Education, division & derivative: Putting a Sky above a Field or a Meadow

Comments on the field, meadow, dynamic quotient and derivative, as seen from research in mathematics education (elementary, highschool & matricola)

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Abstract

The sky is a tentative suggestion for extension group theory, that has supervariables with flexible domains, that would allow the formulation of dynamic rational functions, such that division by zero is prevented by manipulation of the domain. This would allow an algebraic approach to the derivative. Meadows are no alternative to such an approach. The world needs Academic Schools in which teaching is merged with empirical research on didactics.

Summary

(1) A sky is a tentative suggestion for group theory to use a field or meadow as foundation, and then include supervariables and expressions, such that a supervariable has a flexible domain. These variables are symbols and not just placeholders for numbers. The sky has functions with domains and ranges that use such supervariables. The dynamic quotient is defined with supervariables. Its outcome and range require both the algebra of expressions and the manipulation of the domain for the denominator. The dynamic quotient allows an algebraic definition of the derivative. For school mathematics, this allows an algebraic middle ground between Mathematical Analysis (with limits) and Calculus (mainly technique). Though the dynamic quotient is embedded in a sky in the algebra of groups in research mathematics, the algebra that is required in practice in school mathematics is the common algebra of expressions, though with the manipulation of domains.

(2) It has been reported in the literature that advantages of a meadow over a field are: (i) it has a function name for the inverse rather than the mere statement of existence, (ii) there is less testing on zero values, with faster computer calculation, (iii) there is an outcome 1 / 0 = $\hat{a}$ ("additional", e.g. Undefined or Indeterminate). However, the dynamic quotient still has its own exception switch. Since it is not useful to have all derivatives equal to $\hat{a}$, this outcome of division in a meadow does not present an alternative to the standard derivative or the algebraic approach via the dynamic quotient.

(3) The analysis supports the alert in the AMS Notices by H. Wu (2011:372) of "(...) the mathematics community to the urgent need of active participation in the education enterprise." For improvement in education in elementary school, highschool and matricola, and its research, we would rather see more attention for empirics and computer algebra than that sky. Like Academic Hospitals have research for cure and not dissection, the world needs Academic Schools in which teaching is merged with empirical research on didactics.

Keywords

division, quotient, dynamic quotient, simplify, algebraic approach to the derivative, calculus, supervariables, field, meadow, division by zero, abstract data types, computer algebra, rational functions, Pierre van Hiele, levels of abstraction, didactics of mathematics, mathematics education research, research mathematics, academic school

Contents

1. Introduction

1a. A viewpoint from the research in mathematics education

Communication between research mathematics (RM) and school mathematics and its research (SM) seems wise. Wu (2011:372) alerts "(...) the mathematics community to the urgent need of active participation in the education enterprise." This paper intends to make the SM - RM communication gap a bit smaller on *division* and the *derivative*, by replacing limits by *dynamic* division. We consider the education in mathematics in elementary school (division) and highschool and matricola (derivative), and the research on that education. The matricola is the first year of college or university, with an emphasis on non-math-majors.

This paper must be seen in the context of my books *Elegance with Substance* (EWS 2009) and *Conquest of the Plane* (COTP 2011b) (PDFs on the website). These books agree with Wu that there is a huge problem in education of mathematics indeed. They propose a remedy that involves a re-engineering of what is taught and a reconstruction of the mathematics industry. Improvement in education in mathematics requires our rational faculties but also the much neglected empirics of didactics. The world needs Academic Schools in which teaching is merged with empirical research on didactics, like Academic Hospitals have research for cure and not dissection of patients. Empirics are crucial. The didactic suggestions in EWS and COTP themselves are only indicative, since it is empirical research that must show what students understand best. This holds also for the discussion in this paper.

Communication between SM and RM is difficult because they use different languages, even though they seem to use both English and mathematics itself. Some readers may not be aware of the work by Van Hiele (1973, 1986) though he is by far the most important theorist and empirical analyst on the didactics of mathematics of the last century, see Colignatus (2014e). The Van Hiele theory of levels of abstraction and understanding holds that the same words can mean different things depending upon the level of abstraction. Teachers may be aware of RM-subtleties but still must communicate with students in their language. This also warrants the distinction between SM and RM. It may well be that RM has difficulty to return to the former state of understanding. There is also the Gray & Tall (1994) "procept" notion. SM may opt for deliberate ambiguity whereas RM might regard it as the objective to remove it. For example $\sqrt{2}$ may be an instruction to grab the calculator or may be used as a symbol for a number. There is a risk that RM is too abstract so that SM should rather look for help in engineering that is more sensitive to empirics. See Colignatus (2011c) for a short umbrella overview on SM that also responds to Wu (2011).

1b. Notation H = -1 (eta)

We will use -1 as an operator, inspired by Harremoës (2000), using the symbol H (eta) (half turn, i^2). Thus $H = -1$, on the main formula line as the negation operator ($a + H\, b = a - b$), and in superscript as the indicator for an inverse, as $x^H = 1 / x$. For SM we have exponentiation, but for the group theory of skies, fields, commutative rings and meadows the superscript position is not necessarily intended as exponentiation but may merely express the inverse.

1c. Division and the derivative

For real-valued functions f the limit $\Delta x \rightarrow 0$ on the quotient $\Delta f / \Delta x$ gives the derivative. My book *A Logic of Exceptions* (ALOE, 2007:236-7, 2011a:241) on logic, for matricola, looked at the paradoxes of division by zero too, defined the dynamic quotient $y // x$, and as corollary developed an algebraic approach to the derivative with direct substitution of $\Delta x = 0$. The dynamic manipulation of the domain under consideration prevents the actual division by 0. By this algebra we mean the original algebra of expressions, and not the algebra of groups or the derivation algebra. This quotient allows an algebraic middle ground between Mathematical Analysis (with limits) and Calculus (mainly technique). Colignatus (2011b) elaborates for SM.

There are two issues for this paper:

(1) Fitting with the Van Hiele theory of levels, the dynamic quotient is defined in terms that highschool students (SM) should be able to understand. How will RM manage ?

(2) Could there be alternative ways to divide by zero without manipulation of the domain ? Bergstra & Tucker (2007) presented the notion of a Meadow for the algebra of groups. In a Commutative Ring (CR) there is no multiplicative inverse $x^H = 1 / x$. In a Field (such as the real numbers) there is a multiplicative inverse $1 / x$ except for $x = 0$. In a "common" meadow an outcome would be $1 / 0 = â$, for some $â$ ("additional"). Example values are Undefined or Indeterminate. A meadow that has $â = 0$ is called "involutive". While $0^H = 0$ is counter-intuitive, a more natural choice is $â$ = Indeterminate. Could such a meadow be used for the alternative to the derivative ? There is much to say for a generalised inverse, like in matrix algebra, and thus we wonder whether the meadow might be useful for the dynamic quotient.

The reported advantages of a meadow over a field are: (i) it has a function name for the inverse rather than the mere statement of existence, (ii) there is less testing on zero values, with faster computer calculation, (iii) there is an outcome $0^H = 1 / 0 = â$ ("additional").

On close consideration, the idea of using a meadow as an alternative for the dynamic quotient bears no fruit. Direct substitution of $\Delta x = 0$ would generate derivatives that are $â$ everywhere. We thus reject the usefulness of a meadow as an alternative for the dynamic quotient and the algebraic approach to the derivative. The $0^H = 1 / 0 = â$ property appears to involve error handling and the Bergstra & Tucker (2007) paper intends to lift this from Computer Science to group theory. For some instances of computing, it appears useful to skip the testing on 0 and return a result rather than leaving the issue undefined. Thus, overall, there is no useful link to the notion of the derivative in Mathematical Analysis. This answers our main question on the meadow. A short discussion about the meadow might still be helpful to understand the situation. Part of the discussion remains in the body of the paper because of division and the derivative. Most is put in the **Appendices, A** for meadow[0] and **B** for meadow[$â$]. Overall it will be clear that 0^H is semantically not really division by zero. It is advisable to use quotes like "0^H" to indicate that the issue is one of error handling and rewriting that error.

Wolfram (1988, 1996), in presenting *Mathematica,* takes a Computer Algebra approach with pattern recognition that differs from the approaches studied below. *Mathematica* uses Indeterminate. Open source packages are Sage and R. It seems that computer algebra is more robust anyway, see also some other aspects that we meet below.

1d. Algebra of expressions versus algebra of groups

The dynamic quotient uses the algebra of expressions with variables that have flexible domains. Does this concept fit with the standard approaches in the algebra of groups ?

In the following we will take the real numbers as a Field and then put a Sky above it. This sky will also use (super-) variables that have a variable domain in that field, and the dynamic quotient that uses such variables. This will allow an algebraic definition of the derivative.

Is there some interaction w.r.t. the choice between field or meadow ? It turns out that the issues are independent. The sky and the dynamic quotient can be defined for either field or meadow. In some ways it might be just as clear to leave 0^H or $1 / 0$ as undefined, since the dynamic quotient has its own exception switch at zero. But one might prefer a meadow for its properties. Below we will define the sky more traditionally for the field only.

A key point is that the algebraic group theory tends to focus on groups with numerical values only, while the dynamic quotient uses algebraic expressions (with symbols along numbers). In *Mathematica* there is a symbolic Simplify[y / x] that may inspire a more symbolic view.

1e. Syntax and semantics of division

We regard as the *meaning* of division that $x\, x^H = 1$. Three per three is $3^H + 3^H + 3^H = 3\ 3^H = 1$.

Van Hiele (1973) (and possibly in English in Van Hiele (1986) but I have no access to it) suggested x^H as the format to use for elementary school, and to abolish much of the tedium of fractions. Thus *div*[*y, x*] = *y* / *x,* currently a ratio of two quantities, becomes to be seen as the product of a quantity with the inverse of another quantity: $y\ x^H$. Thus a quotient is the numerator-multiple of the inverted-denominator. Compare this to Wu (2011:373) on the US experience on fractions. Also, for the first year of junior highschool Van Hiele wondered whether a small Peano type of introduction to formal arithmetic might be possible. These issues in education are different from the formal approaches considered here, but it is useful to indeed look at some requirements from theory.

It turns out that the pronunciation of fractions is problematic. The rank order names "first, second, third, fourth, fifth, ..." are abused to call 3^H or 1 / 3 "one-third". Colignatus (2014abc) – thanking Harremoës for a discussion on this – suggests calling 3^H "per three" so that 2 3^H becomes "two per three". Thus 21 3^H becomes "two-ten-one per three". (And 7 3 3^H = 7.) Also: a mixed number 2 + 2^H (two *and* a half) should not be written as 2½ (two *times* a half).

A key point is that division as an operator is often just a rewriting. While 2^H is a half, putting that superscript there doesn't amount to much effort in "calculation". Similary $(n\ m^H)^H = n^H\ m$ is just algebra and not really "calculation". A fraction 125 / 500 is rather un-informative but the information surfaces by finding 25%, which is the true "calculation" that we intend that word for. It is known as Long Division. In this case first multiply both terms by 2.

