

Existence of Antiparticles as an Indication of Finiteness of Nature

Felix M. Lev

Artwork Conversion Software Inc., 1201 Morningside Drive, Manhattan Beach, CA 90266, USA (Email: felixlev314@gmail.com)

Abstract:

It is shown that in a quantum theory over a Galois field, the famous Dirac's result about antiparticles is generalized such that a particle and its antiparticle are already combined at the level of irreducible representations of the symmetry algebra without assuming the existence of a local covariant equation. We argue that the very existence of antiparticles is a strong indication that nature is described by a finite field rather than by complex numbers.

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1 Problem statement

A well-known fact of particle physics is that a particle and its antiparticle have equal masses. The explanation of this fact in quantum field theory (QFT) follows.

Irreducible representations (IRs) of the Poincare and anti-de Sitter (AdS) algebras by Hermitian operators used for describing elementary particles have the property that for each IR the Hamiltonian is either positive definite or negative definite. In the first case, the energy has the spectrum in the range $[mass, \infty)$, while in the second case it has the spectrum in the range $(-\infty, -mass]$.

However, for constructing Lagrangians one needs to work not with IRs but with local fields satisfying covariant equations (e.g. the Klein-Gordon equation, the Dirac equation etc.). Those fields are described by non-unitary representations of the Poincare or AdS groups induced from the Lorentz group. Each local field is a combination of two IRs with positive and negative energies called a particle and its antiparticle, respectively. Then, as follows from the CPT theorem, a particle and its antiparticle have the same masses. The problem of negative energies is then solved by quantization, after which the energies of both, the particle and its antiparticle become positive definite.

One might pose the following question. If locality is only approximate then the masses of a particle and its antiparticle should remain equal or can differ

each other? This question is legitimate because the physical meaning of locality is not quite clear. The matter is that since local fields are described by non-unitary representations, their probabilistic interpretation is problematic. As shown by Pauli [1], in the case of fields with an integer spin there is no subspace where the spectrum of the charge operator has a definite sign while in the case of fields with a half-integer spin there is no subspace where the spectrum of the energy operator has a definite sign. Local fields is only an auxiliary tool for constructing operators describing unitary representations of a system as a whole (momentum, energy, angular momentum etc.).

In the present paper we investigate the status of particles and antiparticles in a quantum theory over a Galois field (GFQT) proposed first in Refs. [2, 3]. The motivation and a detailed description of GFQT can be also found in Refs. [4, 5]. In GFQT quantum states are elements of linear spaces over a Galois field of characteristic p and operators of physical quantities are linear operators in such spaces. Since any Galois field is finite, in GFQT infinities cannot exist in principle. At the same time, when p is rather large, GFQT recovers predictions of standard quantum theory.

The idea of correspondence between GFQT and standard theory follows. If p is prime then the Galois field F_p with p elements can be represented as a set of elements $\{0, \pm i\}$ ($i = 1, 2, \dots, (p-1)/2$). Let f be a function from F_p to the ring of integers Z such that $f(a)$ in Z has the same notation in Z as a in F_p . Then for elements $a \in F_p$ such that $|f(a)| \ll p$, addition, subtraction and multiplication are the same as in Z . In other words, for such elements we do not feel the existence of p . Indeed, if the elements a_j ($j = 1, 2, \dots, n$) are such that $|f(a_j)| < [(p-1)/2]^{1/n}$ then

$$f\left(\sum_{j=1}^n a_j\right) = \sum_{j=1}^n f(a_j), \quad f\left(\prod_{j=1}^n a_j\right) = \prod_{j=1}^n f(a_j)$$

which shows that if F_p is treated as a ring then f is a local isomorphism between F_p and Z . When p increases, the bigger and bigger part of F_p becomes the same as Z . This important observation implies that *Z can be treated as a special case of F_p in the formal limit $p \rightarrow \infty$.*

In the general case, division in F_p is not the same as in standard mathematics. For example, $1/2$ in F_p equals $(p+1)/2$, i.e. a very large number if p is large. However, this does not mean that mathematics modulo p cannot describe physics. The matter is that spaces in quantum theory are projective.

