BLACK HOLES, THERMODYNAMICS AND ENTROPY

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Abstract.
Black holes, thermodynamics and entropy are three topics which both separately and together raise several quite deep and serious questions which need to be addressed. Here an attempt is made to highlight some of these issues and to indicate a possible linkage between the accepted entropy expression for a black hole and the paradox linked to black holes and information loss.

Key Words: thermodynamics, statistical mechanics, information theory, entropy, black holes.

Introduction.

The recently published article *Comment on 'What Information Loss is Not'*[1] proved to be an interesting and informative read but it also provoked several thoughts concerning the inter-relation of the three topics mentioned in the title to this note– black holes, thermodynamics and entropy. All three separately provide much food for thought but, when considered together, the thoughts and queries become overwhelming – what is a black hole?, do black holes exist?, how does thermodynamics fit into the picture?, what is entropy?, and possibly most important of all, are the entropies alluded to in thermodynamics, in statistical mechanics and in information theory identical functions? The purpose of this note is to highlight these questions but not necessarily provide concrete answers to all, as well as to point out that the paradox linked with black holes and information loss may be linked to the commonly accepted entropy expression for such bodies.

Black Holes.

In 1784, John Michell [2] first derived an expression, using Newtonian mechanics, for the mass-radius ratio of a spherical body having an escape speed equal to, or greater than, the speed of light. Such a body as Michell envisaged has erroneously been termed a black hole in the past but it might more accurately be termed a dark body since, if such a body exists, it would be simply a very dense body which could be approached and, in fact, viewed from a suitable distance, unlike the modern notion of a black hole. Obviously, this latter comment is in accordance with the usual meaning of a so-called 'escape speed'.

However, towards the middle of the last century, the modern idea of a black hole appeared. Such a body occurs as a consequence of a singularity apparently appearing in the form of the Schwarzschild solution to Einstein's field equations of general relativity for the case of a spherically symmetric mass point which appears in most textbooks on general relativity and cosmology [3]. Normally, this solution is stated as being either

$$ds^2 = \left\{1 - \frac{2Gm}{rc^2}\right\}c^2dt^2 - \left\{1 - \frac{2Gm}{rc^2}\right\}^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

or more usually

$$ds^2 = \left\{1 - \frac{2m}{r}\right\}dt^2 - \left\{1 - \frac{2m}{r}\right\}^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where the universal constant of gravitation, G, and the speed of light, c, have both been put equal to unity. Here r, θ, and ϕ appear to be taken to be normal polar co-ordinates.

In the above expressions, a mathematical singularity is seen to occur when $r = 0$, as might be expected for polar co-ordinates. However, due to the form of the coefficient of dr^2, it follows that a second mathematical singularity occurs when, in the first of the above equations, $rc^2 = 2Gm$ or, in the second, $r = 2m$. The first singularity is regularly dismissed as being merely a property of polar co-ordinates and, therefore, of no physical significance. The second singularity, however, tends to have a physical interpretation attributed to it - namely that it is said to indicate the existence of a black hole. If this interpretation were valid, it would imply that, for an object of mass m and event horizon radius r to be a black hole, it would need to satisfy the inequality

$$m/r \geq c^2/2G = 6.7 \times 10^{26} \text{ kg/m}.$$

It is of interest to note that, for Michell's dark body, the ratio of mass to actual radius, rather than radius of the event horizon, formally gives exactly the same result [2].

These days, claims for the identification of black holes appear fairly regularly in the scientific literature. Quite often, the supposed existence of massive black holes - is invoked to explain some otherwise puzzling phenomenon. However, so far, on no occasion has the postulated object satisfied the requirement mentioned earlier that, for a black hole, the ratio of the body's mass to its radius - or more specifically in general relativistic language, the radius of its event horizon - must be subject to the restriction

$$m/r \geq 6.7 \times 10^{26} \text{ kg/m}.$$

Also, what some regard as the defining feature of a black hole – its event horizon – has never been positively identified.

Black Holes and Thermodynamics.