There is some danger in education that one regards x^H as the operator and 1 / *x* as the result. In that case "calculation" becomes the rewriting from superscript *H* to the expression with the division-slash or division-line. For example, not 2^H but only ½ would be regarded as the number on the number line. This would be very unfortunate. It leads to all kinds of exercises for poor children to determine the relation between ½ and ¾, such as locating them on the number line. Instead, proper didactics would be that 2^H and 3 4^H are simultaneously algebraic expressions and numbers, can be compared with a common factor, 1 or 100%, and thus can be found on the number line as 0.25 and 0.75. For this comparison Long Division is the key operation. For unknown reasons, education in elementary school has not yet taken full advantage of the invention by Simon Stevin (pronouce STAY-vin) in 1585 AD of the decimal point notation. See Colignatus (2011) for a discussion on exact and approximate values in decimal point notation. See Colignatus (2009:21) (2011b:207-210) on proportion space {*x, y*} and the notion of *number* by looking at both the axes and the *x* = 1 line. For example ½ / 1 gives the slope of the ray through {1, 2^H}.

Thus, there is much to say for Van Hiele's suggestion of using the x^H format.

1f. The structure of the paper

We arrived at this subject via the education on division and the derivative. For SM, clarity on the dynamic quotient has been given in ALOE, EWS and COTP. Sections 2d, 3b and 4 below explain this SM definition to RM. The Sky in Section 7 is intended to lift the definition to group theory for RM. Interlaced are connecting sections. We close with some remaining comments. Some more discussions on the Meadow are in the **Appendices**. Since the division exception switches drive this paper, we start with them.

2. Division exception switches

2a. The basic conundrum when Bhaskara invented the zero

If $(0\ .\ 0^H) = 1$ then $(0 = 0) \Rightarrow (0\ .\ 2) = (0\ .\ 5) \Rightarrow (2 = 5) \Rightarrow$ *falsum.* Thus we might form expression "0^H " but it requires exceptional treatment. Could we set $0^H = 0$? This does not solve the exception, as is shown by the "fundamental conundrum theorem" (FCT). Henceforth we will assume the separation axiom (SEP) that 0 ≠ 1.

$$(\text{SEP} \ \&\ (0^H = 0)) \ \Rightarrow\ (x\ x^H = 1 \Rightarrow x \neq 0) \qquad \text{(FCT)}$$

Proof: If $x\ x^H = 1$ also for $x = 0$ then $0 = (0\ .\ 0) = (0\ .\ 0^H) = 1$, which is forbidden by SEP. QED.

A choice of $0^H = 0$ doesn't make 0^H a proper inverse - see the meaning of division. It is error handling, or a neutralisation of an error. Dealing with more involved issues of 'division by zero' requires other methods, such as traditionally limits or alternatively the dynamic quotient.

2b. Conventional response to the conundrum

There is a subtlety of either leaving something undefined or assigning a value "undefined". There is a subtlety of either merely stating the existence of some inverse value or providing a function to calculate it. A bit less subtle is the distinction between finite and infinite fields.

Here we are interested in the real numbers IR, but may look at meadows for the rational numbers $\mathbb{Q}$. Thus we look at the ordered field, with an infinite number of elements.This means that group theory properties strictly for finite cases are not relevant for us.

The division exception switch for a Field, or *General Inverse Law* (GIL), leaves the expression 1/0 undefined. A field has $x / x = 1$ but protected by $x \neq 0$ (see the FCT).

$$If(x \neq 0)\ then\ (x \cdot x^H = 1) \qquad \text{GIL in a field, leaving 1/0 undefined}$$

Let us substitute 2^H. Thus we must test whether ($2^H \neq 0$). There is no way yet how we can decide this. We simply cannot do the test. Would we be allowed to say that the expressions are not the same, whence the inequality does not hold ? Are we allowed to say that clearly a half is not zero ? No. A roundabout is necessary. We first find that $2\ 2^H = 1$. We also derive $(2^H = 0) \Rightarrow (2\ 2^H = 0)$. Since $2\ 2^H \neq 0$ we conclude that $2^H \neq 0$. In general:

$$(x \neq 0) \Rightarrow (x^H \neq 0) \qquad \text{NZI, nonzero inverses}$$

Finally substituting 2^H we derive that $2^H\ (2^H)^H = 1$. Now multiply by 2 and find $2\ 2^H\ (2^H)^H = 2$ which gives $(2^H)^H = 2$. This is the proper approach. Thus the "value" of 2^H is a difficult concept here, and formally we look at the expression 2^H.

The assumptions for a Commutative Ring (CR) are in **Appendix A.** There we will also define the *Strong* Inverse Properties (SIP). SIP3 would be in principle sufficient to derive all SIP.

$$(x^H)^H = x \qquad \text{SIP3}$$

A field can be generated from SEP & CR & GIL. It allows the derivation of "protected properties" such as $(x \neq 0) \Rightarrow ((x^H)^H = x)$. Let us call this "SIP3". Namely:

$$(x \neq 0) \Rightarrow (z = x^H) \Rightarrow (z \neq 0) \Rightarrow \{\ (x^H)^H = ((x^H)^H\ x^H\ x) = (z^H\ z\ x) = x\ \} \quad \text{"SIP3"}$$

2c. Supplementing the if-statement with a meadow

One might read GIL as a *prohibition* of trying 1 / 0. But we "have" both 0 and ^{H} and thus might create the expression 0^H. When we choose $0^H = 0$, the field is not extended with a new element. While a field would not have functions but essentially only states existence, then there is still no change in the assertion of the existence of 0. One can make a strong case that the notion of a field does not change if the following is adopted:

$$If(x \neq 0)\ then\ (x \cdot x^H = 1)\ else(x^H = 0) \qquad \text{GIL[0]}$$

If $0^H = 0$ then $(0^H)^H = 0$ by mere substitution. Then $(x^H)^H = x$ also holds for $x = 0$. Thus SEP & CR & GIL[0] generate (unprotected) SIP3. Conversely if we want $((0^H)^H = 0)$ then we must take $(0^H = 0)$, see below on the properties of 0.

A main idea is that logical tests can be turned into equational or algebraic forms that might cause faster calculation. The following algebraic expression without an explicit protection clause will be called the *Restricted* Inverse Law (RIL). Though GIL is called "general" it is actually RIL that is the "generalised" version. If $0\ 0^H = 0$ then case 0 is trivial and GIL ⇒ RIL.

$$x * x / x = x \quad \text{or} \quad x * (x / x - 1) = 0 \qquad \text{RIL (Von Neumann)}$$

Bergstra & Tucker (2007:22): "A model of CR & SIP & Ril [RIL] is called a *meadow*." [italics added] In the distinction between field and meadow a key question is whether *either* GIL prohibits GIL[0], so that they are alongside, *or* that one is a subcase, *or* that both are the same (i.e. in terms of group algebra and not of datatype with explicit functions).

2d. Dynamic quotient and the derivative

The *dynamic* quotient simplifies the discussion of ratio's without sacrificing exactness. The dynamic quotient assumes the normal *"static"* quotient $div[N, D] = N\ D^H = N / D$ as used in SM, with numerator N and denominator D. The explanation for RM requires a recall of school mathematics. Real numbers are constants, and can be represented symbolicly too. A function has a domain and range. A quotient can be seen as a function of two arguments N and D, that are chosen as either constants with single values or variables with domains. In proportion space $\{D, N\}$ the eye movement is easier when associating D with a horizontal axis and N with a vertical one. See section 3b.3 for variable, function, domain and range in SM.

ALOE (above) defines the "unless" exception switch and the dynamic quotient $N\ D^\# = N // D$ by the following process or program, using brackets to show the program steps, and using "a variable" (see Sections 3b.5 & 4d below for "(a) variable", and see also Section 7):

> $N\ D^\# \equiv$ { $N\ D^H$, unless D is a variable and then: assume $D \neq 0$, simplify the expression $N\ D^H$, declare the result valid also for the domain extension $D = 0$ }

$N // D$ reads in words as: {The dynamic quotient is equal to the normal static quotient, unless the denominator is a variable: in that case, assume that the denominator will not be zero (restrict the domain associated with it), simplify the ratio of numerator to denominator, and declare the result valid also for the domain extension that the denominator would be zero.} Thus alongside variables that run over a domain there are also (super-) variables that associate with a manipulation of the domain.

A quick test of understanding is $(v - v) // (v - v)$: using $z = v - v$ thus $z // z$. In general $x // x = 1$ but only for variables x. It turns out that $z = 0$, so that the denominator has been chosen a constant and is no variable, so that $z // z = 0 // 0 = 0 / 0$ (undefined or Indeterminate).

E.g. for $v \in$ IR: $(v - 1) // ((v - 1)(v + 6)) = 1 / (v + 6)$. The division itself is a function with domain and range. In the domain of the function we find D running over IR. D has been chosen as a variable and is made a function of v: $D = D[v] = ((v - 1)(v + 6))$. Assuming $D \in$ IR \ {0} gives an underlying domain for $v \in$ IR \ {-6, 1}. After simplification the assumption is dropped, the domain associated with D is restored to IR, and this comes with $v \in$ IR as well. Starting out with $v \in$ IR, the final result is ruled by standard division: $1 / (v + 6)$ is undefined for $v = -6$.

An equation like $x^2 - x = x$ would still require the switch that either $x = 0$ or not, with division only in the second case. The dynamic quotient comes in use when setting up a ratio.

Since the notion will tend to be used when domains must be manipulated, the definition focuses on variables. Students better develop such a focus on variables and their domains too. The test on being a variable should not be complex. Thus syntactic "*D* is a variable" is the proper reminder. This syntactic test is formulated in descriptive language, with English "is", rather than with formulas. While learning to focus on variability, students must on occasion still be reminded of cases of constancy with which they would already be familiar. A variable is not a constant. Their understanding of what a constant is will grow. First consider this semantic rule, in two alternative formulations:

If v would be a "variable" with the same value everywhere, then v is a constant; or:
For any constant c: if $v = c$ then v is a constant

We rather not use this rule. The counterfactual blurs the distinctions, as if a constant without a domain still should be tested on some domain. The notion “the same value everywhere” would cause students to think that they have to check all values from -∞ to +∞. We rather use a direct test on syntax. A SM syntactic rule for D on constancy is the following, has been used above in handling $v - v = 0$, and is formally a base for testing on being a variable too:

If D is some constant then D has been chosen to be a constant

For the derivative (in SM) the limit with ε and δ appears to be superfluous. The essence lies in the manipulation of the domain: first excluding 0, then including 0, then restricting it to 0:

$$f'[x] = \mathbf{d}f \,/\, \mathbf{d}x \equiv \{\Delta f \,//\, \Delta x, \text{ then choose } \Delta x = 0\}$$

The dynamic quotient relies essentially on algebra understood as the manipulation of expressions. The notion of “numerical continuity” is replaced by the “continuity contained in the algebraic formula”. The information in the formula is used and not neglected. In terms of didactics, the improvement of the dynamic quotient lies in regarding division from the angle of statics versus dynamics, noun versus verb, or the Gray & Tall (1994) “procept” notion. In terms of mathematics there is the understanding that limits may sometimes be superfluous.

2e. Properties of 0

For CR: $0 \,.\, 0^H = 0$ since $0.x = 0$. See **Appendix B** (B3), also for critique. For us now quickly:

$$(0 = 0 - 0) \Rightarrow (0.x = (0.x - 0.x) = 0) \quad \text{(NUL) (critique } 0 \,.\, 0^H \text{ in B3)}$$

With RIL: $(x = 0) \Leftrightarrow (x \,.\, x^H = 0)$. Namely multiply the LHS by x^H and the RHS by x.

Above we already had: If $(0^H = 0)$ then $((0^H)^H = 0)$ by mere substitution.
With RIL conversely: if $((0^H)^H = 0)$ then $0 = 0^H \,.\, (0^H \,.\, 0)$ (by repeat NUL) $= 0^H \,.\, 0^H \,.\, (0^H)^H = 0^H$.
Thus RIL gives: $(0^H = 0) \Leftrightarrow ((0^H)^H = 0)$. Thus also: SIP3 $\Rightarrow (0^H = 0)$.

