CONVERGENCE RATE OF EXTREME OF SKEW NORMAL DISTRIBUTION UNDER POWER NORMALIZATION

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Abstract: Let $\{X_n,\ n \geq 1\}$ be independent and identically distributed random variables with each X_n following skew normal distribution. Let $M_n = \max\{X_k,\ 1 \leq k \leq n\}$ denote the partial maximum of $\{X_n,\ n \geq 1\}$. Liao et al. (2014) considered the convergence rate of the distribution of the maxima for random variables obeying the skew normal distribution under linear normalization. In this paper, we obtain the asymptotic distribution of the maximum under power normalization and normalizing constants as well as the associated pointwise convergence rate under power normalization.

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1 Introduction

The skew normal distribution is introduced by Azzalini[1]. Because of being able to model skew data, it has more widely applied areas than normal distribution. The probability density function (pdf) of the standard skew normal distribution is

$$f_\lambda(x) = 2\phi(x)\Phi(\lambda x), \ \ -\infty < x < +\infty,$$

where $\phi(\cdot)$ stands for the standard normal pdf and $\Phi(\cdot)$ stands for the standard normal cumulative distribution function (cdf). It is well-known that if $\lambda = 0$, the the standard skew normal distribution deduces to the standard normal distribution.

One interesting problem in extreme value theory is to investigate the rate of uniform convergence of $F^n(\cdot)$ to its extreme value distribution function. For the uniform convergence

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rate under linear normalization, see Hall[2], Peng et al.[3], Liao and Peng[4], Liao et al[5]. and Chen and Huang[6]. For the uniform convergence rate under power/nonlinear normalization, see Chen et al[7]. and Chen and Feng[8]. For the rate of convergence under second regular condition, see Peng and Nadarajah[9] and Liu and Peng[10].

The aim of this paper is to derive the asymptotic distribution of the distribution of the maxima for random variables following the skew normal distribution with parameter λ under power normalization and the associated pointwise convergence rate under power normalization. Our main result could be used to measuring the error committed by substituting the exact distribution of the maximum with the asymptotic distribution in analytical studies.

2 Preliminary Lemmas and Main results

In this section, we give some lemmas which need to prove our main results.

Let $F_\lambda(x)$ and $f_\lambda(x)$ denote the cdf and pdf of the skew normal distribution, respectively. Liao et al.[5] obtained the Mills ratios of the skew normal distribution as following lemma:

Lemma 2.1 For all $x > 0$, we have

(a) if $\lambda < 0$,

$$\frac{1 - F_\lambda(x)}{f_\lambda(x)} \sim \frac{1}{(1+\lambda^2)x}, \tag{2.1}$$

as $x \to +\infty$;

(b) if $\lambda > 0$,

$$\frac{1 - F_\lambda(x)}{f_\lambda(x)} \sim \frac{1}{x}. \tag{2.2}$$

as $x \to +\infty$.

Liao et al.[5] also established the distributional tail representation of the skew normal distribution as following lemma:

Lemma 2.2 Let $F_\lambda(x)$ denote the cdf of the skew normal distribution. Then

$$1 - F_\lambda(x) = c(x) \exp(-\int_1^x \frac{g(t)}{f(t)} dt), \tag{2.3}$$

where

(a) if $\lambda < 0$,

$$c(x) \to \frac{\exp\{-(1+\lambda^2)/2\}}{(-\lambda)(1+\lambda^2)\pi}, \text{ as } x \to +\infty,$$

$$f(x) = \frac{1}{(1+\lambda^2)x}, \quad g(x) = 1 + \frac{2}{(1+\lambda^2)x^2}; \tag{2.4}$$

(b) if $\lambda > 0$,

$$c(x) \to \left(\frac{2}{\pi e}\right)^{\frac{1}{2}}, \text{ as } x \to +\infty,$$

$$f(x) = \frac{1}{x}, \quad g(x) = 1 + \frac{1}{x^2}. \tag{2.5}$$

Remark 2.1 By Lemma 2.2 combining with Corollary 1.7 of Resnik[11] we have $F_\lambda \in D_l(\Lambda)$ and the associated linear norming constants a_n and b_n.

Next we cite some results which come from Mohan and Ravi[12]. Let $r(F) = \sup\{x|F(x) < 1\}$ represent the right end point of a distribution F.

Lemma 2.3 Let F be one distribution function. If $F \in D_l(\Lambda)$ and $r(F) = +\infty$, then

$$F \in D_p(\Phi_1)$$

with the power normalizing constants α_n and β_n are defined by

$$\alpha_n = b_n, \beta_n = a_n/b_n, \tag{2.6}$$

where $\Phi_1(x) = \exp(-1/x)$.