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In retrospect, it seems that it was inevitable that the analogy between an area theorem for black holes, published by Hawking in 1972, which asserted that, in any process involving black holes, the total area of the event horizon may only increase, and the established increase in entropy due to thermal interactions, was one that could not go unnoticed for long. If a connection was to be established, the question remaining was what function of the area was to be identified with the entropy of a black hole? The simplest choice compatible with Hawking's theorem is to set the black hole entropy proportional to the area of the event horizon itself.

Black holes are said to obey a 'no–hair' theorem. This states that black holes cannot be distinguished except for their mass, charge and angular momentum. In the simplest case of a Schwarzschild black hole, which is uncharged and non-rotating, the area of the event horizon is proportional to the so-called 'irreducible', or 'inextractable', part of the mass of the black hole. Actually, the entropy is postulated to have the form

$$S = kM^2 \big/ \sigma_m^2,$$

where M is the 'irreducible' mass of the black hole and $\sigma_m = (ch/2\pi G)^{1/2} = 2 \times 10^5$ gm is the Planck mass.

Actual criticism of the established view has been minimal. However, it has been pointed out that, in conventional thermodynamics, the entropy is a first-order homogeneous function in all the extensive variables and this is not the case for this commonly accepted black hole entropy expression. (Here extensive variables, such as internal energy, volume and number of particles, are those which depend on the size of the particular system; all other variables, such as temperature and pressure, are termed intensive variables.) This might seem a somewhat trivial point to many people but it is, in fact, a feature which has several important consequences. In orthodox thermodynamics, one very useful formula is the so-called Gibbs-Duhem equation, which is a relation linking all the intensive variables of a system and shows that these variables are not all independent of one another. This formula has many important consequences and features in the derivation of many other formulae. However, the derivation of the Gibbs-Duhem relation itself depends critically on the extensive nature of the entropy of the system. Since the proposed black hole entropy expression is certainly not extensive in nature, it follows that there is no Gibbs-Duhem equation for such a system [4]. Hence, formulae derived by using the Gibbs-Duhem relation must be excluded from use also when discussing such systems. It is possible that this is a technical point, which may be appreciated fully only by the theoretician but it is an important point which cannot be over-emphasised. The same argument may be employed when considering the derivation of the well-known Einstein – Boltzmann formula for the probability of spontaneous fluctuations. This derivation holds no longer also. This follows because the Einstein formula implies that the entropy is an additive function; that is, if two systems are considered, the entropy of the combined system equals the sum of the entropies of the individual systems. Alternatively, this may be viewed as meaning that the joint probability of different events reduces to the product of the individual probabilities, implying statistical independence; in other words, the product of probability densities is tantamount to the sum of the entropies, which is Boltzmann's principle. Quite clearly, this is simply not possible for the present case because of the precise nature of Hawking's area theorem, from which it may be concluded that, if two black holes are combined, the entropy of a combined black hole is always greater than the sum of the entropies of the individual black holes, excluding the case where equality may hold. Hence, the Einstein – Boltzmann formula for a spontaneous fluctuation from equilibrium may not be used when considering thermodynamic black hole fluctuations. At the very least, this point has not been fully appreciated on a number of occasions and the said formula has been applied in a number of situations where its use is simply not permissible.

The fact that the sum of the areas before collision is not equal to the area after collision means that thermodynamic equilibrium may not be achieved. Consider two isolated systems at different temperatures. Suppose they are placed in thermal contact with one another but isolated from everything else. Eventually, in accordance with the zeroth law of thermodynamics, they will arrive at a common temperature. During this process, there will have been an increase in entropy. However, if the two separate systems had initially been at the same temperature, the entropy would not have increased. The above mentioned Bekenstein -Hawking expression for the entropy of a black hole is unable to cope with this particular, but very important, case since, if M_1 and M_2 are the masses of the two black holes, then the mass after the collision is given by

$$(M_1 + M_2)^2 > M_1^2 + M_2^2.$$

Another important consequence of the presently accepted black hole entropy expression is that the heat capacity of the system is negative. Although such heat capacities are no strangers in astrophysics, inevitably they refer to one component, or phase, of a multicomponent, or multiphase, system. In reality a black hole must be an open system but it is always treated as a closed system. The mass could be written as the product $M = Nm$, where N is the number of 'particles' in the black hole having mass m, but, if N is not conserved, it would then be necessary to specify the second phase. Further, it has been shown possible for a negative heat capacity in a closed system to lead to a violation of the second law of thermodynamics and so, such heat capacities cannot be permissible. This point has been strengthened even more by work which indicates that it is the mathematical property of concavity of the entropy which embodies the essence of the second law.