For meadow[0] we find:

(i) A field can be generated from SEP & CR & GIL, and it will generate protected properties as $(x \neq 0) \Rightarrow ((x^H)^H = x)$ or "SIP3".

(ii) If we want $(0^H)^H = 0$ then that nonzero clause in GIL must be dropped.

(iii) We had SIP3 $\Rightarrow (0^H = 0)$. Thus (SEP & CR & GIL & SIP3) $\Rightarrow$ (SEP & CR & SIP & RIL), where the latter is the meadow[0].

(iv) We had $(0^H = 0) \Rightarrow ((0^H)^H = 0)$ by mere substitution. Thus (SEP & CR & GIL[0]) $\Rightarrow$ (SEP & CR & SIP & RIL), where the latter is the meadow[0].

2f. Cancellation rules

Relevant is also the zero product rule:

$$a\, b = 0 \text{ iff } a = 0 \text{ or } b = 0 \qquad \text{ZPR}$$

In a field it is easy to prove GIL ⇒ ZPR: Assuming $a \neq 0$ then $a^H\, a\, b = b = 0$, and so on. In a meadow we cannot assume $a^H\, a = 1$. However RIL & ZPR ⇒ GIL. Merely taking the roots in RIL gives $x = 0$ or $x \,/\, x = 1$. This is equivalent to: if $x \neq 0$ then $x \,/\, x = 1$.

A meadow might in general not satisfy the zero product rule (ZPR) that we would wish to have. But we will use the infinite field that satisfies that ZPR. Conceivably that field might be replaced by a meadow merely for its reported advantages.

Bergstra & Ponse (2014a:p3&8) discuss cancellation rules and hold that GIL is equivalent to both ZPR and CL:

$$((x \neq 0)\ \&\ (x.y = x.z)) \Rightarrow (y = z) \qquad \text{CL}$$

For the latter: if $x\ 1 = x\ z$ then $z = 1$, and then substitute $z = x\ /\ x$. Thus RIL & CL ⇒ GIL[0], Bergstra, Bethke & Ponse (2013) speak about a "cancellation meadow". For all practical purposes such a meadow still produces the real numbers as an ordered field, but the formulation as a meadow would have its reported advantages (i) – (iii).

2g. For rationals and reals: the field and meadow "overlap"

Bergstra & Tucker (2007:22): "All fields are clearly meadows but not conversely (as the initial meadow is not a field)." For us, with a vested interest in the reals, it is however relevant that the choice between GIL and RIL appears rather immaterial, except perhaps in (computer) calculation. In **Appendix A** there is a longer proof for the ordered field of the rationals. This proof is is superfluous for our current understanding, but it can be used in education research:

(ℚ, <): (SEP & CR & SIP) ⇒ (RIL ⇔ GIL)

Bergstra (communication 2014-09-10) responded with a short proof for all ordered cases:

> (GIL ⇒ RIL): Obvious. (Not GIL ⇒ not RIL): Directly: (($x \neq 0$) & ($x/x \neq 1$)) ⇒(($x > 0$ or $x < 0$) & (($x/x < 1$) or ($x/x > 1$)) ⇒ (($x.\ x/x > x$) or ($x.\ x/x < x$)) ⇒ ($x.\ x/x \neq x$).

Thus the choice for a meadow would reside in its reported advantages over a field. For us, those advantages are no key issue. A value 1 / 0 = *â* and the calculation speed on testing on zero are no key issues here. The sky uses an explicit inverse function and exception switch. It can be useful that GIL makes the switch between $x\ /\ x = 1$ and $x = 0$ explicit. The RIL has the disadvantage or being implicit. A conceptual problem is that if one would accept ZPR and CL with the logical test on nonzero x then it would be inconsistent to reject GIL because of having it. This supports the decision to put the discussion on meadows in the **Appendices**.

3. School mathematics (SM) doesn't have a formal development.

3a. Distinction between SM and RM

In addition to the earlier distinctions, school mathematics doesn't have a formal development. This can make for complex communication between research mathematics (RM) and teaching and its research (SM). For example, RM may develop a formal system and introduce terms like 2 = 1 + 1, 3 = 2 + 1 etcetera, see Bergstra (2006) in Dutch or Bergstra (2014) in English. But children have had this in elementary school. There is no need in highschool to restate this, unless we teach such formal methods. Other such points are more subtle. See Wu (2010) (2011) and his website for more papers on the distinction between RM and SM.

As SM can be deficient, we might call it textbook SM (TSM), or we can speak about improving SM (ISM), which is what this article is about, since it is not the intention to formalise current TSM confusions. However, once this is clear, we can resort to just using the label SM.

3b. Syntax versus semantics, expression versus value

3b.1 Ambiguous "is": element, assignment, identity, equality, expression

The distinction between syntax and semantics gives the distinction between expression and value. In computer algebra these distinctions may be clear, for example in *Mathematica* "=" stands for assignment (value or actually anything), "==" for equation, and "===" for identity of expression. Thus "1 + 1 == 2" gives True and "1 + 1 === 2" gives False. Thus "x = 2" sets x to that value, turning it into a constant, so that "f[x] = 5" now means that f[2] is set to 5.

However, in SM, only "=" is used, thus with ambiguous meaning, and it is left to the student to understand what meaning applies. In SM it is custom that 1 + 1 = 2 and that the one may be replaced by the other. When John says "1 + 1" then, however, he hasn't said "2", whence it follows that there still is a (hidden) distinction between value and expression. A student is supposed to understand that (1 + 1) / 2 must be simplified to 1, or that 3 x + 5 = 4 can be solved, even though one might argue, seen from the viewpoint of value rather than expression, that the values involved don't change, so that nothing is accomplished by simplification or solution, and one might as well leave the expressions as they are. In SM we say that we calculate a value but we often (if not always) mean that we rewrite expressions into some standard form (with stronger associations with value).

For the following it might be possible to distinguish =, == and ===. However, readers from the world of SM might hold that this isn't done in SM whence the discussion would not be relevant for them. The topic of discussion would shift towards this issue. For that reason, this paper sticks to the definition of the dynamic quotient that uses ambiguous "is" and "equality".

Incidently, in *Mathematica* "{*a*1, ..., a*n*}" is a list, and "statement1; ... ; statement*n*" would be a program with executable statements. Each statement will have a result but this is not shown in output, as the printing of output is suppressed by the semi-colon. In "{statement1, ... , statement*n*}" the program steps are still evaluated consecutively but the intermediate results are available in that list, and they then can be used to check the program. The latter format is used in the definition of the dynamic quotient. Or one may hold that commas read better than semi-colons.

3b.2 What is an expression ?

There may be an ambiguity in the term "expression". A string such as "1 + 1" could contain an expression. But there are also formula's like 1 + 1 that some will want to call an expression. We will do no tests on strings and henceforth we can use "expression" also for formulas.

One might use the term "meta-variable" for a variable which domain consists of expressions. This term is not used in SM. In SM there are formulas and functions. Since a variable can be set to a function, like $D = f[z] = a\ z + b$, there is similar control over expressions.

In *Mathematica* there is the Hold attribute. If the argument in function f has been given the Hold attribute, then the call f[1 + 1] holds 1 + 1 as it is, and one might e.g. check whether 1 occurs in the expression. Otherwise the call will first evaluate 1 + 1 = 2 and continue with f[2].

Expressions are open or closed. If v is a variable then "$v – v$" or expression v - v is open w.r.t. having variables. Also $v - v = 0$ while 0 is closed w.r.t. having variables (assuming that v is not undefined so that we can find $v – v$ to be 0 indeed). A test whether an expression is 0 should not be confused with a test whether the expression contains variables. An expression without variables can still be zero, see 1 – 1. See Section 6a.

3b.3 Variable, function, domain and range

In some areas of RM it might not be so common to associate variables with domains too, so that the notion of a domain strictly applies to functions. In that case one could formulate a definition of N // D such that N & D are functions only. The following explains the SM situation.

Functions have domains and ranges. The range of one function can be the domain of another function. The notions are conditional upon the perspective. For $g[f[x]]$:

> f: $A \rightarrow B$ then A is the domain of f and B is the range of f.
> When also g: $B \rightarrow C$ then the range of f is the domain of g.

This formulation uses sets and avoids the use of variables. However SM doesn't have set theory. Set theory was introduced after the Sputnik satellite in 1957, badly implemented, and abolished again in the 1970s. For SM the variable becomes the bridge to understand functions. (For more on functions in COTP, see the Reading Notes, Colignatus (2011-2014).)

Variables are assigned domains too. Variables belong to the natural numbers, integers, rationals or reals, and for some students complex numbers. The domain for a variable can become more specific when it is used as an argument in a function, as in $y = f[x]$. Then x runs over A and y runs over B. Here we write "domain of" a function and "domain for" a variable.

For example, one may say in the classroom that the square root is only defined for nonnegative reals, which associates the domain with the function. When a variable is used as an argument, however, then for $\sqrt{x}$ one will specify that $x \geq 0$, and in this expression a variable has become the argument, and the focus is on the variable and not on the function. When discussing x and $f[x]$ then A will tend to be called the domain for x and B will be called the range for $f[x]$. To say that A is the domain of f is superfluous when it already has been said that there is a domain for x, its argument. In this way there is a custom to associate domains with variables – the arguments – and ranges with the functions (results) on those variables.

While it is said that A is the domain for x it is less common to say that B is the domain for y, since with $y = f[x]$ the attention is for the range for $f[x]$. However, when considering the composition of two functions, then it is consistent to say that B is not only the domain of g but that B is also the domain for its argument y.

For $div[N, D] = N / D$ with $N = f[z]$ and $D = g[z]$ we see the switch in perspective for numerator and denominator. For *div* and as a variable D has a domain, and as a function g has a range.

While "set" is ambiguous, "choose" is better. For $div[N, D] = N / D$ one might hold that numerator N and denominator D are metavariables that can be expressions like constants, (meta-) variables or functions. Since SM doesn't have this notion of "metavariables", the method adopted here is that N and D are *chosen* to be either constants or variables. When they are variables then they may be equal to functions, like $N = f[z]$ and $D = g[z]$. Though the denominator D can be (==) equal to a functional expression, it will syntactically still be a variable with a domain.

ALOE, EWS and COTP introduced the dynamic quotient using the form $y // x$. Their definition is formally identical to the above using $N // D$. A motivation to use N and D here is: (i) Some discussants substituted expressions like $x + 1$ in denominator x, but did no full substitution to $x + 1 \neq 0$ but kept $x \neq 0$. (ii) On occasion this substitution however was done in the syntactic test "x is a variable" where it is not allowed, see 3b.5. The use of D might better indicate that it is a syntactic test on the denominator. (iii) The use of upper case N and D might be somewhat suggestive of metavariables even though SM doesn't have that notion. (iv) The choice of $y // x$ was motivated by both the axes in proportion space $\{x, y\}$ and the differential quotient $\Delta f / \Delta x$. Some discussants used $x // y$ which doesn't honour that motivation and can cause confusion with now $y \neq 0$. In that case it is better to split the introductions of $N // D$, proportion space and differential quotient. It is still advisable to use $y // x$ and avoid $x // y$.

3b.4 What is a constant ?