By analogy with standard quantum theory, it is natural to define the elementary particle in GFQT as a system described by an IR of a Lie algebra over a Galois field. Representations of Lie algebras in spaces with nonzero characteristic are called modular representations. There exists a well developed theory of such representations. One of the known results is the Zassenhaus theorem [6] that any modular IR is finite dimensional.

It is well-known that any Galois field contains p^k elements where p is prime and k is natural. We use F_{p^k} to denote such fields. In standard theory one considers

representations of real Lie algebras in complex Hilbert spaces. Modular analogs of such representations are representations of Lie algebras over F_p in spaces over F_{p^2} . However, the following remark is in order.

Consider when the elements of F_{p^2} can be represented as $a + bi$ where $a, b \in F_p$ and i is a formal element such that $i^2 = -1$. The division in F_{p^2} can be defined as $(a + bi)^{-1} = (a - bi)/(a^2 + b^2)$ if $(a^2 + b^2) = 0$ in F_p implies that $a = 0$ and $b = 0$. As explained in textbooks on number theory, this is the case only if $p = 3 \pmod{4}$. Therefore F_{p^2} with such values of p can be treated as analogs of complex numbers.

Since we treat standard theory as a special case of GFQT in the formal limit $p \rightarrow \infty$, it is desirable not to postulate that GFQT is based on F_{p^2} with $p = 3 \pmod{4}$ because standard theory is based on complex numbers but vice versa, explain the fact that standard theory is based on complex numbers since GFQT is based on F_{p^2} . Therefore we should find a motivation for the choice of F_{p^2} with $p = 3 \pmod{4}$. Arguments in favor of such a choice are discussed in Refs. [3, 7, 5]. In this paper we will not use the restriction that the representation space is over F_{p^2} with $p = 3 \pmod{4}$ and will consider a general case when it is over F_{p^k} .

In standard quantum theory, Poincare symmetry is a special case of de Sitter (dS) or AdS symmetries in the procedure called contraction. As shown in Refs. [3, 5], in GFQT there is no analog of Poincare symmetry but analogs of dS and AdS symmetries are well defined. In the present paper we consider modular analogs of IRs of the AdS algebra while the case of IRs of the dS algebra is mentioned in Sec. 4.

The paper is organized as follows. In Sec. 2 we explicitly construct modular IRs of the $sp(2)$ algebras. Such IRs play an important auxiliary role for constructing modular IRs of the AdS algebra in Sec. 3. The results show that the status of particles and antiparticles in GFQT considerably differs from that in standard theory. Finally Sec. 4 is a discussion.

2 Modular IRs of the $sp(2)$ algebra

The key role in constructing modular IRs of the $so(2,3)$ algebra is played by modular IRs of the $sp(2)$ subalgebra. They are described by a set of operators (a', a'', h) satisfying the commutation relations

$$[h, a'] = -2a', \quad [h, a''] = 2a'', \quad [a', a''] = h \quad (1)$$

The Casimir operator of the second order for the algebra (1) has the form

$$K = h^2 - 2h - 4a''a' = h^2 + 2h - 4a'a'' \quad (2)$$

We first consider representations with the vector e_0 such that

$$a'e_0 = 0, \quad he_0 = q_0e_0 \quad (3)$$

where $q_0 \in F_p$ and $f(q_0) > 0$. Denote $e_n = (a'')^n e_0$. Then it follows from Eqs. (2) and (3), that

$$h e_n = (q_0 + 2n) e_n, \quad K e_n = q_0(q_0 - 2) e_n, \quad (4)$$

$$a' a'' e_n = (n + 1)(q_0 + n) e_n \quad (5)$$

One can consider analogous representations in standard theory. Then q_0 is a positive real number, $n = 0, 1, 2, \dots, \infty$ and the elements e_n form a basis of the IR. In this case e_0 is a vector with a minimum eigenvalue of the operator h (minimum weight) and there are no vectors with the maximum weight. The operator h is positive definite and bounded below by the quantity q_0 . For these reasons the above modular IRs can be treated as modular analogs of such standard IRs that h is positive definite.

