# New Meanings of the Fine Structure Constant

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**Abstract**

*In this paper I shall develop the simplest relativistic model of the hydrogen atom. On one hand, this model is incomplete since it neglects the spin of the electron. On the other hand, it incorporates relativistic effects which are not taken into account by the atomic models of Bohr and Schrodinger. In the light of this simple theory it would seem that the Rydberg constant, as far as the spectral lines is concerned, is just the result of the non-relativistic approach used by Bohr and not a true indication of the dependence of the spectral lines on this constant. This formulation emphasizes the significance of the fine-structure constant in the atomic structure of hydrogen and reveals two new meanings of this constant and its connection with life. Additionally this investigation introduces three new constants of Nature a) the minimum speed of light for the existence of hydrogen,* $c_H$ *, b) the energy-distance constant,* $u_d$ *, and c) the quantum energy-distance constant,* $u_q$ *.*

***Keywords****: Hydrogen atom, spectral lines, Lynman series, Balmer series, Paschen series, Planck's constant, Balmer formula, Rydberg formula, Rydberg constant, fine-structure constant, Sommerfeld's atomic theory.*

## 1. Introduction

Spectroscopic experiments carried out in the late 19th century showed that atoms emitted radiation at specific frequencies. Moreover, it was found that these frequencies were quantized.

In 1885 Johann Balmer discovered an empirical formula describing the spectral lines emitted by hydrogen. This emission happens when an electron makes a transition between energy levels *n* and energy level *2*. Balmer formula is

$$\frac{1}{\lambda}=R_\infty\left(\frac{1}{2^2}-\frac{1}{n^2}\right)$$

Where $R_\infty$ is the Rydberg constant. This constant is known with greatest accuracy (also referred as precision) and, according to CODATA 2010, its value is

$$R_\infty = 10\ 973\ 731.568\ 539(55)\,m^{-1.}$$

In 1889, Johannes Rydberg generalized Balmer's formula by proposing another relationship to describe the hydrogen emission lines. This emission takes place when an electron makes a transition between energy levels $n_i$ and $n_f$ . His formula is

$$\frac{1}{\lambda}=R_\infty\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)$$

Rydberg formula becomes identical to Balmer's formula when $n_f$ equals 2.

The explanation of the frequencies of the radiation emitted by the atom remained a mystery until 1913 when Niels Bohr introduced the first quantum hybrid theory of the hydrogen atom. This theory successfully explained not only the spectral lines predicted by Rydberg's formula but also the spectral lines of ionized Helium (and other hydrogenoid) and the size of the hydrogen atom.

Bohr's idea was that when an electron makes a transition from a higher energy state ( $n_i$ ) to a lower energy state ( $n_f$ ) the excess energy must be given out as a photon of energy $hf$. Then, and according to Einstein, he was able to write the following relationship

$$hf = E_i - E_f$$

Bohr also postulated the electron angular momentum was an integer multiple of $\hbar$ . Mathematically this means that

$$mv\, r = n\hbar$$

Where *n* is a quantum number (*n*=1, 2, 3, …). Furthermore, he demanded the electron did not lose energy in its orbit around the proton. In the present formulation we use the de Broglie condition which states that the circumference of the electron's orbit must be an integer multiple of the de Broglie wavelengths of the electron. Mathematically this can be expressed as follows

$$2\pi\, r = n\lambda$$

Both postulates are equivalent since they are the same equation written in a slightly different way. The present formulation is relativistic in nature but unlike Sommerfeld's theory it only considers circular orbits and therefore uses only one quantum number. Thus this is the simplest relativistic atomic theory we can think of. This theory has almost all the limitations found in the Bohr theory, except that the fine-structure constant, $\alpha$ , emerges as a natural feature of the formulation. In Bohr's theory the fine-structure constant is “visible” only in the expression for the wavelengths of the spectral lines (see table 2) and, as far as I know, nobody realized about the existence of this constant until 1916 when Sommerfeld developed his atomic theory. The relativistic treatment of the formulation presented in this paper is what allows us to unveil the $\alpha$ -dependent formulas for the “allowed” atomic radii, the “allowed” energy levels and the “allowed” frequencies/wavelengths of the spectral emission lines of hydrogen.

The significance of this theory is, therefore, based on the fact that this formulation is the simplest relativistic model of the hydrogen atom that is capable of predicting, explicitly, the fundamental relationship between the fine-structure constant and the atomic structure of hydrogen.

According to this formulation the radius of the atom (as we shall see in section 2.3) depends on the fine-structure constant, as opposed to Bohr's theory. This is a profound conceptual difference between the two theories.