It might be argued that the second law, as popularly known, does not hold for such exotic objects as black holes. This is not a totally unreasonable point of view since the said law, although it might be said to have stood the test of time, is really a statement of fact based on worldly experience. For the hundred and fifty years or so since it was first proposed, people have sought to find violations of the second law of thermodynamics, just as they have striven to find violations of the first law. The reason for this preoccupation is the lure of 'getting something for nothing', while making massive inroads into the problem of solving the world's energy requirements. It goes without saying that, so far, all these efforts have been in vain. However, as pointed out by Planck, if units of time, length, and mass that may be constructed from the fundamental constants of nature "necessarily retain their significance for all times and for all cultures, including extraterrestrial and nonhuman ones, these 'natural units' would retain their natural significance as long as the laws of gravitation and the propagation of light in vacuum, and the two laws of thermodynamics retain their validity" [5]. Therefore, according to Planck, to question universality and the fundamental constants is tantamount to questioning the two laws of thermodynamics. Although it might be argued that it is not concavity but rather the property of super-additivity that is the true stamp of entropy, it only requires one single exception to disprove this possibility. That exception is provided by black body radiation which possesses a sub-additive entropy.

Since black body radiation has been mentioned, it seems worth considering, at this point, what happens when a black hole is bathed in black body radiation in a closed container. In the Bekenstein-Hawking entropy expression, the original dependence is on M, the so-called 'irreducible' mass. It is only via use of the relation

$$E = Mc^2$$

that the dependence of the entropy on the energy is established. Hence, for a so-called Schwarzschild black hole, the entropy is given by

$$S_{bh} = \frac{2\pi kGM^2}{hc} = \frac{2\pi kGE_{bh}^2}{hc^5},$$

while that of black body radiation is given by

$$S_{bb} = \frac{4}{3} k\sigma^{1/4} E_{bb}^{3/4} V^{1/4},$$

where σ is the radiation constant. It needs a little imagination to achieve it but, given that, it might be possible to become convinced that the total entropy in the container is given by the sum of these two expressions; that is,

$$S = S_{bh} + S_{bb} = k\left\{\frac{2\pi G E_{bh}^{2}}{hc^{5}} + \frac{4}{3}\sigma^{1/4} E_{bb}^{3/4} V^{1/4}\right\}.$$

The constancy of the total energy, $E = E_{bh} + E_{bb}$, means that $dE_{bh} = - dE_{bb}$ or, in other words, any changes in the black hole entropy must be exactly balanced by corresponding changes in the black body entropy. Again, the condition for thermal equilibrium demands that any change in the total entropy vanishes for arbitrary variations of energy. Hence,

$$1/T_{bh} = 1/T_{bb},$$

where, in an obvious notation, T_{bh} and T_{bb} represent the black hole and black body temperatures respectively.

Further, following earlier work, it might seem that

$$\frac{\partial^2 S_{bh}}{\partial E_{bh}^2} + \frac{\partial^2 S_{bb}}{\partial E_{bb}^2} < 0$$

is a condition for thermodynamic stability in a system comprising two bodies; in this case the two bodies being a black hole and black body radiation. From this last inequality, which expresses the concavity of the total entropy, there would result

$$E_{bb} < E_{bh}/4.$$

In Hawking's words, "in order for the configuration of a black hole and gravitons to maximise the probability, the volume, V, of the box must be sufficiently small that the energy E_{bb} of the black body gravitons is less than ¼ the mass of the black hole". However, thermodynamics can never place limits on the size of the volume or energy above which the system would be unstable. Thermodynamics is a 'black box' that provides no specific information about the system under consideration. An explicit physical model is necessary if actual numerical values are to be obtained.

