Notes on the Proof of Second Hardy-Littlewood Conjecture

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Résumé

In this paper a slightly stronger version of the Second Hardy-Littlewood Conjecture (see [1]), namely that inequality $\pi(x)+\pi(y) > \pi(x+y)$ is examined, where $\pi(x)$ denotes the number of primes not exceeding x. It is shown that the inequality holds for all sufficiently large x and y. It has also been shown that for a given value of $y \geq 55$ the inequality $\pi(x)+\pi(y) > \pi(x+y)$ holds for all sufficiently large x. Finally, in the concluding section an argument has been given to completely settle the conjecture.

1 Introduction

The original version of the conjecture is $\pi(x) + \pi(y) \geq \pi(x + y)$ for all x , $y \geq 2$. It had been suggested by E. Landau that $\pi(2x) < 2\pi(x)$ for all $x \geq 3$. This was subsequently proved by Rosser and Scholenfeld in [8]. There are some well-known inequalities that are similar in spirit to that of this conjecture. For example, C. Karanikolov in [3] showed that if $a \geq e^{\frac{1}{4}}$ and $x \geq 364$ then we have,

$$\pi(ax) < a\pi(x)$$

V. Udrescu in [4] proved that if $0 < \epsilon \leq 1$ and $\epsilon x \leq y \leq x$ then $\pi(x) + \pi(y) > \pi(x + y)$ for x and y sufficiently large. L. Panaitopol made these two results sharper by proving that if $a > 1$ and $x > e^{4(\ln a)^{-2}}$ then $\pi(ax) < a\pi(x)$ and if $a \in (0, 1]$ and $x \geq y \geq ax$, $x \geq e^{9a^{-2}}$, then $\pi(x) + \pi(y) > \pi(x + y)$. However, as may be noted that the inequalities has been proved under some hypothesis on the ranges of the values of y that depends on the values of x. So, these inequalities are not *unconditional* in ordinary sense. In the same paper Panaitopol proved an unconditional inequality which is true for all positive integers x and y such that x , $y \geq 4$. The result is,

$$\frac{1}{2}\pi(x + y) \leq \pi\left(\frac{x}{2}\right) + \pi\left(\frac{y}{2}\right)$$

However, it is commonly believed that this conjecture is false. The reason for this belief the paper (see [2]) Hensley and Richards which shows that the First Hardy-Littlewood Conjecture (or k-tuple Conjecture) and Second Hardy-Littlwood Conjecture are incompatiable with each other.

In this paper we prove that for all sufficiently large x and y there is no exception of the *Hardy-Littlewood Inequality* (the term that will be used instead of *Second Hardy-Littlewood Conjecture*).

In fact, we prove something more. We show that for all $y \geq 55$ there exists a real number M such that for all $x \geq M$ we will have $\pi(x) + \pi(y) > \pi(x + y)$.

For this purpose we will examine the inequality $\pi(ky) + \pi(y) > \pi((k + 1)y)$ and try to find out the range of values of y for which the inequality holds for all $k > 1$.

2 Methodological Remarks

Before going into the messy details of the arguments of the theorems that is going to be proved, let us first elaborate the goal of this paper and the results that is going to be used in the paper once again. In this paper our final goal is to outline a method that, at least in principle, shows a way to check the validity of the Hardy-Littlewood Inequality.

For this purpose we will be using the result that is immediate from de la Vallée-Poussin's proof of Prime Number Theorem, namely the result that,

$$\frac{x}{\ln x - (1 - \epsilon)} < \pi(x) < \frac{x}{\ln x - (1 + \epsilon)}$$

for all $\epsilon > 0$ and for all sufficiently large x. Since we will be referring to this inequality more than once, from now on let's call it *Poussin's Inequality*. From this inequality and from the examination of the inequality $\pi(ky)+\pi(y) > \pi((k+1)y)$ we will try to find a bound on the value of ϵ such that all x and y will satify the Hardy-Littlewood Inequality. That will be our goal in our first theorem.

Having done this we will try to prove that for all $y \geq 55$ satisfying

$$\frac{y}{\ln y} < \pi(y) < \frac{y}{\ln y - 4}$$

there exists a real number M such that for all $x \geq M$ we will have $\pi(x)+\pi(y) > \pi(x+y)$. , i.e., we will prove that even if for our choosen value of y satisfying *Poussin's Inequality* the corresponding value of ϵ doesn't satisfy the bound, we needn't worry because even then the inequality holds for all sufficiently large x. Proving this will be the objective of our second and final theorem.

3 The Theorems

Theorem 1

For all $k > 1$ and for all $y \geq 2e$ satisfying $\frac{y}{\ln y - (1 - \epsilon)} < \pi(y) < \frac{y}{\ln y - (1 + \epsilon)}$ for all $0 < \epsilon \leq \ln \sqrt{2}$ we will have $\pi(ky) + \pi(y) > \pi((k + 1)y)$.

