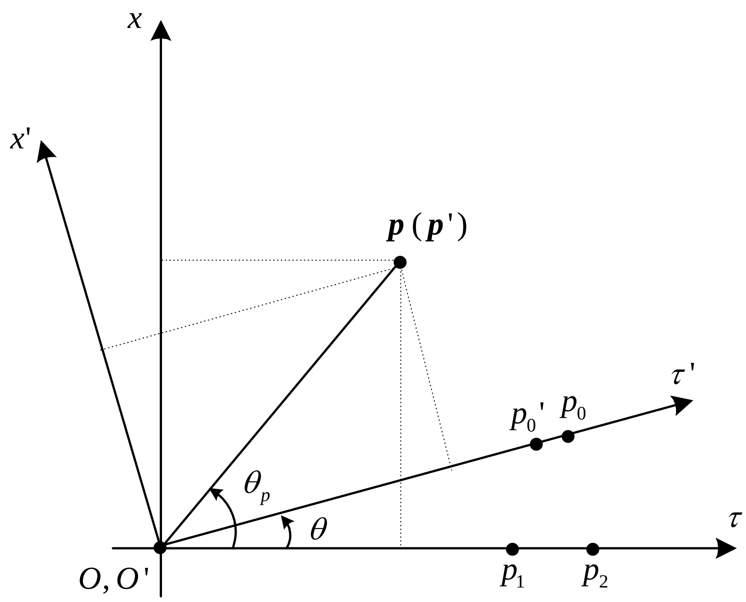

Fig. 1. Coordinate systems S and S' in CES.