## 2. The Relativistic Model of the Hydrogen Atom

In order to improve the structure, reading and understanding of this paper, the development of this theory is divided into the following subsections:

## 2.1. Nomenclature and Equations

We shall use the following nomenclature for the constants and variables used in this paper

**Nomenclature**

$n =$ principal quantum number
$i =$ another symbol for the principal quantum number $n$
(used in frequency and wavelength formulas)
$j =$ another symbol for the principal quantum number $n$
(used in frequency and wavelength formulas)
$h =$ Planck's constant
$\hbar =$ Planck's constant over $2\pi$ (reduced Planck's constant)
$k =$ proportionality constant of Coulomb law
$\epsilon_0 =$ permittivity constant or permittivity of free space
$\alpha =$ fine-structure constant (electromagnetic coupling constant) or alpha
$q_1 =$ charge 1 (e.g. proton charge)
$q_2 =$ charge 2 (e.g. electron charge)
$e =$ elementary charge
$m =$ relativistic mass of the electron
$m_0 =$ electron rest mass
$p =$ relativistic momentum of the electron
$\beta =$ velocity ratio (electron orbital velocity over the speed of light in vacuum)
$\lambda =$ de Broglie wavelength of the electron
$\lambda_{Ce} =$ Compton wavelength for the electron
$\bar{\lambda}_{Ce} =$ Compton wavelength for the electron over $2\pi$ (reduced Compton wavelength)
$\lambda_s =$ wavelength of a given spectral line of hydrogen
$\lambda_{ij} =$ wavelength of the spectral line corresponding to the electron transition between
the energy levels $i$ and $j$
$\lambda_{ij}(A) =$ wavelength of the spectral line corresponding to the transition between
the energy levels $i$ and $j$ in Angstroms
$f =$ frequency of the spectral lines
$f_{ij} =$ frequency of the spectral line corresponding to the electron transition between
the energy levels $i$ and $j$
$r =$ hydrogen atomic radius
$r_n =$ hydrogen atomic radius
$r_1 =$ hydrogen atomic radius (for $n = 1$)
$a_0 =$ hydrogen atomic radius (for $n = 1$) (Standard name)
$v =$ orbital velocity of the electron
$v_n =$ orbital velocity of the electron for level $n$

$v_1 =$ orbital velocity of the electron for the ground level ($n$=1)
$c_H =$ orbital velocity of the electron for the ground level ($n$=1) which is equal to the the minimum value of the speed of light necessary for the hydrogen atom to exist
$c =$ speed of light in vacuum
$F =$ attractive electrostatic force between the proton and the electron
$F_C =$ centripetal force
$F_{JeV} =$ conversion factor from Joules to electron-volts
$E_{rel} =$ total relativistic energy of the hydrogen atom
$E =$ energy levels of the hydrogen atom
$E_n =$ energy levels of the hydrogen atom
$E_1 =$ energy of the ground level of the hydrogen atom
$K =$ kinetic relativistic energy of the electron of a hydrogen atom
$K_p =$ kinetic relativistic energy of the proton of a hydrogen atom (neglected)
$U =$ potential energy of the electron in a hydrogen atom
$u_d =$ energy-distance constant (J-m)
$u_q =$ quantum energy-distance constant (J-m)

## Equations

The equations given below are the equations we shall use in the development of this theory (the are other equations, such as the Compton wavelength of the electron, that are not included here because they are not essential to the development of this theory). I have not included the equation for the fine-structure constant in this section as I assume that we do not know of its existence yet. This constant will emerge explicitly in the next sections.

Thus, we shall assume that

I) the circumference of the electron's orbit equals a whole number of wavelengths (pilot wavelength) of the electron. Mathematically this means that

$$2\pi r = n\lambda \qquad \text{de Broglie standing wave condition} \qquad (2.1\text{-}1)$$

where

$n = 1, 2, 3, 4, \ldots$

II) the wavelength and the relativistic momentum of the electron are related through the de Broglie's relationship

$$\lambda = \frac{h}{p} \qquad \text{de Broglie equation (pilot wavelength)} \qquad (2.1\text{-}2)$$

III) the momentum of the electron is

$$p = mv \qquad \text{Momentum} \qquad (2.1\text{-}3)$$

IV) the mass of the electron is given by the Einstein's relativistic mass

$$m = \frac{m_0}{\sqrt{1-\beta^2}}$$ Relativistic mass (2.1-4a)

$$\beta = \frac{v}{c}$$ Velocity ratio (2.1-4b)

V) the electrostatic force between the proton and the electron is described by the Coulomb's law

$$F = k\frac{q_1 q_2}{r^2}$$ Coulomb's law (2.1-5a)

$$k = \frac{1}{4\pi\epsilon_0}$$ Electric constant in terms of the permittivity constant (2.1-5b)

VI) the centripetal force exerted on the electron is relativistic

$$F_C = \frac{m v^2}{r}$$ Relativistic centripetal force (2.1-6)

VII) the potential energy of the electron at a distance $r$ from the proton is given by the following law

$$U = -k\frac{q_1 q_2}{r}$$ Potential energy (2.1-7)

we shall assume that the potential energy depends on the spatial coordinates and does not depend on time.

VIII) the kinetic energy of the electron is given by its relativistic kinetic energy

$$K = \sqrt{(pc)^2+(m_0 c^2)^2} - m_0 c^2$$ Kinetic energy of the electron (2.1-8)

The kinetic energy of the proton, $K_p$ , is neglected in this theory.