In SM, the expression $f[z] = a\ z + b$ can use names a and b also when they are constants. When a is a constant then "a is a variable" is false. When denominator D is chosen to be a constant then "D is a variable" is false. There is no law against choosing also symbol x as a constant, though it might be a bit confusing since x is often used for variables. For example in *Mathematica*: NumberQ[x] gives False, set $x = 2$ then NumberQ[x] gives True.

When the denominator D is chosen to be constant c, then one cannot argue that the equation $D == c$ (*Mathematica*) shows that D still is a variable. The true form is $D = c$ (*Mathematica*) (or in Algol $D := c$). In that case $D == c$ collapses to True (and it is no longer an open expression).

Reconsider the example above that the denominator D is (has expression) $v - v$. Apply the syntactic test whether the denominator is a constant. We find $v - v = 0$ (see Section 6a), and 0 is a constant. Since the denominator is a constant it apparently has been chosen to be a constant. Thus it is not a variable. A way to see this is to consider the statements $D = 0$ (set) and $0 == v - v$ (equation). Thus the denominator is chosen as a constant, namely 0, that is equal to an expression that also has the value 0. The different roles must be kept apart.

In stronger formal development the testing on a particular numerical value would also become an operation of syntax, and this would be generalised to testing on constancy. In that case the semantic rule ("the same value everywhere") might be developed into a syntactic form. In an axiomatic system or computer program we can look how a symbol has been been declared. SM cannot rely on that and has to find another way. If the quotient were defined in a semantic manner then above semantic test on constancy could be used, but this would require a change in perspective (say "constant function"). The present approach fits the current SM conventions and develops a novel application (supervariables).

3b.5 "A variable" versus "variable"

We say (in *syntax*) that x is *a variable* and (in *semantics*) $x[z]$ is *variable*. But as $x = x[z]$, are you allowed to substitute it in "x is a variable", to create "$x[z]$ is a variable" ? Teachers in SM would reject the latter. Such substitution does not respect the syntactic form. If x is a variable & $x = x[z]$ then $x[z]$ is not a variable. For some students it may help to switch from syntactic "x is a variable" to semantics: "x is variable" that allows substitution. This however also shifts the focus to the range for $x[z]$. The manipulation of a range sounds less reasonable than to do this for a domain, even when they are supposed to be equal. The dynamic quotient thus asks for a domain that can be manipulated. Below the option of "(a) variable" is mentioned but this has not been developed further.

3b.6 In sum

RM has different objectives than SM, and will be inclined to handle these issues with rigor. But the dynamic quotient has to deal with SM. Definitions provided in ALOE, EWS and COTP, Colignatus (2007, 2011a) (2009)(2011b), are targetted at SM and must not be confused with RM. (The claims on some new insights for logic and the infinitesimal calculus do not prohibit a formulation in SM format.) Also, these books explicitly presume much of SM, like the theory of functions and ranges and variables and domains, and do not re-explain this, as now has done a bit in the above. If RM finds something unclear in these books then a first step is to refresh on SM. There are also the Reading Notes for COTP (2011-2014).

3c. Hurdles in understanding between SM and RM

The following problem arises: when one proposes an improvement in SM, and then asks RM whether the point is seen, RM may say "your proposal isn't formally developed", hence "we don't understand what you are doing". This can even happen at the level of a definition – rather than theorem or proof – since the context of the definition may be neglected. The consequence may well be that the improvement for SM is rejected, not because it would not be an improvement, but because RM doesn't understand that SM isn't formally developed anyway.

From the point of view of SM, RM can be *both* 'silly' in restating the obvious (1 + 1 = 2) *and* be too abstract for use in SM. Will help from RM be useful for resolving issues in education ? It might very well be hopeless. RM may also have a subculture way of stating things. On occasion RM might employ the phrase "I do not understand it, thus it is wrong". Properly though, a statement "it is wrong" would be backed up by an explanation why it would be wrong, and thus would imply some form of understanding. When RM do not understand what is happening in SM and reject to study it then this kind of subculture language might cause them to send the wrong messages to others.

RM might think that it is RM that determines what mathematics is and what should be taught. RM then forgets that SM has its own responsibility and competence in education: with its amalgam of mathematics, didactics, cognitive psychology, upbringing (such as avoiding stress from asking too much from a certain level of development), and not to forget: the empirical methods in didactics that RM tend to be rather abstract about.

The mentioning of cognitive psychology and empirics in SM might cause another instance of the SM – RM communication gap. RM might hold not to be interested in "psychology" and only be interested in what the definition of the dynamic quotient would mean, as it would not fit RM and thus “cannot be understood”. This response does not reckon with the Van Hiele levels of abstraction. RM concerns the highest level of abstraction and might not see that a lower level of abstraction still employs logic and mathematics. There are different languages here, ideally perhaps a language of a people of a newly discovered island in the Pacific, but in reality consisting of that part of the population that uses SM rather than RM. Since the dynamic quotient has been formulated for SM, the RM who want to understand it will have to learn the language of SM, thus at a lower level of abstraction. It may come as a surprise that the natives still do logic and mathematics. Learning a new language will cause understanding of what was incomprehensible before.

Some communication and help could still be useful. Perhaps a way to resolve the communication gap is to introduce a little bit of formality w.r.t. SM. This would turn out to be succesful if the RM would say “yes, I understand what you are doing”. They might even add “interesting”. The most important “yes, that would improve education” ought to be judged by SM though. Overall we shouldn’t expect too much. Academic Hospitals study patients to cure them, external science wants to dissect them. Education requires Academic Schools.

Obviously, it will not do to define the dynamic quotient both at the level of RM and present this for SM, since SM will hold that this would be much too abstract to use in school. One should also be very careful in looking for formulations in RM, for a wrong translation creates error and confusion (and indirectly perhaps eventually also for SM). Thus the suggestion below in Section 7 is only tentative. Overall Section 7 better be studied by RM that also has a firm re-training in SM and Van Hiele didactics, and RM not willing to learn the language should regard it as off-limits.

But do we need a full formalisation and acceptance in RM before the dynamic quotient can be used in SM ? Admittedly, in RM, there can be a formal development of variables and expressions, and rules about how an expression can be equated to a variable. If we assume however that students in highschool know about division then it ought to be possible to use above definition of the dynamic quotient without getting involved in RM on formal developments. Overall RM can help but only under strong caveats.

4. "Conquest of the Plane" is about didactics

4a. A general context of didactics of mathematics but an eye on mathematics

Colignatus (2011b)(COTP) is intended for didactics of mathematics for SM. It claims to contribute to a bit better mathematics too. The distinction between “didactics” and “mathematics” is subtle. For example, traditional plane geometry measures angles using 360 degrees or Θ (pronounced "Archi") = 2π radians for a full circle. COTP proposes to take the plane itself as the unit of account, thus 1. This doesn’t change anything about angles. But it can be seen as better mathematics since it relates directly to what measuring angles is about. A right angle is 4^H plane or ¼ or 25% of a plane, and this directly communicates what it is, and doesn’t require the additional calculation from 90 360^H or 90 / 360 (“Ah, 4^H !”) to understand what a right angle is.

In the same way, COTP develops the notion of a “dynamic quotient”. It doesn’t change anything in mathematical results, but it contributes to better didactics, and in that respect changes some mathematics. On close inspection it however also amounts to an essential redefinition of the infinitesimal calculus. It uses notions from mathematical logic and set theory that were unavailable to its originators 150-250 years ago, such as notably Langrange (1736-1813) who also tried for an algebraic approach, see Grabiner (2010).

To develop the dynamic quotient, COTP takes quite some pages (p48-58). Readers will be very busy and may not take the time to read those pages. They thus might concentrate on page 57 where the definition is given that students in highschool can understand.

4b. Definition of the dynamic quotient (continued from 2d)

The following requires Sections 2d & 3b above. Let us reformulate the definition with “(a)”.

Definition of the dynamic quotient for SM (COTP (2011b:57), now with "(a) variable")

N // *D* ≡ { *N* / *D,* unless *D* is (a) variable and then: assume *D* ≠ 0, simplify the expression *N* / *D*, declare the result valid also for the domain extension *D* = 0 }.

Observe the transfer of concepts that is internal to SM.

Definition of the division function, and arguments numerator and denominator, in SM

In the definition of the dynamic quotient *N* // *D*, there is a transfer of concept from standard division *N* / *D*, such that numerator *N* and denominator *D* can be expressions, but can be treated as variables themselves (see 3b.3).

The 2007-2011 & 2d definition uses "a variable" rather than this "(a) variable", see 3b.5. The proper SM context is:

(1) There are functions (with domains and ranges), variables (with domains) and constants (with single values).
(2) Both the normal static quotient and the dynamic quotient are functions with a range and a domain with two positions, the numerator and the denominator.
(3) The numerator *N* and denominator *D* are constants or variables (with domains).
(4) Variables are symbols rather that placeholders or "arbitrary numbers".
(5) The test "unless *D* is a variable" is a syntactic test on the denominator. When *D* = *expr* then one doesn't substitute to create "unless *expr* is a variable". (See Section 3b.5.)
(6) When the denominator is a constant, then *D* is a constant, and “*D* is a variable” is false.
(7) If *D* is a variable, *N* / *D* is treated as a symbolic expression rather than a numerical value.
(8) The simplification is an algebraic simplification that uses symbolic manipulation.
(9) The manipulation of the domain for *D* with first assuming *D* ≠ 0 and later including *D* = 0, can only be done since there is a domain, and that can only be so when *D* is a variable. (If one would make that error of substituting *expr* in (5) then one would look for a range.)

RM need not be aware of this SM context (with imperfect recollection of the highschool years or without highschool teaching and study of didactics). It may be that RM forget about possible objections to normal static division as already used in SM, such as simplifying and determining when outcomes are defined. For RM the reasoning on the dynamic quotient might only be acceptable within specific formal systems, that need to be specified.

4c. Possible questions from RM

Questions from RM are for example: (a) what is “simplify” ? (b) is it a field ? (c) are it variables or expressions (open (with variables) or closed (without them)) ? (d) how do you establish whether something is zero (for example w.r.t. Goldbach’s conjecture) ?

Such questions had already been considered in SM for plain *N* / *D*, and there are practical answers. The SM definition now for *N* // *D* may cause so many problems for RM that the subsequent discussion takes much more time than if they had read pages 48-58 first and had taken a refresher on SM. One way to understand this present paper is that we discuss why RM may reject above definition as a proper definition, how this can be remedied, such as with the sky, and why it always was a proper definition for SM anyway. It are students who must understand it and not necessarily RM who are not aware of Van Hiele levels of understanding. Still, an effort at communication is advisable. Once the community of mathematicians can accept the definition of the dynamic quotient, and help out, SM will see a great advance in both elegance and substance, also for education in physics, economics, biology and such.

Note that the definition uses concepts that highschool students must understand if they want to graduate. Given below possible objections, one wonders why some students still manage to graduate (like RM once did). The obvious answer to the question on Goldbach's conjecture is of course that it isn't required for highschool graduation (yet).

4d. Syntax versus semantics, expression versus value (continued from 3b)

In the definition of *dyndiv*[*N, D*] = *N* // *D, N* stands for the numerator and *D* stands for the denominator. Numerator and denominator can be taken as variables or constants. It is customary in SM to assign domains to variables. When the domain of the function *dyndiv* is manipulated, the definition uses the denominator as a variable. Therefor the original definition has "a variable" and not "variable". The numerator and denominator, regarded as variables, can of course be equal to expressions that are not constants. When denominator *D* = *expr* that is not constant then we say that "*expr* is variable", but *D* remains a variable. The test is a syntactic test and not a semantic test on the value of the variable. The denominator *D* is not a variable when *D* is chosen to be a constant.