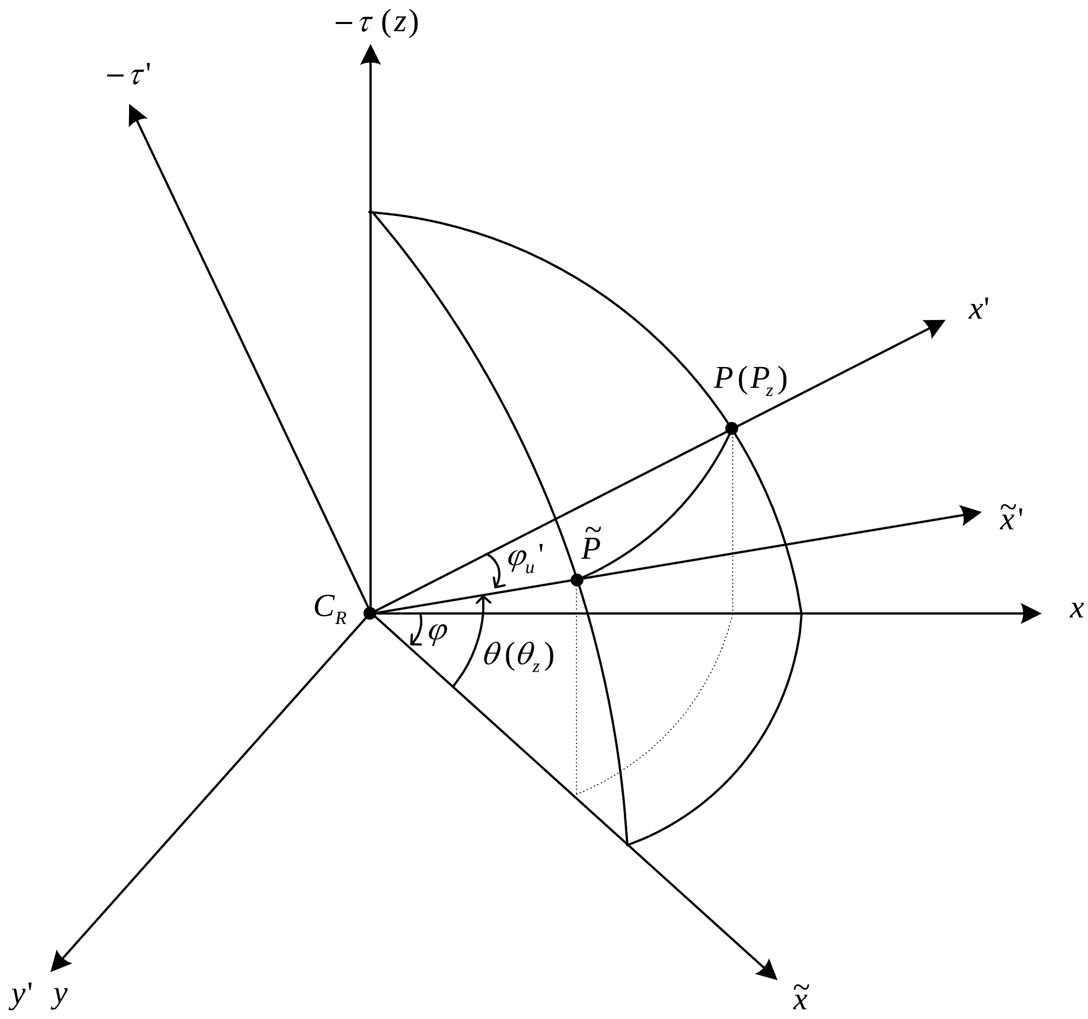

Fig. 2. Coordinate systems S, $\tilde{S}$, $\tilde{S}'$, and S' for the relativistic transformation from the unprimed to the primed.

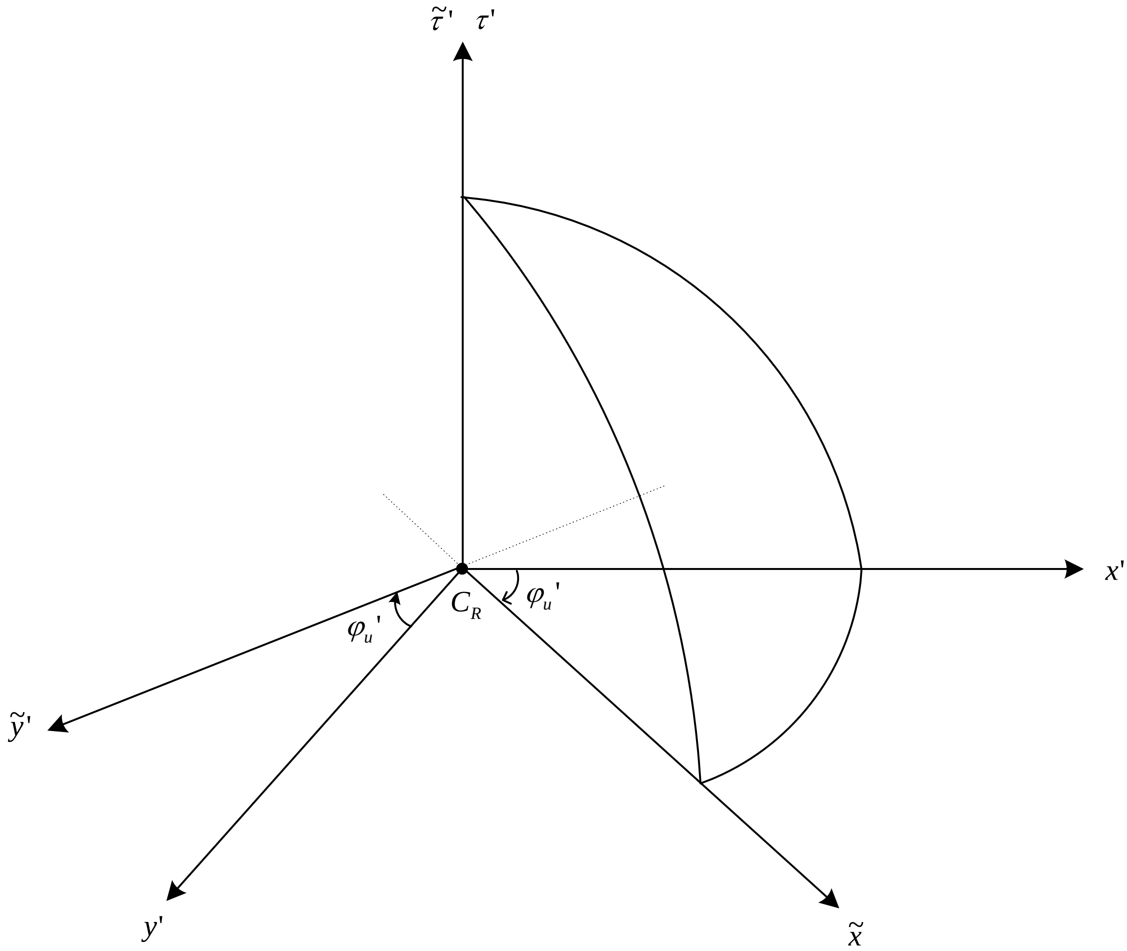

Fig. 3. The orientation of $\tilde{S}'$ with respect to S'.