Analogously, one can construct modular IRs starting from the element e'_0 such that

$$a'' e'_0 = 0, \quad h e'_0 = -q_0 e'_0 \quad (6)$$

and the elements e'_n can be defined as $e'_n = (a')^n e'_0$. Such modular IRs are analogs of standard IRs where h is negative definite. However, in the modular case one can easily prove the following statement.

Theorem 1: *Eqs. (3) and (6) define the same IR with the dimension $p - q_0 + 1$.*

Proof. The set (e_0, e_1, \dots, e_N) will be a basis of IR if $a'' e_i \neq 0$ for $i < N$ and $a'' e_N = 0$. These conditions must be compatible with $a' a'' e_N = 0$. As follows from Eq. (5), N is defined by the condition $q_0 + N = 0$ in F_p . As a result, if q_0 is one of the numbers $1, \dots, p - 1$ then $N = p - q_0$ and the dimension of IR equals $p - q_0 + 1$ (in agreement with the Zassenhaus theorem [6]). The element e_N satisfies Eq. (6) and therefore it can be identified with e'_0 .

3 Modular IRs of the so(2,3) algebra

Standard IRs of the AdS so(2,3) algebra relevant for describing elementary particles have been considered by many authors. The description in this section is a combination of two elegant ones given in Ref. [8] for standard IRs and Ref. [9] for modular IRs. In standard theory the representation operators of the so(2,3) algebra in units $\hbar/2 = c = 1$ are given by

$$[M^{ab}, M^{cd}] = -2i(g^{ac} M^{bd} + g^{bd} M^{cd} - g^{ad} M^{bc} - g^{bc} M^{ad}) \quad (7)$$

where a, b, c, d take the values $0, 1, 2, 3, 4$ and $M^{ab} = -M^{ba}$. The diagonal metric tensor has the components $g^{00} = g^{44} = -g^{11} = -g^{22} = -g^{33} = 1$. In these units the spin of fermions is odd, and the spin of bosons is even. If s is the particle spin then the corresponding IR of the su(2) algebra has the dimension $s + 1$.

If a modular IR is considered in a linear space over F_{p^2} with $p = 3 \pmod{4}$ then Eq. (7) is also valid but in the general case it is convenient to work with another

set of ten operators. Let (a'_j, a_j'', h_j) ($j = 1, 2$) be two independent sets of operators satisfying the commutation relations for the $\text{sp}(2)$ algebra

$$[h_j, a'_j] = -2a'_j, \quad [h_j, a_j''] = 2a_j'', \quad [a'_j, a_j''] = h_j \quad (8)$$

The sets are independent in the sense that for different j they mutually commute with each other. We denote additional four operators as b', b'', L_+, L_- . The operators $L_3 = h_1 - h_2, L_+, L_-$ satisfy the commutation relations of the $\text{su}(2)$ algebra

$$[L_3, L_+] = 2L_+, \quad [L_3, L_-] = -2L_-, \quad [L_+, L_-] = L_3 \quad (9)$$

while the other commutation relations are

$$\begin{aligned} [a'_1, b'] &= [a'_2, b'] = [a_1'', b''] = [a_2'', b''] = [a'_1, L_-] = [a_1'', L_+] = \\ [a'_2, L_+] &= [a_2'', L_-] = 0, \quad [h_j, b'] = -b', \quad [h_j, b''] = b'', \quad [h_1, L_\pm] = \pm L_\pm, \\ [h_2, L_\pm] &= \mp L_\pm, \quad [b', b''] = h_1 + h_2, \quad [b', L_-] = 2a'_1, \quad [b', L_+] = 2a'_2, \\ [b'', L_-] &= -2a_2'', \quad [b'', L_+] = -2a_1'', \quad [a'_1, b''] = [b', a_2''] = L_-, \\ [a'_2, b''] &= [b', a_1''] = L_+, \quad [a'_1, L_+] = [a'_2, L_-] = b', \\ [a_2'', L_+] &= [a_1'', L_-] = -b'' \end{aligned} \quad (10)$$

At first glance these relations might seem rather chaotic but in fact they are very natural in the Weyl basis of the $\text{so}(2,3)$ algebra.