However, an alternative is to neglect above context in SM and instead use equation *D* = *expr* and then use substitution. One might e.g. substitute: (*v* + *v*) // (*v* + *v*) = { (*v* + *v*) / (*v* + *v*), unless (*v* + *v*) is a variable and ....}. However, (*v* + *v*) or 2*v* is variable but not *a* variable, whence the definition with "a variable" seems to break down.

This substitution is the error in 3b.5: with syntactic "*x* is a variable" then substitute *x* = *x*[*a*] to create "*x*[*a*] is a variable". This violation of a syntactic test is a wrong reading of the original definition and could be seen as a misplaced idea of applying RM rigor to SM, in which one focuses on the idea of substitution and neglects the definition and its explanation for SM. This substitution neglects that the definition gives a program using descriptive language, and that the phrase "*D* is a variable" is a syntactic test formulated in language instead of formulas open to substitution. Of course substitution occurs in SM too, but the original definition of the dynamic quotient does not use that context in SM. If the context of substitution had been selected then there would be mention of a range for *expr* instead of a domain (see the explanation steps in 4b).

Who wants to use substitutions in this manner, would use "variable" in above (reformulated) definition rather than "a variable". Bergstra (in a communication 2012-07-07) requires to see substitution and suggests to define *D* and *N* as "metavariables" that can also be expressions and then test on a constant value. Translating this:

> *Substitutionable form of the dynamic quotient*
>
> $N\ D^S$ ≡ { If *D* has a constant value then *N* / *D* else assume *D* ≠ 0, simplify *N* / *D*, declare the result valid also for *D* = 0 }

This is no different than reading "(a) variable" as "variable", except for:

> (i) This formally may require metavariables and meta-domains and ranges rather than have just functions. SM would have to accept this, or have no dynamic quotient.
>
> (ii) The distinction between range and domain has become implicit.
>
>> For variables $x\ //^S\ x$ there would be domains but for $(2x)\ //^S\ (x + x)$ there would be ranges. A student is provided no anchor. The original definition of the dynamic quotient intends to point students to locating the domain for the denominator. When *D* will be replaced by some *expr* then the domain for the denominator is out of focus, and the focus is on the range for *expr* = 0. It is still adequate, but students might forget that this range is actually the domain for the denominator regarded from the view from the quotient itself.
>
> (iii) The suggested metavariables have conventionally a given domain and no possibility to change it. The true message and crux of the dynamic quotient lies with

the flexibility of the domain for (super-) variables. It is better to adapt a domain than a range (even when they are the same). This form thus occludes the true innovation.

Of course, also students might misread the definition and perform such a substitution. They however would be less bothered by the difference between "a variable" and "variable", and react more practically to an observation that *D* is not constant. It is a matter of proper teaching to alert them to the syntactical test. Replacing "a" with "(a)" might be included in a next edition of COTP though if it helps communication between SM and RM. It would however also require some nomenclature on the domain vs range issue. Of course, for SM it is an empirical issue what students best understand, and hopefully this would be respected by RM.

5. A key example case of (non-) division

The following example is taken from Alders (1965:23), “Algebra III”. It is in Dutch but English readers will get the point. It is from an old textbook since it still gives the proper development with ε and δ. Dutch “teller” is numerator and “noemer” is denominator. Alders observes that they are zero (“nul”) where their ratio is undefined. Note the hole in the graph of *f* at *x* = 2.

Figure: Quote from Alders (1965:23), “Algebra III”.

§ 27. Lim $f(x)$ voor $x \to a$.

We beschouwen de functie

$$f(x) = \frac{2x^2 - 4x}{x - 2} = \frac{2x(x-2)}{x-2}.$$

Voor $x \neq 2$ is $f(x) = 2x$; voor $x = 2$ is $f(x)$ niet gedefinieerd, omdat de teller en de noemer dan nul zijn. — Nu is

$$f(2 + \delta) = 4 + 2\delta \text{ en } f(2 - \delta) = 4 - 2\delta.$$

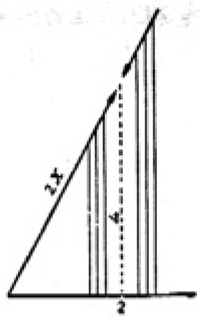

Fig. 11.

Door nu het (positieve) getal δ voldoende klein te kiezen kan men het verschil $|f(x) - 4|$ kleiner maken dan *elk* (positief) getal ε voor elke x uit het interval $2 - \delta < x < 2 + \delta$ (behalve voor $x = 2$).

Kiezen we b.v. $\varepsilon = 0{,}2$, dan is $\delta = 0{,}1$. Dus voor *elke* x uit het interval $1{,}9 < x < 2{,}1$ (behalve voor $x = 2$) is

$$|f(x) - 4| < 0{,}2.$$

We schrijven nu:

$$\lim_{x \to 2} \frac{2x^2 - 4x}{x - 2} = \lim_{x \to 2} \frac{2x(x-2)}{x-2} = \lim_{x \to 2} (2x) = 4.$$

De grafiek van $f(x)$ is de rechte lijn uit fig. 11 met uitzondering van het punt (2, 4).

DEFINITIE: *$f(x)$ heeft voor $x \to a$ de limiet c, als er bij elk gegeven positief getal ε een δ bestaat zó, dat $|f(x) - c| < \varepsilon$ als $0 < |x - a| < \delta$.*

Is $f(x)$ voor $x = a$ wèl gedefinieerd, en gelijk aan $f(a)$, dan kan men de voorwaarde $|x - a| > 0$ weglaten. Zo is $\lim (2x - 3)$ voor $x \to 2$ gelijk aan 1.

Comparing this example with above definition of the dynamic quotient, we see that the latter allows great simplification and clarity. If *f* would be defined with the dynamic quotient then there are no limits and it is a matter of substitution only, giving both *f*[*x*] = 2*x* and {2, 4}. It is a crucial step to see that the approach with limits also uses algebraic simplification and substitution to determine the position where the limit is taken and what the limit value would be. We can abolish the need to wrap this into a limit structure.

We may also restate what is implied in above traditional algebra & analysis, such as for the "(a) variable" issue. This is not a historical question about Alders but on content. Since he allows that numerator and denominator are zero separately, implicitly they are equal to separate functions. He may not make the distinction between choosing a variable or a constant, but this is implicit in say $D = D[x] = x - 2$ where D is a variable. He may not have an algebraic view, and regard N / D only as defined for N and D as chosen constants, for any value in their domain. However, "choosing any value" is generally understood as having a variable. He doesn't take limits in the no-problem area (only suggests that it could be done), and thus he allows for algebraic simplification. Actually, even if limits were used for a proof that $f[x] = 2x$ for all $x \neq 2$, then this still uses algebraic simplication, thus it is there anyway. All in all, what is missing is only the manipulation of the domain for the denominator.

It might well be the purpose in the above that this particular f is undefined at $x = 2$. If we need a continuous function then tradition requires some h that uses f and an exception at $x = 2$. The dynamic quotient makes the creation of such a function more elegant. There is a similar manipulation of the domain but included in a single operation. Dynamic division only rewrites and supplements the available tools to express or create continuity. Static division of course remains available. Limits remain important for school education, but rather for infinity.

When the derivative is taught in highschool or matricola, the dynamic quotient will have been trained on fractions and rational functions and will not be regarded as a new topic for the derivative itself. The discussion on the derivative can focus on the real issue why one would look at the (dynamic) ratio of Δf and Δx.

Colignatus (2011b), *Conquest of the Plane,* contains examples of application of the dynamic quotient. See p48-58 for the definition, p79-80 for a graph, and p149-159 for the first steps in the application for the derivative, for polynomials. The subsequent parts of the book treat the exponential function and trigonometry. Please observe also the Reading Notes (2011-2014). One can use these examples to see what it means and how it works. One should feel invited to work out the steps for say the derivative of x^2 with in one column the normal approach with limits and parallel the use of the dynamic quotient. See also the website for a video, though with slow speed of presentation.

6. Some research questions for RM and answers for SM

6a. What is "simplify" ?

Highschool students must be able to simplify $x^2 / x = x$ for $x \neq 0$. This is a shift from arithmetic with numbers to algebra with variables. A problem in SM is that such rules apparently are not fully formalized. There are rules and regulations, yet it may well be that these do not fully cover situations that teachers regard as obvious.

For example, it may well be that there is no formal rule that $\text{Sin}[\alpha] / \sqrt{(1 - \text{Sin}[\alpha]^2)}$ requires Tan[α] as the proper answer, while there is no need for specification of the domain since this is already provided in the definition of Tan. [1] Interestingly, such lists of rules could be provided by Computer Algebra systems. However some CAS are commercial and do not list their rules.

Thus we can provide the following definition (empirical, or more abstractly for SM):

Definition of simplify

E.g. simplify[*x*] uses rules in schools (country, year).

[1] Another example: http://mathforum.org/library/drmath/view/52280.html

E.g. for practical discussions it might suffice to refer to some CAS, such as *Mathematica* (version ...), or open source Sage (version ...).

This dependence upon empirical circumstance causes a hesitance w.r.t. providing examples. An example might generate debate, violate the common belief that mathematics is universal and should provide unique answers, and cause readers to think that the dynamic quotient would be worthless since it relies on undefined or arbitrary notions. However, the definition of the dynamic quotient is sound, it is conditional on decent mathematics on simplification, this present paper is not about simplification, and this subsection only explains the conditionality.

With reference to Section 3b.4 above, we may hold that students should be able to determine that $x - x = 0$, so that $(x – x)$ // $(x – x)$ is undefined. However, some RM only recognize “open” (with variables) versus “closed” (no variables) expressions, whence they lack the tools to deduce that $x- x = 0$. In this case, however, we can use the bridge that Simplify[$x - x$] = 0. We thus have a tentative answer to the question “how do you establish whether something is zero (for example Goldbach’s conjecture) ?”.

Definition of the test x = 0

$x = 0$ iff Simplify[x] = 0

Thus: $(x – x)$ // $(x – x)$ = { $(x – x) / (x – x)$ which is Indeterminate, unless $D = x - x$ is a variable which we test by D = Simplify[$x – x$] = 0 so that D is a constant and thus no variable, then ... } = Indeterminate.

A quick application: $(x – x)$ // $(x – x)$ = { Simplify [2] [Simplify[$x – x$] / Simplify[$x – x$]]= Simplify[0 / 0] = Simplify[Indeterminate] = Indeterminate, extend domain if relevant } = Indeterminate.

6b. What does a flexible domain mean ?

In the definition of the dynamic quotient the denominator x is defined for some domain, say S. We assume that 0 is in S so that we have a problem. One dynamic step is that we assume that $x \neq 0$, thus we consider $S' = S \setminus \{0\}$. Does a variable remain the same if its domain is changed ? We might use the variable x' too. Subsequently, we extend the domain again. Does this mean that we return to the same variable ? In steps:

(1) x defined for S
(2) x' defined for $S' = S \setminus \{0\}$, and $x' = x$ whenever x' is defined
(3) x'' defined for $S = S' \cup \{0\}$, and $x'' = x$ whenever x is defined

Thus, one could replace the variables and argue that they are essentially different because the domains are adjusted, but one could also agree that the final values fully overlap, so that there is no difference materially.