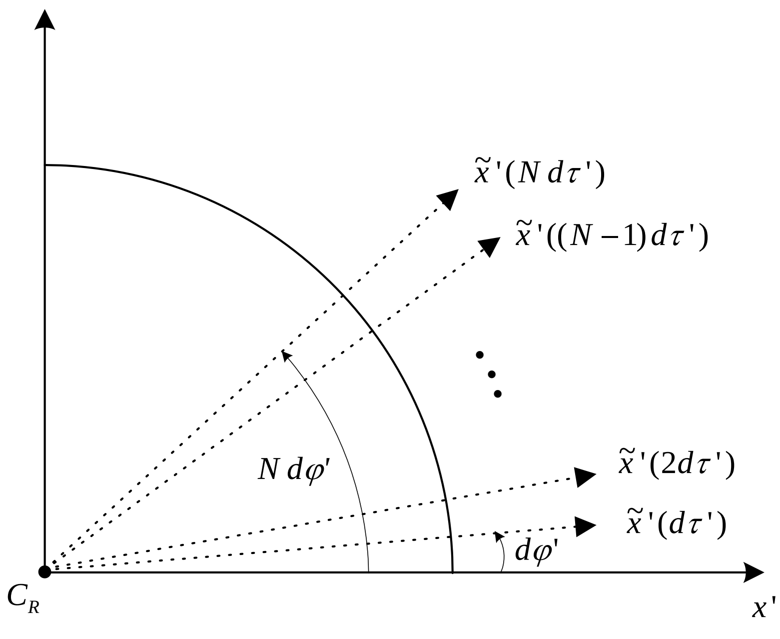

Fig. 4. Rotation of the $\tilde{x}'$-axis with time progress.