In spaces over F_{p^2} with $p = 3 \pmod{4}$ the relation between the above sets of ten operators is

$$\begin{aligned} M_{10} &= i(a_1'' - a'_1 - a_2'' + a'_2), \quad M_{14} = a_2'' + a'_2 - a_1'' - a'_1, \\ M_{20} &= a_1'' + a_2'' + a'_1 + a'_2, \quad M_{24} = i(a_1'' + a_2'' - a'_1 - a'_2), \\ M_{12} &= L_3, \quad M_{23} = L_+ + L_-, \quad M_{31} = -i(L_+ - L_-), \\ M_{04} &= h_1 + h_2, \quad M_{34} = b' + b'', \quad M_{30} = -i(b'' - b') \end{aligned} \quad (11)$$

which is why the sets are equivalent. The relations (8-10) are more general since they can be used when the representation space is a space over F_{p^k} where k is arbitrary.

We use the basis in which the operators (h_j, K_j) ($j = 1, 2$) are diagonal. Here K_j is the Casimir operator (2) for algebra (a'_j, a_j'', h_j) . For constructing IRs we need operators relating different representations of the $\text{sp}(2) \times \text{sp}(2)$ algebra. By analogy with Refs. [8, 9], one of the possible choices is

$$\begin{aligned} A^{++} &= b''(h_1 - 1)(h_2 - 1) - a_1'' L_-(h_2 - 1) - a_2'' L_+(h_1 - 1) + a_1'' a_2'' b', \\ A^{+-} &= L_+(h_1 - 1) - a_1'' b', \quad A^{-+} = L_-(h_2 - 1) - a_2'' b', \quad A^{--} = b' \end{aligned} \quad (12)$$

We consider the action of these operators only on the space of "minimal" $\text{sp}(2) \times \text{sp}(2)$ vectors, i.e. such vectors x that $a'_j x = 0$ for $j = 1, 2$, and x is the eigenvector of the operators h_j . If x is a minimal vector such that $h_j x = \alpha_j x$ then $A^{++} x$ is the minimal

eigenvector of the operators h_j with the eigenvalues α_j+1 , $A^{+}x$ - with the eigenvalues $(\alpha_1 + 1, \alpha_2 - 1)$, $A^{-}x$ - with the eigenvalues $(\alpha_1 - 1, \alpha_2 + 1)$, and $A^{-}x$ - with the eigenvalues $\alpha_j - 1$.

By analogy with Refs. [8, 9], we require the existence of the vector e_0 satisfying the conditions

$$a'_j e_0 = b' e_0 = L_+ e_0 = 0, \quad h_j e_0 = q_j e_0 \quad (j = 1, 2) \quad (13)$$

where $q_j \in F_p$, $f(q_j) > 0$ and $f(q_1 - q_2) \geq 0$. It is well known (see e.g. Refs. [8, 5]) that $M^{04} = h_1 + h_2$ is the AdS analog of the energy operator. As follows from Eqs. (8) and (10), the operators (a'_1, a'_2, b') reduce the AdS energy by two units. Thus e_0 is an analog of the state with the minimum energy which can be called the rest state, and the spin in our units is equal to the eigenvalue of the operator $L_3 = h_1 - h_2$ in that state. For these reasons we use s to denote $q_1 - q_2$ and m to denote $q_1 + q_2$. In the standard classification [8], the massive case is characterized by the condition $q_2 > 1$ and the massless case — by the condition $q_2 = 1$. There also exist two exceptional IRs discovered by Dirac [10] (Dirac singletons). As shown in Refs. [2, 5], the modular analog of Dirac singletons is simple and the massless case has been discussed in detail in Ref. [11]. For these reasons in the present paper we consider only the massive case.