The use of both simplification and a flexible domain forms a safety valve to prevent division by zero. For Simplify[N / D] the domain might often not be used (and working out what it means for subvariables would only be relevant after simplification) but in some cases the idea is that it would. The domain would seem to be relevant for e.g. Sqrt[x] // x. It is not inconceivable that there might be false variables that seem nonzero but actually are. This notion of a false variable might not be relevant for an application in SM oriented at continuous functions. For RM we could subsume it under the formal development of simplification. Regard for example x // x for x in $S = \{0\}$. If a student or experimental CAS simplification routine looks merely at “there is a domain S” without considering what it actually is, then 0 // 0 = 1, and this would also be a general result, for constant 0 (and not for a variable expression). But $S' = \{ \}$ and this is thought to block Simplify[x' / x']. Conceivably, one might insert another safety valve in the definition: After assuming $D \neq 0$ check that D still is a variable. It is thought that this makes the algorithm needlessly complex for SM, and that students should recognise that x in {0} gives a false variable. But it is not precluded that there are other complexities in RM.

[2] This might require a test or subroutines to prevent infinite loops.

6c. Creation of the ordered field of the reals

One would generally be in favour of some number theory in highschool and actually also in elementary school. It will help pupils to understand that 0.999… = 1, and what it means to work with a calculator and approximate 3^H, see Colignatus (2011c). SM is rather sloppy with writing $3^H = 0.333333$ to suggest an infinite expansion instead of $3^H \approx 0.333333$. Or check the many instances that we see "Archi" $\Theta = 2\pi = 6.28$ suggesting exactness.

A development of a field seems part and parcel of number theory. We thus would consider some group theory, reduced and translated to what highschool students can understand, and to what is useful for them to know, see Van Hiele (1973) on division, above. Thus, we would end up explaining 2 = 1 + 1, 3 = 2 + 1 etc. in highschool again anyway. This seems okay up to a degree, which is that we want to educate students that mathematics is essentially about definitions, theorems and proofs – but not overdo this because of other goals.

While the natural numbers and thus the rationals $\mathbb{Q}$ can be defined recursively, the infinite decimals in say the interval [0, 1] require a somewhat different treatment. A useful approach for SM is Gowers (2003). See also the abstraction operator "@" in Colignatus (2012).

Depending upon how such a SM concoction would look, some RM might hold that this would not be the proper real numbers. We might agree with RM and call this system P (for "practical" or Greek "R"). However, it is a convention in education to not overly burden students with different names for common concepts if the differences are educationally negligible. When students are working on the path of transitions to higher Van Hiele levels of abstraction then it can suit them to have the same name with increasing accuracy rather than introduce new names. As said, depending upon those levels, the same word may mean different things. Also SM concoctions might be called real numbers, and hopefully this can be understood by RM.

Thus we have the reals $\mathrm{P} = \{0, 1, +, -, \cdot, {}^H, <, @\}$ with rules SEP & CR & GIL & (<) & (@) or conventionally ($\mathbb{R}$, +, ·, <) for which we define the dynamic quotient and apply it for calculus.

7. Creation of a sky

This section intends to lift the dynamic quotient from SM to RM attention level, and to indicate some of its theoretical requirements.

While the former section has a caveat on SM, a caveat in this section is that the present author is only a novice in algebraic group theory so that this section must be taken with a grain of salt. This paper concentrates on (i) presenting the dynamic quotient and handling the SM-RM gap on it, and (ii) getting a clear view on the meadow in relation to the dynamic quotient. The remaining 1% of attention in composing this paper has been for (iii) putting a sky above field and meadow. The sky can only be regarded as a rough sketch and indication for RM. Mention of the sky in the title of the paper is motivated by that the meadow is in group theory. Rejection of the meadow seemed too simple and it was felt that one should at least indicate an alternative that might help RM on the way.

Division N / D already has got some formal theory in the rational functions in abstract algebra. A transcendental element (say variable X) is used to create the extension field $\mathbb{R}[X]$. Polynomial N and D can have real parameters, and presumably also the Taylor series are allowed for exp[X] and trigonometry. Apparently D must be nonzero but X is transcendental and may not have a domain. Perhaps this already gives a formulation for the *result* of using *dynamic* rational functions, and the problem is rather inverse, to create *"static"* cases that threaten to become zero (such as $\Delta F / \Delta X$) so that they must be manipulated to prevent it. Also, possibly there is a formal definition available for a decent computer algebra package; I do not know whether authors have tried that. For this paper it suffices to indicate some elements.

A sky $\Sigma \equiv \Sigma_8$ while Σ_i is called a cumulative stage, with Σ_1 = { (1) } = { IR } and Σ_i = {(1), ..., (*i*) } for points (*i*) below:

(1) The reals (IR, +, ·, <) as an ordered field with density
(2) Variables *x*, *y*, *z*, *x'*, *y'*, *z'*, x_1, … that can be used as placeholders or algebraic symbols
(3) Variables have domains *A*, *B*, *C*, *A'*, *B'*, *C'*, A_1, … that may be flexible w.r.t. elimination and inclusion of 0, and that may be symbolic too ((2) & (3): "supervariables")
(4) Other primitives such as "=", program {..., ..., ...} and switch If {...} then {...} else {...}
(5) Expressions *P*, *Q*, *R*, *P'*, *Q'*, *R'*, P_1, … consisting of variables and operators
(6) Parallel with (5): Functions *f*, *g*, *h*, f_1, … with domains and ranges, defined on variables and using expressions, so that *f*[*x*] could be a formal expression and *f*[3] could be a numerical value, defined recursively on (5) so that *f*[*x*] might be said to consist of an expression of other functions ((5) is the form and (6) the domain-range relation, so that ranges arise from expressions in (5) and domains in (3))
(7) A Simplify algorithm (e.g. reducing expression 1+1 to basic expression 2 ("value"))
(8) The dynamic quotient *P* // *Q*, defined in the body of the text, Section 2d (or 4b with "(a)").

Thus *P* // *Q* in (8) is defined on domain $\Sigma_7 \times \Sigma_7$ and has range Σ_7 (potentially recursively Σ). Assuming that (1) - (7) is clear, there can be discussion with and within research mathematics (RM) whether (8) is sufficiently clear for RM.

Let us try at a definition (8') that uses Section 6b rather than supervariables in (8):

1. *P* /:/ *Q* is an expression like in (6) (possibly recursively containing / and /:/)
2. *R* = Simplify[*Q*] (so that *R* = 0 iff *Q* = 0) (e.g. *R'* – *R'* and 1 - 1 reduce to 0)
3. If *R* is a closed expression (containing no variables):
   3.1. If *R* = 0 then *P* /:/ *Q* = Indeterminate
   3.2. If *R* ≠ 0 then *P* /:/ *Q* = Simplify[*P* / *Q*]
4. If *R* is an open expression (containing variables), then *Q* and *R* must be functions, then *P* /:/ *Q* must have a domain w.r.t. *Q* and let this be *A*. Then *A* is also the range of *Q* w.r.t. its subvariables that have their own subdomains:
   4.1. If *R* can be 0 in *A* (or 0 is an element in *A* that can apply) then:
      4.1.1. Choose *Q'* with *A'* = *A* \ {0}, such that *Q'* = *Q* where *Q'* is defined (i.e. *A'*). Make this choice effective by using subvariables and their subdomains in *Q* to create subvariables and subdomains in *Q'*.
      4.1.2. Adapt *P* to *P'* for relevant subvariables in the change from *Q* to *Q'*
      4.1.3. *P* /:/ *Q* = Simplify[*P'* / *Q'*] for *A'*
      4.1.4. Choose *Q"* with *A* = *A'* U {0}, such that *Q"* = *Q* where *Q* is defined (with 0)
      4.1.5. For *P* /:/ *Q* found in 4.1.3 adjust the domain where it is defined correspondingly (replacing subvariables in *Q'* with those in *Q"* = *Q*)
   4.2. If *R* cannot be 0 in *A* (or 0 is no element in *A*), then *P* /:/ *Q* = Simplify[*P* / *Q*]

In my perception, (8') is only a more complex rephrasing of (8), but perhaps it helps RM.

One reason why (8') is more complex is possibly because of the distinction between variables in (2) and expressions in (5). We use *R* = Simplify[*Q*] instead of the more natural *x* = Simplify[*Q*]. There is no reason why a variable cannot be used to represent the denominator, or why the denominator cannot be regarded as a variable, as in fact is shown in the SM definition of the dynamic quotient. It seems to be a convention in some realms of RM to make the distinction anyhow, and prefer *R* as a metavariable. In this area of RM one might argue that SM implicitly uses such metavariables, without calling them such.

8. Some remaining remarks from the research in mathematics education

Some remaining comments from the viewpoint of research in mathematics education are:

(1) We came to this subject from the paradox of the 'division by zero' and the derivative as a corollary, see Colignatus ALOE (2007:236-327, 2011a:241), and restated in EWS (2009) and COTP (2011b). Recently also the operator H = -1 got included in that story, both for subtraction and the inverse, see Colignatus (2014abc). The positional system for numbers is of key importance for the education in arithmetic and later algebra, which again is important for the proposed algebraic approach to the derivative. It all hangs together. The decimal system is rather simple, with a natural number $x = d[0] + d[1] * 10 + d[2] *10^2 + ... + d[n] * 10^n$, voor $d[i]$ een digit en $n \geq 0$. For decimals see Gowers (2003) and Colignatus (2011c)(2012). It would be useful when higher theory on the decimal system remains closer to the requirements from education.

(2) Harremoës (2000) caused me: (a) to adopt the Harremoës operator H = -1 for the half turn on the unit circle, along i for the quarter turn, (b) to suggest notation -123 = [-1][-2][-3] to open up the positional system for various internal operations and local handling of overflow and underflow (not only for negative numbers). It was a pleasant surprise to see that Bergstra and Ponse (2014b:11) also recognise such possibility, and label it as a tree format. They actually wonder whether the position handling of overflow and underflow is so useful, but in my impression it very likely would be so for education. Thus I would advise research to support this kind of use in elementary education. See Colignatus (2014b) that pupils can use text balloons rather than brackets. It would change the American manner of subtraction.

(3) In practice in a computer algebra package like *Mathematica*, Wolfram (1988, 1996), there is in fact already 1 / 0 = Indeterminate. One wonders why some authors do not simply refer to computer algebra. The development of the meadow for group theory with reference to computer science can be understood as providing a theoretical foundation. There doesn't seem yet to exist a theoretical justification that calculation can be faster when logical testing is replaced by arithmetic. Overall, such results do not seem of major importance for our purposes anyhow, since the rewriting of $0^H = \hat{a}$ doesn't generate a real inverse. It differs from the requirements of mathematical analysis and the algebraic approach to the derivative, that has its own exception switch anyhow. The attention for computer algebra and its research would likely have more benefits for education and its research in general.

9. Conclusions

(1) The dynamic quotient has been given more formal shape via the tentatitve notion of a sky.

(2) Meadows are a formalisation of error handling. They are no useful avenue for finding an alternative definition for the derivative, if we look for a division by zero at the point where the derivative is taken. Error handling should rather use "0^H " (with quotes).

(3) The question whether the sky would be created above an ordered field or an ordered meadow is not so material for our purposes. It are independent issues. Analytically, for our purposes that require order, infinite elements and the zero product rule (ZPR), the *Restricted Inverse Law* (RIL) and *General Inverse Law*(GIL) are equivalent. Conceptually the GIL seems more advantageous since it makes explicit what otherwise is implicit. If the RIL has the advantage of allowing for faster calculation then it can be used in those steps of course.