Lemma 2.4 Let F be a distribution function, then $F \in D_p(\Phi_1)$ if and only if

(i)$r(F) > 0$, and

(ii) $\lim_{t\to r(F)} \frac{1-F(t\exp(yf(t)))}{1-F(t)} = \exp(-y)$, for some positive valued function f.

If (ii) holds for some f then $-\int_a^{r(F)}(1-F(x))/x\,\mathrm{d}x < \infty$ for $0 < a < r(F)$ and (ii) holds with the choice $f(t) = 1/(1-F(t))\int_t^{r(F)}(1-F(x))/x\,\mathrm{d}x$. The power normalization constants may be chosen as $\alpha_n = F^{\leftarrow}(1-1/n)$ and $\beta_n = f(\alpha_n)$, where $F^{\leftarrow}(x) = \inf\{y : 1-F(y) \geq x\}$.

Now we give the asymptotic distribution of the distribution of the maxima for independent and identically distributed random variables obeying the skew normal distribution.

Theorem 2.1 Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with common distribution F which follows the skew normal distribution. Then $F \in D_p(\Phi_1)$, i.e.

$$\lim_{n\to\infty} P\left(\left|\frac{M_n}{\alpha_n}\right|^{\frac{1}{\beta_n}} \operatorname{sign}(M_n) \leq x\right) = \lim_{n\to\infty} F^n(\alpha_n|x|^{\beta_n}\operatorname{sign}(x)) = \Phi_1(x),$$

and the normalizing constants can be chosen as $\alpha_n = b_n,\ \beta_n = a_n/b_n$, where

(i) for $\lambda < 0$,

$$\begin{aligned}\alpha_n &= \left(\frac{2\log n}{1+\lambda^2}\right)^{\frac{1}{2}} - \frac{\log\log n + \log(-2\lambda\pi)}{(1+\lambda^2)^{\frac{1}{2}}(2\log n)^{\frac{1}{2}}} + o((\log n)^{-\frac{1}{2}})\\ \beta_n &= \frac{1}{(1+\lambda^2)\alpha_n^2} \sim \frac{1}{2\log n}.\end{aligned} \tag{2.7}$$

(ii) for $\lambda > 0$,

$$\begin{aligned}\alpha_n &= (2\log n)^{\frac{1}{2}} - \frac{\log\log n + \log\pi}{2(2\log n)^{\frac{1}{2}}} + o((\log n)^{-\frac{1}{2}})\\ \beta_n &= \frac{1}{\alpha_n^2} \sim \frac{1}{2\log n}.\end{aligned} \tag{2.8}$$

We provide another main result. Theorem 2.2 shows the convergence rate of $F^n(\alpha_n x^{\beta_n})$ to its limit is proportional to $1/\log n$.

Theorem 2.2 Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with common distribution F following skew normal distribution. For $\lambda \in \mathbb{R}$, $x > 0$ and sufficiently large n,

$$|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| \sim \exp(-x^{-1})x^{-1}\beta_n, \tag{2.9}$$

where power norming constants α_n and β_n are defined by (2.7) and (2.8).

3 Proofs

Proof of Theorem 2.1**.** We just prove the case of $\lambda < 0$, as for the case of $\lambda > 0$ is similar.

By Remark 2.1 and Lemma 2.3, we have $F \in D_p(\Phi_1)$. By (2.1) and Lemma 2.4, we can choose the power noming constant α_n that make it satisfy the equation $1 - F_\lambda(\alpha_n) = 1/n$. Let

$$\frac{2\phi(\alpha_n)\Phi(\lambda\alpha_n)}{(1+\lambda^2)\alpha_n} = \frac{1}{n}. \tag{3.1}$$

By some elementary calculations similar to the of proof Proposition 3 of Liao et al., we have

$$\alpha_n = \left(\frac{2\log n}{1+\lambda^2}\right)^{\frac{1}{2}} - \frac{\log\log n + \log(-2\lambda\pi)}{(1+\lambda^2)^{\frac{1}{2}}(2\log n)^{\frac{1}{2}}} + o((\log n)^{-\frac{1}{2}}).$$

By (2.4) and Lemma 2.4, we have

$$\beta_n = \frac{f(\alpha_n)}{\alpha_n} = \frac{1}{(1+\lambda^2)\alpha_n^2} \sim \frac{1}{2\log n}. \tag{3.2}$$

The proof of the result is complete. □

Proof of Theorem 2.2**.** In this we just give the proof of the case of $\lambda < 0$, as for the proof of the case of $\lambda > 0$ is similar.