As follows from the above remarks, the elements

$$e_{nk} = (A^{++})^n (A^{-+})^k e_0 \quad (14)$$

represent the minimal $\mathfrak{sp}(2) \times \mathfrak{sp}(2)$ vectors with the eigenvalues of the operators h_1 and h_2 equal to $Q_1(n, k) = q_1 + n - k$ and $Q_2(n, k) = q_2 + n + k$, respectively. It can be shown by a direct calculation that

$$A^{-} A^{++} e_{nk} = (n + 1)(m + n - 2)(q_1 + n)(q_2 + n - 1) e_{nk} \quad (15)$$

$$A^{+-} A^{-+} e_{nk} = (k + 1)(s - k)(q_1 - k - 2)(q_2 + k - 1) e_{nk} \quad (16)$$

As follows from these expressions, in the massive case k can assume only the values $0, 1, \dots, s$ and in standard theory $n = 0, 1, \dots, \infty$. However, in the modular case the following results are valid.

Theorem 2: *The full basis of the representation space can be chosen in the form*

$$e(n_1 n_2 n k) = (a_1^{-})^{n_1} (a_2^{-})^{n_2} e_{nk} \quad (17)$$

*The value of n is in the range $n = 0, 1, \dots, n_{max}$ where n_{max} is the first number for which the r.h.s. of Eq. (15) becomes zero in F_p , i.e. $n_{max} = p + 2 - m$. As follows from **Theorem 1**, Eq. (8) and the properties of the A operators,*

$$\begin{aligned} n_1 &= 0, 1, \dots, N_1(n, k), & n_2 &= 0, 1, \dots, N_2(n, k), \\ N_1(n, k) &= p - q_1 - n + k, & N_2(n, k) &= p - q_2 - n - k \end{aligned} \quad (18)$$

As a consequence, the representation is finite dimensional in agreement with the Zassenhaus theorem [6]. Moreover, it is finite since any Galois field is finite.

In standard Poincare and AdS theories there also exist IRs with negative energies. They can be constructed by analogy with positive energy IRs. Instead of Eq. (13) one can require the existence of the vector e'_0 such that

$$a_j'' e'_0 = b'' e'_0 = L_- e'_0 = 0, \quad h_j e'_0 = -q_j e'_0 \quad (j = 1, 2) \quad (19)$$

where the quantities q_1, q_2 are the same as for positive energy IRs. It is obvious that positive and negative energy IRs are fully independent since the spectrum of the operator M^{04} for such IRs is positive and negative, respectively. However, the following theorem indicates to a crucial difference between standard theory and GFQT.

Theorem 3: *The modular analog of the positive energy IR characterized by q_1, q_2 in Eq. (13), and the modular analog of the negative energy IR characterized by the same values of q_1, q_2 in Eq. (19) represent the same modular IR.*

Proof. Let e_0 be a vector satisfying Eq. (13). Denote $N_1 = p - q_1$ and $N_2 = p - q_2$. Our goal is to prove that the vector $x = (a_1'')^{N_1} (a_2'')^{N_2} e_0$ satisfies the conditions (19), i.e. x can be identified with e'_0 .

As follows from the definition of N_1, N_2 , the vector x is the eigenvector of the operators h_1 and h_2 with the eigenvalues $-q_1$ and $-q_2$, respectively, and in addition it satisfies the conditions $a_1'' x = a_2'' x = 0$. Let us prove that $b'' x = 0$. Since b'' commutes with the a_j'' , we can write $b'' x$ in the form

$$b'' x = (a_1'')^{N_1} (a_2'')^{N_2} b'' e_0 \quad (20)$$

As follows from Eqs. (10) and (13), $a_2' b'' e_0 = L_+ e_0 = 0$ and $b'' e_0$ is the eigenvector of the operator h_2 with the eigenvalue $q_2 + 1$. Thus, $b'' e_0$ is the minimal vector of the $\text{sp}(2)$ IR which has the dimension $p - q_2 = N_2$. Therefore $(a_2'')^{N_2} b'' e_0 = 0$ and $b'' x = 0$.

