Conjecture that states that a Mersenne number with odd exponent is either prime either divisible by a 2-Poulet number

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Abstract. In this paper I make a conjecture which states that any Mersenne number (number of the form 2^n – 1, where n is natural) with odd exponent n, where n is greater than or equal to 3, also n is not a power of 3, is either prime either divisible by a 2-Poulet number. I also generalize this conjecture stating that any number of the form P = ((2^m)^n – 1)/3^k, where m is non-null positive integer, n is odd, greater than or equal to 5, also n is not a power of 3, and k is equal to 0 or is equal to the greatest positive integer such that P is integer, is either a prime either divisible by at least a 2-Poulet number (I will name this latter numbers Mersenne-Coman numbers) and I finally enunciate yet another related conjecture.

Note:

For a list of 2-Poulet numbers see the sequence A214305 which I posted on OEIS. For a list of Mersenne numbers see the sequence A000225 in OEIS.

Conjecture 1:

Any Mersenne number 2^n – 1 with odd exponent n, where n is greater than or equal to 3, also n is not a power of 3, is either prime either divisible by a 2-Poulet number.

Verifying the conjecture:
(For the first thirteen such n)

: 2^3 – 1 = 7, prime;
: 2^5 – 1 = 31, prime;
: 2^7 – 1 = 127, prime;
: 2^11 – 1 = 2047 = 23*89, a 2-Poulet number;
: 2^13 – 1 = 8191, prime;
: 2^15 – 1 = 32767 = 7*31*151, which is divisibe by 4681 = 31*151, a 2-Poulet number;
: 2^17 – 1 = 131071, prime;
: 2^19 – 1 = 524287, prime;
: 2^21 – 1 = 2097151 = 7^2*127*337, which is divisibe by 42799 = 127*337, a 2-Poulet number;

: 2^23 – 1 = 8388607 = 47*178481, a 2-Poulet number;
: 2^25 – 1 = 33554431 = 31*601*1801, which which is divisibe by 1082401 = 601*1801, a 2-Poulet number;
: 2^29 – 1 = 536870911 = 233*1103*2089, which which is divisibe by 256999 = 233*1103, a 2-Poulet number;
: 2^31 – 1 = 2147483647, prime.

Conjecture 2:

Any Mersenne-Coman number of the form P = ((2^m)^n – 1)/3^k, where m is non-null positive integer, n is odd, greater than or equal to 5, also n is not a power of 3, and k is equal to 0 or is equal to the greatest positive integer such that P is integer, is either a prime either divisible by at least a 2-Poulet number.

Verifying the conjecture:
(For m = 2 and the first twelve such n)

: (4^5 – 1)/3 = 341 = 11*31, a 2-Poulet number;
: (4^7 – 1)/3 = 5461 = 43*127, a 2-Poulet number;
: (4^11 – 1)/3 = 1398101 = 23*89*683, which is divisibe by:
  : 2047 = 23*89, a 2-Poulet number;
  : 15709 = 23*683, a 2-Poulet number;
  : 60787 = 89*683, a 2-Poulet number.
: (4^13 – 1)/3 = 22369621 = 2731*8191, a 2-Poulet number;
: (4^15 – 1)/3^2 is divisibe by:
  : 341 = 11*31, a 2-Poulet number;
  : 4681 = 31*151, a 2-Poulet number;
  : 10261 = 31*331, a 2-Poulet number;
  : 49981 = 151*331, a 2-Poulet number.
: (4^17 – 1)/3 = 5726623061 = 43691*131071, a 2-Poulet number;
: (4^19 – 1)/3 = 91625968981 = 174763*52487, a 2-Poulet number;
: (4^21 – 1)/3^2 divides 5461, 14491, 233017, 42799, 688213 and 1826203, all of them 2-Poulet numbers;
: (4^23 – 1)/3 = 23456248059221 = 47*178481*2796203, which is divisibe by:
  : 8388607 = 47*178481, a 2-Poulet number;
  : 131421541 = 47*2796203, a 2-Poulet number;
  : 499069107643 = 178481*2796203, a 2-Poulet number.
: (4^29 – 1)/3 = 96076792050570581 = 59*233*1103*2089*3033169, which is divisibe by:
  : 13747 = 59*233, a 2-Poulet number;
  : 65077 = 59*1103, a 2-Poulet number;
  : 123251 = 59*2089, a 2-Poulet number;
  : 178956971 = 59*3033169, a 2-Poulet number;
  : 256999 = 233*1103, a 2-Poulet number;
  : 486737 = 233*2089, a 2-Poulet number;

: 706728377 = 233*3033169, a 2-Poulet number;
: 2304167 = 1103*2089, a 2-Poulet number;
: 3345585407 = 1103*3033169, a 2-Poulet number;
: 6336290041 = 2089*3033169, a 2-Poulet number.

: (4^31 - 1)/3 = 1537228672809129301= 715827883*2147483647, a 2-Poulet number;
: (4^37 - 1)/3 = 6296488643826193618261= 223*1777*25781083*616318177, which is divisibe by 396271 = 223*1777 and other 2-Poulet numbers.

Verifying the conjecture:
(For m = 3 and the first four such n)

: 8^5 - 1 = 32767 = 7*31*151, which is divisibe by 4681 = 31*151, a 2-Poulet number;
: 8^7 - 1 = 2097151 = 7^2*127*337, which is divisibe by 42799 = 127*337, a 2-Poulet number;
: 8^11 - 1 = 8589934591 = 7*23*89*599479, which is divisibe by 2047 = 23*89, a 2-Poulet number;
: 8^13 - 1 = 549755813887 = 7*79*8191*121369, which is divisibe by 647089 = 79*8191, a 2-Poulet number.

Verifying the conjecture:
(For m = 4 and the first four such n)

: (16^5 - 1)/3 divides 341 = 11*31, a 2-Poulet number;
: (16^7 - 1)/3 divides 5461 = 43*127, a 2-Poulet number;
: (16^11 - 1)/3 divides 2047 = 23*89, a 2-Poulet number;
: (16^13 - 1)/3 divides 8321 = 53*157, a 2-Poulet number.

Note:

The Mersenne-Coman primes (Mersenne-Coman numbers which are primes) seems to be very rare. For m = 2 (*i.e.* 4^n - 1, where n is odd, n ≥ 5) there is no such a prime up to n = 107.

Conjecture 3:

For any prime p greater than or equal to 5 the number (4^p - 1)/3 is either prime either a product of primes $p_1*p_2*...p_n$ such that all the numbers p_i*p_j are 2-Poulet numbers for $1 \le i < j \le n$.

Note:

This Conjecture is verified for p up to 31 (see the Conjecture 2 above).