(4) For improvement in school education and its research it would be advisable to see more attention for empirics and computer algebra. Also the sky would lose in priority.

(5) The analysis supports the H. Wu (2011:372) alert of "(...) the mathematics community to the urgent need of active participation in the education enterprise." Like Academic Hospitals have research for cure and not dissection, the world needs Academic Schools in which teaching is merged with empirical research on didactics.

Appendix A. The meadow

A1. Introduction

From the development of the meadow by Bergstra & Tucker (2007) we take only the main properties. The first batch of properties concerns the Commutative Ring (CR). The second batch of properties is called *Strong* Inverse Properties (SIP) since they do not require that the variables are nonzero. The final property is the *Restricted* Inverse Law (RIL) that actually forms a generalisation of the notion of an inverse, that has no protection clause on $x \neq 0$ and that would also apply to 0 (and allows the creation of 0^H).

CR 1	$(x + y) + z = x + (y + z)$
CR 2	$x + y = y + x$
CR 3	$x + 0 = x$
CR 4	$x + (-x) = 0$
CR 5	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
CR 6	$x \cdot y = y \cdot x$
CR 7	$x \cdot 1 = x$
CR 8	$x \cdot (y + z) = x \cdot y + x \cdot z$
SIP 1	$(-x)^{-1} = -(x^{-1})$
SIP 2	$(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$
SIP 3	$(x^{-1})^{-1} = x$
RIL	$x \cdot (x \cdot x^{-1}) = x$

A set of statements {*a*1, ..., *a*n} can be turned into a logical statement *a*1 or .... or *a*n. Some writers would represent above CR, SIP and RIL as sets, and then take the union for the complete system. However, in that case the relation becomes (*a*1 or .... or *a*n) ⇒ *q*, which would suggest that only one of the *ai* would suffice. The proper form uses &.

Bergstra & Tucker (2007:22) define: A model of CR & SIP & RIL is called a *meadow*.

Bergstra & Tucker (2007:4) mention without proof that (CR & SIP3 & RIL) ⇒ (SIP1 & SIP2)

For a field, the autors also specify the Axiom of Separation (SEP) as $0 \neq 1$. Apparently this is not required for a meadow. However, it is usefully adopted since then it is easier to determine an explicit contradiction. Indeed $0 \neq 1$ would hold for a trivial meadow, but we thus consider non-trivial meadows.

Note that the above CR & SIP & RIL still lacks the successor function of a more traditional development such as the Peano axioms. This extension in computer science rewriting could be found e.g. in Bergstra & Ponse (2014b). Below we take the case of the rational numbers.

Bergstra & Tucker (2007) derive various properties, such as $0 \,.\, x = 0$ (their lemma 2.1a, see also the **Appendix B** below) thus $0 \,.\, 0^H = 0$, and $0^H = 0$. Their derivation is that $0 = 0^H - 0^H = 0^H + (-0)^H = 2\ 0^H$ and inversion gives $0^H = 2^H\ 0 = 0$. See below for the separate steps. However, in Section 2 we already found SIP3 ⇒ ($0^H = 0$).

A2. Proof that 1^H = 1 and H^H = H and SIP => 0^H = 0

Apply RIL to 1: $1 * 1 * 1^H = 1$. Apply CR 7 twice to get $1^H = 1$.

At the line level $H = -1$ thus $H\ H = 1$. Thus RIL $H\ H\ H^H = H$ collapses tot $H^H = H$.

A proof that $0^H = 0$, fully quoting from Bergstra & Tucker (2007:7).

We know from (a) that $0 = 0 \cdot 0^{-1}$ is valid in a commutative ring. On adding the axioms *SIP* to *CR*, we force a value for 0^{-1}:

THEOREM 2.2. *The following equation is provable from $CR \cup SIP$:*

$$0^{-1} = 0.$$

PROOF. First observe that:

$$\begin{aligned} 0 &= 0^{-1} + -(0^{-1}) && \text{by } CR4 \\ &= 0^{-1} + (-0)^{-1} && \text{by } SIP1 \\ &= 0^{-1} + 0^{-1} && \text{by Lemma 2.1(d).} \end{aligned}$$

Now we calculate:

$$\begin{aligned} 0^{-1} &= (0^{-1} + 0^{-1})^{-1} && \text{by applying }^{-1} \\ &= (1 \cdot 0^{-1} + 1 \cdot 0^{-1})^{-1} && \text{by } CR6 \text{ and } CR7 \\ &= ((1+1) \cdot 0^{-1})^{-1} && \text{by } CR8 \\ &= (1+1)^{-1} \cdot (0^{-1})^{-1} && \text{by } SIP2 \\ &= (1+1)^{-1} \cdot 0 && \text{by } SIP3 \\ &= 0 && \text{by Lemma 2.1(a) and } CR2. \quad \square \end{aligned}$$

A3. Understanding RIL

Define $U[x] = x \,.\, x^H$ (read as "unit" that hopefully is 1). Note that $U[x]^H = U[x]$. The crucial point is that we do not have a general rule that $U[x] = x \,.\, x^H = x^H.\ x = 1$. This would cause 1 = 0. The idea is that a weaker rule RIL allows 0^H to be defined (to lose the "undefined") without $U[0] = 1$. The following deduction shows that RIL is a weakened GIL in multiplicative format. Since none of these assumptions is protected, $x \cdot x^H = 1$ might still apply for $x = 0$. However, substitution of $x = 0$ in (u1-u4) still generates a truth since 0 = 0.

(u 1) $x \cdot (x \cdot x^H) = x$	RIL for any x
(u 2) $x \cdot (x \cdot x^H) - x \,.\, 1 = 0$	CR 4 and CR 7
(u 3) $x \,.\, (x \cdot x^H - 1) = 0$	CR 8
(u 4) $x = 0$ or $x \cdot x^H = 1$	only if it is possible to take roots

Bergstra & Tucker (2007:16-17) define $Z[x] = 1 - U[x]$ and prove that $U[x]\ U[x] = U[x]$. Directly:

(v 1) $x \cdot (x \cdot x^H) = x$	RIL for any x
(v 2) $x^H \,.\, x \cdot (x \cdot x^H) = x^H \,.\, x$	multiply with x^H
(v 3) $U[x]\ U[x] = U[x]$ or $U[x] * (U[x] - 1) = 0$	CR 6 and $U[x] = x \cdot x^H$
(v 4) $U[x] = 0$ or $U[x] = 1$?	However RIL & ZPR ⇒ GIL.
(v 5) $x \,.\, x^H = 0$ or $x \,.\, x^H = 1$?	$U[x] = x \cdot x^H$

Since $U[x]$ depends upon x, there is a difference between $U[0] = 0$ and $U[x] = 1$, and it is necessary to be aware of that. At some stage I wrote $y = U[x]$, used $y\ y = y$ thus $y\ (y - 1) = 0$, then ZPR and SEP, and derived contradictions $1 * 1 = 0$ and $0 * 0 = 1$. But it is invalid to use a general $y\ y = y$. Who is getting used to the RIL thus might be warned.

A4. For ($\mathbb{Q}$,<): (SEP & CR & SIP) ⇒ (RIL is equivalent to GIL)

Note that this was already proven in Section 2. This is my own proof that Bergstra (communication 2014-09-10) refers to. It remains useful for math education research when it is discussed how to show students where the ordering in rationals comes from.

For the rationals $\mathbb{Q}$, assume SEP & CR & SIP. With their theorem 2.2: (CR & SIP) ⇒ ($0^H = 0$).

Definitions (in the body of the text N is the numerator but here it gives the natural numbers):

$N^+ = \{ n \mid n = 1 \text{ or } (n - 1) \in N^+ \}$	$\mathbb{Q}^1 = \{ n^H \mid n \in N^+ \}$	$\mathbb{Q}^+ = \{ n\ .\ m^H \mid n \in N^+\ \&\ m \in N^+ \}$
$N^- = \{ n \mid -n \in N^+ \}$	$\mathbb{Q}^0 = \{ n^H \mid n \in Z \}$ (0^H !)	$\mathbb{Q}^- = \{ x \mid -x \in \mathbb{Q}^+ \}$
$N = \{0\} \cup N^+$	$Z = N^- \cup \{0\} \cup N^+$	$\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$
$x > 0$ iff $x \in \mathbb{Q}^+$	$x < 0$ iff $x \in \mathbb{Q}^-$	$x > 0$ iff $-x < 0$
$x > y$ iff $(x - y) \in \mathbb{Q}^+$	$x < y$ iff $(x - y) \in \mathbb{Q}^-$	$x < 0$ or $x = 0$ or $x > 0$

On whether 0 is a natural number, see Harremoës (2011) and Colignatus (2014d). It is.

The above does not just give the rationals $\mathbb{Q}$ but also the order. The above two tables can be represented by ($\mathbb{Q}$, <). An expression is that there is a "positive cone". For this ordered field we have automatic cancellation, with the rules ZPR and CL, and thus also GIL. It will be nice to prove this in a somewhat different fashion.

Lemma: For ($\mathbb{Q}$, <): GIL and RIL: for integer $m \geq 1$ it holds that $m.m^H = 1$.
Proof: For GIL: $m \neq 0$.
For RIL: (1) For $m = 1$: $1.1.1^H = 1$ and drop the first 1 because of $1\ x = x$.
(2) For $m > 1$: $n = m - 1 > 0$. Then:

$$m\ m^H = k$$
$$m = m\ k$$
$$(n + 1) = (n + 1)\ k$$
$$n + 1 = n\ k + k$$
$$1 - k = n\ k - n$$
$$1 - k = n\ (k - 1)$$

$k < 1$	sign + = sign -	impossible
$k > 1$	sign - = sign +	impossible
$k = 1$	$0 = 0$	the only remaining possibility

QED

Theorem: For ($\mathbb{Q}$, <): (SEP & CR & SIP) ⇒ (RIL is equivalent to GIL)
Proof:
GIL implies RIL:
Already given is that for $x = 0$ we get $0\ (0\ 0^H) = 0$.
If $x \neq 0$ then $x \cdot x^H = 1$. Multiply both sides by x which gives $x \cdot (x \cdot x^H) = x$.
RIL implies GIL:
For $x < 0$, use $x\ .\ x^H = (-x)\ .\ (-x)^H$ so that we can look at $-x > 0$.
Assume $x > 0$. Then $x = n\ m^H$ for $n \in N^+$ & $m \in N^+$. Above lemma applies to these n and m. Thus $x \cdot x^H = n\ m^H\ n^H\ m = (n\ n^H)\ (m\ m^H) = 1\ .\ 1 = 1$.
Thus $x \neq 0$ implies $(x \cdot x^H) = 1$.
QED

The upshot is that GIL and RIL are logically equivalent for our purposes with the Reals and the Rationals. This does not look at the particular features of the meadow though, such as providing explicit functions and including an adapted-inverse value for 0 (that is not *really* inverse since we do not have $0\ 0^H = 1$).

A5. Inversive vs divisive meadows

Bergstra & Middelburg (2009, 2010) consider "inversive" meadows that use x^H and "divisive" meadows that use 1 / x. I am not sure whether there is really a difference when H is only used as an operator, without a particular value that can be interpreted as exponentiation. Once we define this x^H = 1 / x identity then the difference disappears, and why not define it such ? A formal argument would be that an "inversive" meadow simply does not have the symbol H for such a definition, and thus would be unable to make it. This is overly formal (formalistic).