The next step is to show that $L_- x = 0$. As follows from Eq. (10) and the definition of x ,

$$L_- x = (a_1'')^{N_1} (a_2'')^{N_2} L_- e_0 - N_1 (a_1'')^{N_1-1} (a_2'')^{N_2} b'' e_0 \quad (21)$$

We have already shown that $(a_2'')^{N_2} b'' e_0 = 0$, and hence it suffices to prove that the first term in the r.h.s. of Eq. (21) equals zero. As follows from Eqs. (10) and (13), $a_2' L_- e_0 = b' e_0 = 0$ and $L_- e_0$ is the eigenvector of the operator h_2 with the eigenvalue $q_2 + 1$. Thus, $(a_2'')^{N_2} L_- e_0 = 0$ and the proof is completed.

4 Discussion

The construction in Sec. 3 applies to both, standard IRs of the $\text{so}(2,3)$ algebra and their modular analogs. Consider first standard IRs. Here the element e_0 defined by

Eq. (13) is the state with the minimum energy, i.e. we start from the rest state where, by definition, energy=mass and the value of the energy is positive. When the representation operators act on e_0 one obtains states with higher and higher energies and the energy spectrum is in the range $[mass, \infty]$. Analogously the element e'_0 defined by Eq. (19) is such that the energy in this state is such that energy=-mass while the energy spectrum is in the range $[-\infty, -mass]$. As noted in Sec. 1, in standard theory positive and negative energy IRs are called particles and antiparticles, respectively. Here a particle and its antiparticle are different objects because they are described by fully independent IRs. Then, as noted in Sec. 1, a problem arises why a particle and its antiparticle have equal masses.

Let us now discuss what happens in GFQT. We again start from the state e_0 and one might think that the corresponding IR is the modular analog of the standard positive energy IR with the minimum weight. Indeed, when the operators A^{++} act on e_0 we successively obtain states where the energy increases by two units. However, since the values of the energy now belong not to Z but to F_p then sooner or later we will arrive to states where the energy is "negative" (i.e. in the range $[-(p-1)/2, -1]$) and finally we will arrive to the state where energy=-mass. In mathematical terminology this means that a modular analog of IR with the minimum weight is simultaneously a modular analog of IR with the maximum weight, while from the point of view of physics, one modular IR describes a particle and its antiparticle simultaneously.

As noted in Sec. 1, in QFT a question arises that if locality is only approximate then it is not clear whether the notion of antiparticles is exact or approximate and whether they have equal masses. At the same time, the above construction shows that *in GFQT the existence of antiparticles follows from the fact that any Galois field is finite.*

Consider a simple well-known model of particle theory when electromagnetic and weak interactions are absent. Then the fact that the proton and the neutron have the same masses and spins is irrelevant of locality or nonlocality; it is only a consequence of the fact that the proton and the neutron belong to the same isotopic multiplet. In other words, they are simply different states of the same object - the nucleon. We see that in GFQT the situation is analogous. The fact that a particle and its antiparticle have the same masses and spins is irrelevant of locality or nonlocality and is simply a consequence of the fact that they are different states of the same object since they belong to the same IR.

Note also, that in standard theory, IRs of the dS algebra contain states with both, positive and negative energies and, as shown in Ref. [12], the only possible interpretation of such IRs is that they describe a particle and its antiparticle simultaneously.

In summary, while in standard theory the existence of antiparticles depends on additional assumptions, in GFQT it is inevitable. Therefore, *the very existence of antiparticles is a strong indication that nature is described by a finite field*

rather than by complex numbers.

Strictly speaking, the above construction shows that the very notion of particles and antiparticles is approximate. A set of states where the energy E is such that $f(E) > 0$ and $f(E) \ll p$ can be called a particle while a set of states where $f(E) < 0$ and $|f(E)| \ll p$ can be called an antiparticle. This situation has far reaching consequences. A problem also arises how to treat neutral particles where a particle and its antiparticle are the same. Those problems are discussed in Refs. [11, 12, 4, 5].

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