Noting the following Mills ratio concerning the standard normal distribution provided by Mills[13]: for all $x > 0$, as $x \to +\infty$,

$$\frac{1-\Phi(x)}{\phi(x)} \sim \frac{1}{x},$$

and utilizing the symmetric property of $\Phi(-x) = 1 - \Phi(-x)$, and (2.1), we have

$$1 - F_\lambda(x) = x^{-2}(1+\lambda^2)^{-1}(-\lambda\pi)^{-1}\exp\left(-\frac{1+\lambda^2}{2}x^2\right)(1+O(x^{-2})).$$

So,

$$\begin{aligned}1 - F_\lambda(\alpha_n x^{\beta_n}) &= (\alpha_n x^{\beta_n})^{-2}(1+\lambda^2)^{-1}(-\lambda\pi)^{-1}\exp\left(-\frac{1+\lambda^2}{2}(\alpha_n x^{\beta_n})^2\right)(1+O((\alpha_n x^{\beta_n})^{-2}))\\ &= \alpha_n^{-2}(1+\lambda^2)^{-1}(-\lambda\pi)^{-1}\exp\left(-\frac{1+\lambda^2}{2}\alpha_n^2\right)x^{-2\beta_n}\\ &\times \exp\left(-\frac{(1+\lambda^2)\alpha_n^2}{2}(x^{2\beta_n}-1)\right)(1+O((\alpha_n x^{\beta_n})^{-2})).\end{aligned} \tag{3.3}$$

Noting that $e^x = 1 + x + O(x^2)$, as $x \to 0$, and by (3.2), we have

$$\begin{aligned} 1 + (\alpha_n x^{\beta_n})^{-2} &= 1 + (1+\lambda^2)\beta_n x^{-2\beta_n} \\ &= 1 + (1+\lambda^2)\beta_n (1 - 2\beta_n \log x + O(\beta_n^2)) \\ &= 1 + O(\beta_n). \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) and combining (3.1) with (3.2), we have

$$\begin{aligned} 1 - F(\alpha_n x^{\beta_n}) &= \frac{1}{n} x^{-2\beta_n} \exp\left\{ - \frac{(1+\lambda^2)\alpha_n^2 \beta_n}{2} \frac{x^{2\beta_n} - 1}{\beta_n} \right\} (1 + O(\beta_n)) \\ &= \frac{1}{n} x^{-2\beta_n} \exp\left\{ - \frac{x^{2\beta_n} - 1}{2\beta_n} \right\} (1 + O(\beta_n)). \end{aligned}$$

Observe that

$$\frac{x^{2\beta_n} - 1}{\beta_n} = 2 \log x + 2\beta_n \log^2 x + O(\beta_n^2).$$

Hence, we have

$$\begin{aligned} 1 - F(\alpha_n x^{\beta_n}) &= \frac{1}{n} x^{-2\beta_n} \exp(-\log x - \beta_n \log^2 x + O(\beta_n^2))(1 + O(\beta_n)) \\ &= \frac{1}{nx} \exp(-2\beta_n \log x - \beta_n \log^2 x + O(\beta_n^2))(1 + O(\beta_n)) \\ &= \frac{1}{nx} (1 - \beta_n (2 \log x + \log^2 x) + O(\beta_n)). \end{aligned}$$

Noting $\log(1 - x) = -x + O(x^2)$ and $e^x = 1 + x + O(x^2)$ as $x \to 0$, thus, we have

$$\begin{aligned} F^n(\alpha_n x^{\beta_n}) - \exp(-\frac{1}{x}) &= \left\{ 1 - \frac{1}{nx} (1 - \beta_n (2 \log x + \log^2 x) + O(\beta_n)) \right\}^n - \exp(-\frac{1}{x}) \\ &= \exp\left(-\frac{1}{x} + \frac{1}{x} \beta_n (2 \log x + \log^2 x) + O(\beta_n) \right) - \exp(-\frac{1}{x}) \\ &= \exp(-\frac{1}{x}) (\frac{1}{x} \beta_n (2 \log x + \log^2 x) + O(\beta_n)). \end{aligned}$$

Our desired result follows.

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幂赋范下偏正态分布极值的收敛速度

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摘 要: 令$\{X_n,\ n\geq 1\}$是独立同分布随机变量序列并且每个变量均服从偏正态分布. 再令$M_n=\max\{X_k,\ 1\leq k\leq n\}$表示$\{X_n,\ n\geq 1\}$的部分最大值. 廖昕等人(2014)考虑了线性赋范下随机变量序列同服从偏正态分布最大值分布的收敛速. 在这篇文章中, 我们得到了幂赋范下最大值分布的渐近分布和赋范常数以及幂赋范下相应的逐点收敛速度

关键词: 渐近分布; 最大值; 收敛速度; 偏正态分布

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