Proof

We start by noting that,

$$\pi(ky) + \pi(y) > \frac{ky}{\ln ky - (1 - \epsilon)} + \frac{y}{\ln y - (1 - \epsilon)}$$

and for the same ϵ,

$$\frac{(k+1)y}{\ln(k+1)y-(1+\epsilon)} > \pi\left((k+1)y\right)$$

Hence proving,

$$\frac{ky}{\ln ky-(1-\epsilon)} + \frac{y}{\ln y-(1-\epsilon)} \geq \frac{(k+1)y}{\ln(k+1)y-(1+\epsilon)}$$

Or equivalently,

$$\frac{k}{\ln ky-(1-\epsilon)} + \frac{1}{\ln y-(1-\epsilon)} \geq \frac{k+1}{\ln(k+1)y-(1+\epsilon)}$$

will imply our inequality.

Notice that the above inequality is satisfied if and only if,

$$k\left(\frac{1}{\ln ky-(1-\epsilon)} - \frac{1}{\ln(k+1)y-(1+\epsilon)}\right) \geq \left(\frac{1}{\ln(k+1)y-(1+\epsilon)} - \frac{1}{\ln y-(1-\epsilon)}\right)$$

Now we take $\ln(k+1)y-(1+\epsilon) > 0$. Keeping in mind the bound on ϵ as stated in theorem we note that for all $y \geq \sqrt{2}e$ the inequality holds trivially for all $k > 1$. Consequently, with this bound assumed on y we conclude that the above inequality holds if and only if,

$$k\left(\frac{\ln\left(1+\frac{1}{k}\right)-2\epsilon}{\ln ky-(1-\epsilon)}\right) \geq \left(\frac{2\epsilon-\ln(k+1)}{\ln y-(1-\epsilon)}\right)$$

Now we note that

$$\frac{k}{\ln ky-(1-\epsilon)} \geq \frac{1}{\ln y-(1-\epsilon)}$$

for $y \geq 2e$. Because the above inequality is implied by, $\left(\frac{y}{e}\right)^{k-1} \geq k$ which holds for all $k > 1$ and for all $y \geq 2e$.

Thus we are left with proving

$$\ln\left(1+\frac{1}{k}\right) - 2\epsilon \geq 2\epsilon - \ln(k+1)$$

Or equivalently,

$$(k+1)^2 \geq ke^{4\epsilon}$$

which holds for all $k > 1$ and for all $0 < \epsilon \leq \ln\sqrt{2}$.

Hence the theorem is proved.

Theorem 2

For all $y \geq 55$ there exists a real number M such that for all $x \geq M$ we will have $\pi(x) + \pi(y) > \pi(x + y)$.

Proof

For a proof of this inequality we again use the inequality $\pi(ky)+\pi(y) > \pi((k+1)y)$. The objective is to find a lower bound for k above which for all k the inequality holds.

We know that for all $y \geq 17$ we have $\frac{y}{\ln y} < \pi(y)$ by [7] and for all $y \geq 55$ we have, $\pi(y) < \frac{y}{\ln y - 4}$ by [6]. Combining these inequalities we get for all $y \geq 55$,

$$\frac{y}{\ln y} < \pi(y) < \frac{y}{\ln y - 4}$$

The inequality to be proved is implied by,

$$\frac{ky}{\ln ky} + \frac{y}{\ln y} > \frac{(k+1)y}{\ln(k+1)y - 4}$$

For all $y \geq e^4$ which holds if and only if,

$$\left(\frac{ky}{\ln ky}\right)\left(\ln\left(1 + \frac{1}{k}\right) - 4\right) \geq \left(\frac{y}{\ln y}\right)(4 - \ln(k+1))$$

Now note that the function $f(x) = \frac{x}{\ln x}$ is strictly increasing for all $x > e$ hence we can say that in our case

$$\frac{ky}{\ln ky} > \frac{y}{\ln y}$$

holds for all $k > 1$.

So, we are now left to prove that,

$$\ln\left(1 + \frac{1}{k}\right) - 4 \geq 4 - \ln(k+1)$$

Or equivalently, $(k+1)^2 \geq e^8 k$ and this indeed holds for all sufficiently large k (for example, the inequality holds trivially for all $k > e^8$. Hence the theorem is proved.

4 Conclusion

From what we have shown it can be easily proved that for all x and y greater than or equal to $2e$ for which the bound on respective ϵ satisfies $0 < \epsilon \leq \ln\sqrt{2}$ there can be no exception of the inequality $\pi(x) + \pi(y) > \pi(x + y)$. Then by Theorem 2 we notice that for all x and y such that $\min(x, y) \geq 55$ we have $\pi(x)+\pi(y) > \pi(x+y)$ for all $x > e^8 y$. The cases where $2 \leq \min(x, y) < 55$ is already proved by Gordon and Rodemich in [5]. Thus we now turn our attention to the original form of Second Hardy-Littlewood Conjecture. In view of the two theorems, a way of completely settling the conjecture may be outlined. First of all, we will have to calculate explicit bounds for x and y for which Theorem 1 holds. Assuming this bound to be M_0, next we examine the cases when $M_0 > \min(x, y) > 1731$. The final task would be to check all the remainig cases by computer.

5 References

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