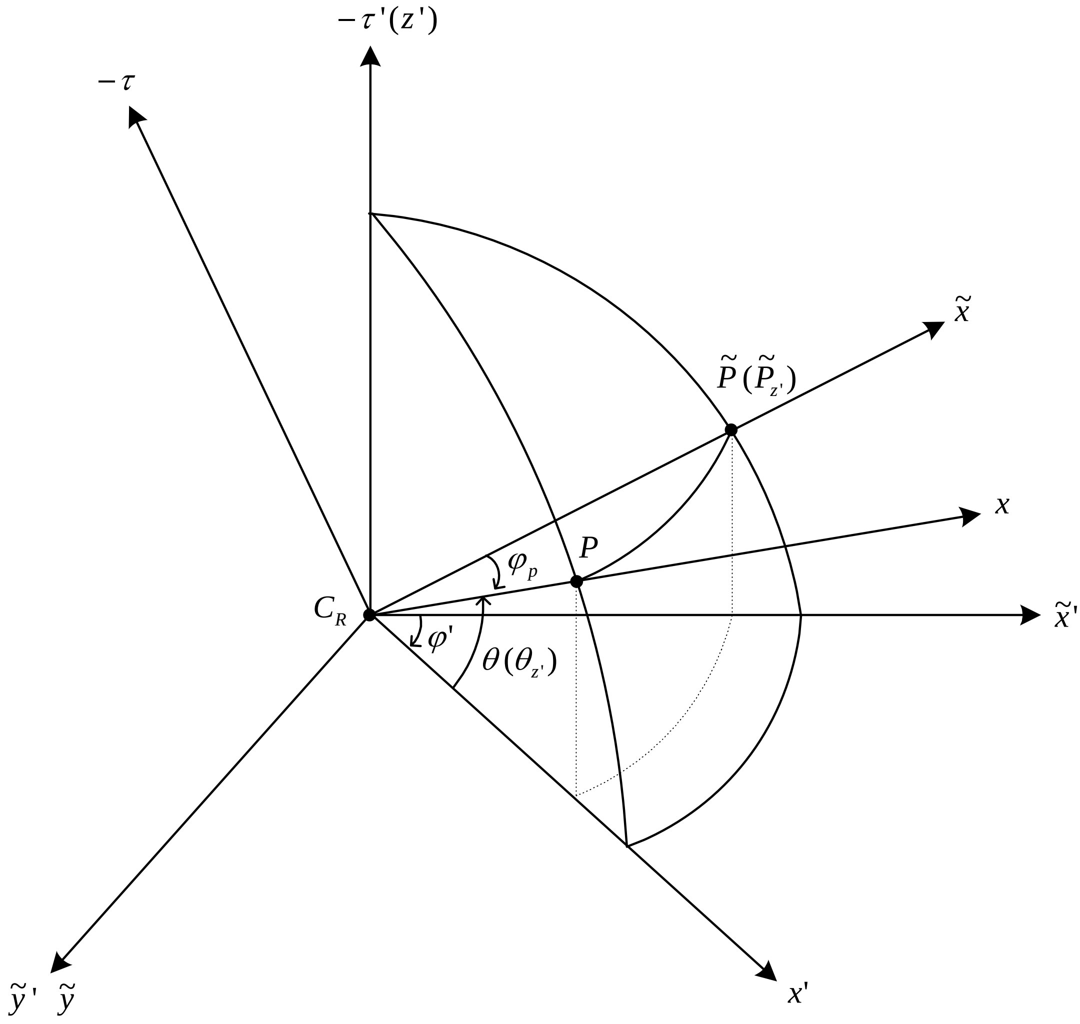

Fig. 5. Coordinate systems S, $\tilde{S}$, $\tilde{S}'$, and S' for the relativistic transformation from the primed to the unprimed.

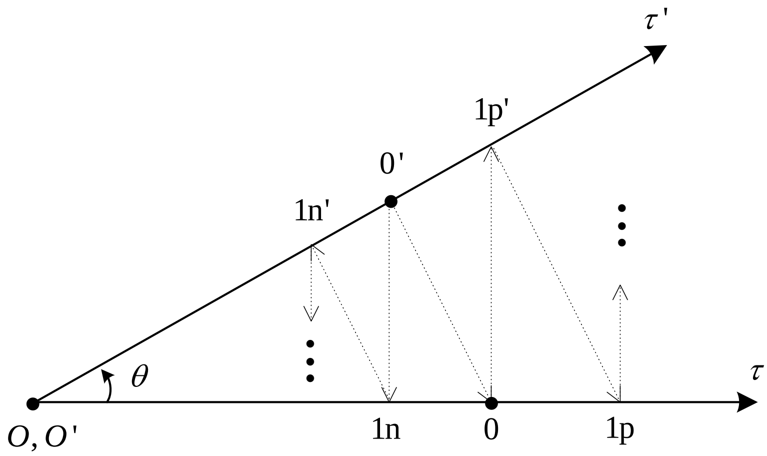

Fig. 6. Observation lines for twin paradox.

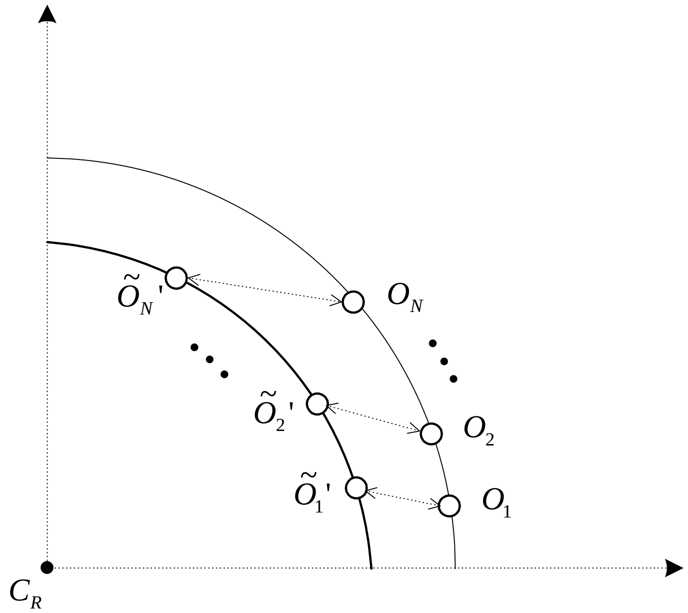

Fig. 7. Objects on a rotating plate that meet respective objects in a linear frame.