In the situation just considered, a system composed of two parts – a black hole and black body radiation – was under examination. However, what precisely is a composite system? The notion of a composite system was introduced by Carathéodory, when he looked at the problems surrounding the foundations of thermodynamics at the beginning of the last century, in order to avoid considering non-equilibrium states. In fact, he compared two states of equilibrium, a more and a less constrained state of thermodynamic equilibrium that is achieved from the former by removing a restrictive partition between the two subsystems. Here the subsystems must necessarily be of the same type and not two different types, such as in the situation considered by Hawking [6]. It was claimed that "although the canonical ensemble did not work for black holes, one can still employ a microcanonical ensemble of a large number of similar insulated systems each with a fixed energy *E*".

It should be noted that all the material contained in the foregoing discussion is well documented [7].

Entropy.

Although thermodynamics is a subject based on phenomena with which all are familiar, it is, nevertheless, a topic which causes many worries and concerns. Much of this centres around the concept of entropy, possibly because it is the one quantity in the introduction to the subject which is not in any way part of people's everyday experience. Hence, as such, an aura of mystery surrounds this quantity for most people. If people are more mathematically inclined, the problem is less severe since, whatever the approach adopted, the entropy is seen to enter the theory merely as the name given to a total derivative, where that total derivative equals an inexact differential of the heat multiplied by its integrating factor which is the reciprocal of the absolute temperature. The problem is exacerbated in all probability by at least two modern occurrences:

(i) the modern tendency to drift away from the origins of the subject and so, cease to stress the importance of cycles in the development

and

(ii) possible confusion caused by the link between thermodynamics and statistical mechanics.

As far as the first of these is concerned, it must always be remembered that the founding fathers of thermodynamics were closely involved with the working of heat engines. The only place in the early development where cycles were not involved was in the observations of Rumford. Apart from that, people like Carnot derived their inspiration from the practical work of men like Watt and Trevithick who were concerned with improving the efficiency of heat engines for use in, amongst other places, the Cornish tin mines. Some of Carnot's inspiration came from a desire to help the French catch up with the British in this area of production of heat engines. Hence, cycles were a vitally important part of the beginnings of thermodynamics and the people who pushed it forward and began to give the topic a firm theoretical foundation – Thomson (Lord Kelvin), Tait, Clausius [8] – based their work on engines working in cycles. It should be noted that the two modern versions of the famous statements of the Second Law, that due to Thomson :

It is impossible to transform an amount of heat completely into work in a cyclic process in the absence of other effects

and that due to Clausius:

It is impossible for heat to be transferred by a cyclic process from a body to one warmer than itself in the absence of other effects,

both stress the notion of cyclic processes as well as the absence of effects other than those specifically mentioned. It might be noted also that these are the fundamental forms of the Second Law. In what follows, mention will be made of the possibility of entropy increasing in an irreversible change. It should be noted that the property of entropy increase, even if true, is *not* a statement of the Second Law of Thermodynamics; at best, it is merely a deduction from that law.

The link between thermodynamics and statistical mechanics can also lead to problems since, normally, the entropy of thermodynamics is immediately equated with the entropy of statistical mechanics. It is obvious to see why such an identification should be made, but a moment's reflection immediately identifies serious problems. When the question is considered, it is realised at once that the backgrounds of the two entropies are somewhat different; that from thermodynamics is purely due to a change in heat, but that in statistical mechanics, at first sight at least, is a function related to the statistical distribution of the particles under consideration. Possibly even more confusion arises due to the modern tendency to link the entropy function associated with information theory with those of thermodynamics and statistical mechanics. At first sight, because of its mathematical form, this does not seem unreasonable but a moment's reflection indicates concerns. While the possibility of a link between the entropy functions of thermodynamics and statistical mechanics can be justified, it is far more difficult to do so with the function of information theory because there is no immediate link with a change in heat in the information theory case. This point cannot be overemphasised; in the thermodynamics, an entropy change is irrevocably linked with a change in heat. It might be noted that this and other associated points have formed the basis of a number of recent articles. [9]

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