On page 19, they state: "(...) the question remains whether the equation 0 / 0 = 0 is natural. The total cost C_n of producing n items of some product is often viewed as the sum of a fixed cost FC and a variable cost VC_n . Moreover, for $n \geq 1$, the variable cost VC_n of producing n items is usually viewed as n times the marginal cost per item, taking VC_n / n as the marginal cost per item. For n = 0, the variable cost of producing n items and the marginal cost per item are both 0. This makes the equation VC_0 / 0 = 0 natural."

This is unusual in economics.The marginal cost would be dVC_n / dn, and normally it would not be zero at n = 0. Even if you take $VC_n = v\, n$, for constant marginal cost v, then $v \neq 0$ at n = 0.

They discuss on p23-25: "(...) that partial meadows together with logics of partial functions do not quite explain how mathematicians deal with 1 / 0 in mathematical works. (...) In the setting of a logic of partial functions, there may be terms whose value is undefined. Such terms are called non-denoting terms. Moreover, often three truth values, corresponding to true, false and neither-true-nor-false, are considered. These truth values are denoted by T, F, and *, respectively. (...) This means that the classical logical connectives and quantifiers must be extended to the three-valued case. (...) In mathematical practice, the truth value of $\forall$x • (x $\neq$ 0) $\Rightarrow$ (x / x = 1) is considered T. Therefore, the truth value of (0 ≠ 0) $\Rightarrow$ (0 / 0 = 1) is T as well. (...) but also the truth value of (0 = 0) $\vee$ (0 / 0 = 1) is T. In our view, the latter does not fit in with how mathematicians deal with 1 / 0 in mathematical works. Hence, we conclude that LRMd, even together with the imperative to comply with the two-valued logic convention, fails to provide a convincing account of how mathematicians deal with 1 / 0 in mathematical works." (brackets added).

It so happens that Colignatus ((1981), 2007, 2011) introduces a threevalued logic that differs from common approaches. However, I do not think that threevalued logic is needed here. There is no need to reduce statements about undefined values to be undefined themselves too. When 0 / 0 is undefined then the statement (0 / 0 = 1) is not true, then (0 = 0) $\vee$ (0 / 0 = 1) still is true, since (0 = 0). This is actually best seen in Wolfram (1988, 1996), in which a test on values (Indeterminate == 1) remains unevaluated but a test on identity (Indeterminate === 1) returns F. The issue is one of predicate logic rather than threevalued logic.

Appendix B. The common meadow

B1. Introduction

Bergstra & Ponse (2014a) generate a "common" meadow that has a general value *â* (for "additional", that might be Undefined or Indeterminate). The GIL is extended to a *Common Inverse Law* (CIL). Bergstra & Tucker (2007:5) use *â* = 0 for an "involutive" meadow[0].

$$If(x \neq 0) \& (x \neq \hat{a})\ then\ (x \cdot x^H = 1)\ else (x^H = \hat{a}) \qquad \text{CIL}[\hat{a}]$$

Application to *â* gives $\hat{a}^H = \hat{a}$. Directly ($\hat{a}^H = \hat{a}$) $\Rightarrow$ ($(\hat{a}^H)^H = \hat{a}^H = \hat{a}$). But ($(0^H)^H = 0$) cannot be derived and thus CIL[*â*] $\Rightarrow$ SIP3 does not hold.

With SIP3 then ($(0^H)^H = 0$) but also ($0^H = \hat{a}$) and thus ($(\hat{a})^H = 0$) and ($\hat{a} = 0$). To prevent this we would require a "SIP3" with longer protection clauses.

While $0^H = 0$ is counter-intuitive, more natural is *â* = Indeterminate (in use in *Mathematica*).

B2. A curious axiom 12

Bergstra & Tucker (2007:6) derive from the CR that 0 . x = 0:

$$\begin{aligned}
0+0 &= 0 && \text{by } CR3 \\
(0+0)\cdot x &= 0\cdot x && \text{multiplying both sides by } x \\
0\cdot x+0\cdot x &= 0\cdot x && \text{by } CR8 \text{ and } CR6 \\
(0\cdot x+0\cdot x)+(-(0\cdot x)) &= 0\cdot x+(-(0\cdot x)) && \text{adding to both sides} \\
0\cdot x+(0\cdot x+(-(0\cdot x))) &= 0 && \text{by } CR1 \text{ and } CR4 \\
0\cdot x+0 &= 0 && \text{by } CR4 \\
0\cdot x &= 0 && \text{by } CR3.
\end{aligned}$$

Thus we also have 0 . *â* = 0.

Bergstra & Ponse (2014a) on the common meadow assume:

axiom (12): $x \,.\, x^H = 1 + 0 \,.\, x^H$.

Given that the axioms for CR apply, 0 . x = 0, and then (12) reduces to $x \,.\, x^H = 1$.

Thus also 0 * *â* = 1.

Thus 0 = 1.

Thus this implementation of a common meadow reduces to a trivial meadow.

B3. Question about 0 * 0^H

Actually, above derivation that 0.x = 0 for all x causes us to wonder about 0 0^H. Are we sure that 0 * Indeterminate is a proper expression ? If 0 0^H = Indeterminate, then above proof uses Indeterminate – Indeterminate = 0 (CR 4). We may wonder whether this is proper too.

The substitution of $x = 0^H$ into 0.x = 0 assumes that 0^H is a well-formed expression, such that the assumptions of CR apply to it. The proof essentially first supposes that 0 0^H is proper and then derives that it is proper and becomes 0. This assumes what needs to be proven. Perhaps the creators of CR did not regard 0^H as such a proper expression for CR 4. It may well be that they intended a protected CR 4*: (x is defined) ⇒ ((x + (-x)) = 0).

B4. Calculus in a meadow

Bergstra, Bethke & Ponse (2013) speak about a "cancellation meadow". Given the cancellation properties it seems that this meadow generates a field (except for the additional properties of a meadow). They introduce differential operators in algebraic fashion, related to Derivative algebra and Derivation (differential algebra) in abstract algebra. The approach differs from the dynamic quotient. In abstract algebra there is a set of rules that you can apply once you have defined them. In highschool we might teach calculus by giving just the derivation rules.

Instead, we need a method that allows to link up with where the notions of derivative and integral come from. If one wants to link up the abstract algebra with the proper derivative then it is not unlikely that conventionally there still would be required an application of the limit. Instead, the dynamic quotient provides an alternative algebraic definition of the derivative.

B5. Cancellation laws

See Section 2 on the GIL and the other cancellation rules.

Bergstra, Hirshfeld & Tucker (2009:6) Lemma 2.2 derive for $w = 1$:

(a) If $x\ y = w$ & $x\ z = w$, then $x\ .\ (y - z) = 0$. (The Zero Product Rule need not apply)

(b) $y - z = 1\ .\ (y - z) = y\ x\ .\ (y - z) = 0$, whence $y = z$ (using $w = 1$, without ZPR)

The derivation using $w = 1$ only allows unicity. The authors suggest $y = z = x^H$, but this is not proper at this stage. However, in Lemma 2.7, p8 there: Als $x\ y = 1$ then $y = x^H$. A full quote is:

To improve readability we denote x^{-1} by $\overline{x}$ and use $1_x = x \cdot x^{-1}$. Recall that $1_x = 1_{\overline{x}}$.

Proposition 2.7. *Implicit definition of inverse:*
$CR + Ril \vdash x \cdot y = 1 \rightarrow x^{-1} = y$

Proof. $\overline{x} = 1 \cdot \overline{x} = x \cdot y \cdot \overline{x} = 1_x \cdot y = (1_x + 0) \cdot y = (1_x + 0 \cdot \overline{x}) \cdot y = (1_x + (x - x) \cdot \overline{x}) \cdot y = (1_x + (x \cdot 1 - x \cdot x \cdot \overline{x}) \cdot \overline{x}) \cdot y = (1_x + (x \cdot x \cdot y - x \cdot x \cdot \overline{x}) \cdot \overline{x}) \cdot y = (1_x + x \cdot x \cdot (y - \overline{x}) \cdot \overline{x}) \cdot y = (1_x + x \cdot (y - \overline{x})) \cdot y = (1_x + x \cdot y - x \cdot \overline{x}) \cdot y = x \cdot y \cdot y = 1 \cdot y = y$ $\square$

The problem now is the find the converse: if we have an x and an x^H generated by the system, prove that their product is 1. Then the meadow collapses to a field (except for its properties). The equivalence of GIL and RIL now must be tested for *â.*

PM 1. For arbitrary w unequal to zero: $x\ y = w$ & $x\ z = w$ associate with $x'\ y = 1$ and $x'\ z = 1$, namely for $w\ x' = x$. In that case $y = z$ too. The problem is that we cannot presume an association with some x' such that $w\ x' = x$ but must show a determination e.g. as $x' = x\ w^H$.

PM 2. One small extension is that x might associate with some z for which we have a proper inverse. This is just to show that one quickly runs into dead ends.

If $z\ y = 1$ then $y = z^H$ and $U[y] = y^H\ y = 1$.
If also $x = w\ z = w\ y^H$ (so that $x\ y = w$) then $x^H = w^H\ y$.
Then $U[x] = w\ z\ w^H\ y = U[w]$.

B6. To what extent is a meadow needed ? Advice to use "0^H"

One might read GIL as a *prohibition* of trying 1 / 0. While GIL leaves 0^H undefined, we may still consider that we have both 0 and ^{H} and thus might create the expression 0^H that might work as Indeterminate. Thus:

$0^H \neq 0$	Assumption. Write 0^H = Indeterminate
$0^H\ (0^H)^H = 1$	GIL: (Indeterminate Indeterminate^H) = 1
$0\ .\ 0^H$ = Indeterminate (?)	See section B3 above. But multiplication of the second line with 0 gives Indeterminate Indeterminate^H = 0. Then 0 = 1
$0\ .\ 0^H = 0$ (?)	Then: 0 Indeterminate = 0
$(0^H)^H \neq 0$	NZI: Indeterminate^H does not reduce
$((0^H)^H)^H = 0^H$	"SIP3": (Indeterminate^H)^H = Indeterminate

It is not clear to me whether this might collapse for some other reasons (too).

Clearly above consequence of NZI is in conflict with SIP3, and it only survives because of the protection in "SIP3". Another example is unprotected SIP2 and line two above from GIL:

$$(0\ 2 = 0\ 5) \Rightarrow (0^H\ \ 2^H\ = 0^H\ \ 5^H) \Rightarrow ((0^H)^H\ 0^H\ \ 2^H\ = (0^H)^H\ 0^H\ \ 5^H) \Rightarrow (2^H = 5^H)$$

A conclusion would be that all SIP require a protection (like CR 4).

Once we allow 0^H then the question arises when the expression can actually be applied and be useful. One might argue that GIL itself generates a symbol 0^H that can work as an indicator for Indeterminate. This might be somewhat useful. As it looks now, however, all expressions containing 0^H or Indeterminate better collapse to Indeterminate, and then the exit, instead of further elaborations on Indeterminate.

Overall, it seems advisable to write "0^H " rather than 0^H. The latter has the meaning that there would be an inverse for 0, while there isn't one. Using "0^H " better shows that: "some calculation runs the danger that this expression is being created whence there is a switch to the exit".

The reported advance in calculation speed with RIL might require a closer look whether programs with logical tests on division by zero are as efficient as they might be. Allowing for $0^H = 0$ and continue calculating with 0 for hours without aborting is not efficient either. If computers are fast in calculating with equalities perhaps we should try to find other ways (than RIL) for handling inequalities.

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