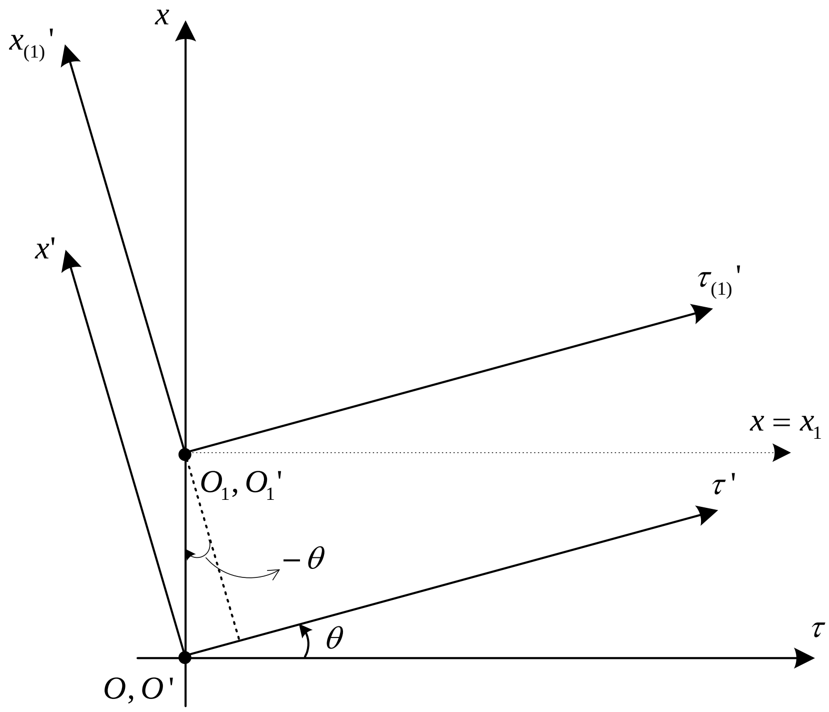

Fig. 8. Coordinate systems of O' and O_1', who meet O and O_1, respectively, at $\tau = 0$.

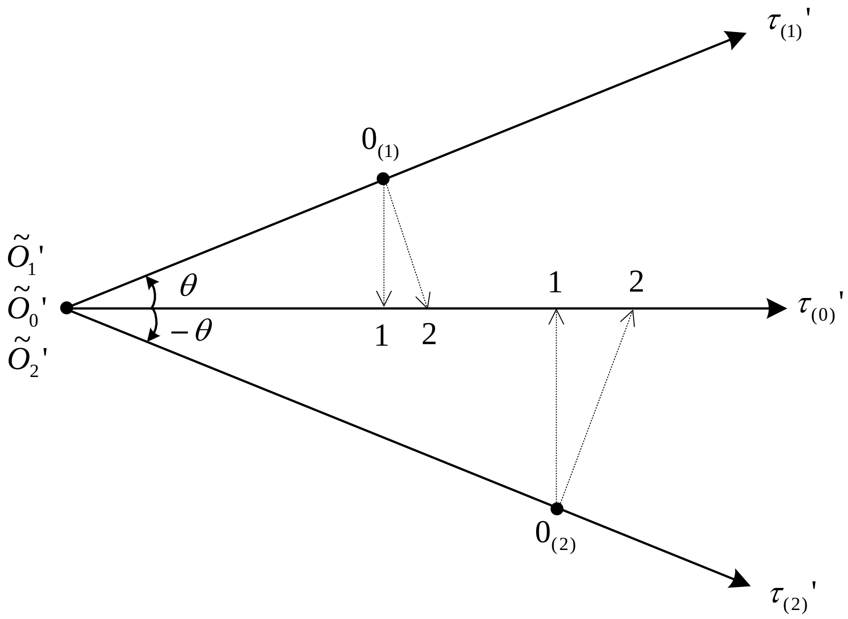

Fig. 9. Equivalent observation lines of $\tilde{O}_0'$, $\tilde{O}_1'$ and $\tilde{O}_2'$ in the HK experiment.

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