IX) the total relativistic energy of the electron is given by the Einstein's relationship

$$E_{rel} = \sqrt{(pc)^2+(m_0 c^2)^2}$$ Relativistic energy of the electron (2.1-9)

X) the total mechanical energy of the atom equals the kinetic energy of the electron plus the kinetic energy of the proton ( $K_p$ is neglected) plus the potential energy between the electron and the proton

$$E = K + U$$ Total energy (2.1-10)

XI) the wavelength of a monochromatic electromagnetic wave as a function of its frequency is

$$\lambda_s = \frac{c}{f}$$ Wavelength of the spectral lines (2.1-11)

## 2.2. Electron Orbital Velocity

We start the analysis from equations (2.1-5a), (2.1-5b) and (2.1-6) observing that

$$q_1 = q_2 = e \tag{2.2-1}$$

Thus we can write

$$k \frac{e^2}{r^2} = \frac{m v^2}{r} \tag{2.2-2a}$$

Now we use equation (2.1-4a)

$$\frac{k e^2}{r} = \frac{m_0 v^2}{\sqrt{1-\beta^2}} \tag{2.2-2b}$$

solving for *r* we get

$$r = \frac{k e^2}{m_0} \frac{\sqrt{1-\beta^2}}{v^2} \tag{2.2-3}$$

From equations (2.1-1), (2.1-2b) and (2.1-3) we get the radius *r i*n terms of the momentum of the electron

$$r = n \frac{h}{2 \pi m v} \tag{2.2-4}$$

Using the relativistic mass of the electron, given by equation (2.1-4a), we get

$$r = n \frac{h \sqrt{1-\beta^2}}{2 \pi m_0 v} \tag{2.2-5}$$

solving for the electron orbital velocity *v* we get the formula

$$v = n \frac{h \sqrt{1-\beta^2}}{2 \pi m_0 r} \tag{2.2-6}$$

Substituting *r* in equation (2.2-6) with the second side of equation (2.2-3) produces

$$v = \left( n \frac{h \sqrt{1-\beta^2}}{2 \pi m_0} \right) \left( \frac{m_0 v^2}{k e^2 \sqrt{1-\beta^2}} \right) \tag{2.2-7}$$

After a simple mathematical work we solve this equation for the velocity *v.* Thus we get

$$v = \frac{2\pi k e^2}{h}\frac{1}{n} \tag{2.2-8}$$

Substituting the electric constant *k* with equation (2.1-5b), the electron orbital velocity, *v*, takes the following shape

$$v_n = \frac{e^2}{2\epsilon_0 h}\frac{1}{n} \tag{2.2-9}$$

$$n = 1, 2, 3, 4, \ldots$$

We have also substituted *v* with $v_n$ to indicate that the orbital velocity of the electron depends on the quantum number *n*.

**Note**

It is worthy to remark that some authors write this equation as follows

$$v_n = \frac{e^2 c}{2\epsilon_0 h c}\frac{1}{n} = \alpha c \frac{1}{n} \tag{2.2-9b}$$

They claim that $v_n$ depends on the fine-structure constant. This is incorrect since if the speed of light were to change, the fine-structure constant would not change because the speed of light *c* is missing in equation (2.2-9). This is an indication that $v_n$ does not depends on the fine-structure constant. But let us forget about this constant for the time being as we have assumed that we don't know about its existence.

Now we define the velocity $c_H$ , for reasons that will be explained later, as

$$c_H = \frac{e^2}{2\epsilon_0 h} \tag{2.2-10}$$

It is worthy to remark that $c_H$ is the orbital velocity of the electron in its lowest orbit. In other words $c_H$ is the orbital velocity of the electron for *n* =1. The reasons of using $c_H$ instead of $v_1$ will become clear later after stepping further into this theory. However it is important to keep in mind that

$$c_H = v_1 \tag{2.2-11}$$

Now we rewrite equation (2.2-9) incorporating this new definition

$$v = c_H \frac{1}{n} \tag{2.2-12}$$

$n = 1, 2, 3, 4, \ldots$

## 2.3. Radii of the Orbits

Let us reconsider equation (2.2-5) and let us write it as follows

$$r = n\,\frac{h}{2\pi}\frac{1}{v}\frac{\sqrt{1-\beta^2}}{m_0} \tag{2.3-1}$$

considering the definition of the variable $\beta$ (equation (2.1-4b)) this equation can be written as

$$r = n\,\frac{h}{2\pi\,m_0}\sqrt{\frac{1}{v^2}-\frac{1}{c^2}} \tag{2.3-2}$$

Substituting $v$ with the second side of equation (2.2-9) we can calculate the radii of the orbits of the hydrogen atom

$$r = n\,\frac{h}{2\pi\,m_0}\sqrt{\left(\frac{n\,2\epsilon_0 h}{e^2}\right)^2-\frac{1}{c^2}} \tag{2.3-3}$$

After some simple mathematical work we get

$$r = n\,\frac{h}{2\pi\,m_0 c}\sqrt{n^2\left(\frac{2\epsilon_0 h c}{e^2}\right)^2-1} \tag{2.3-4}$$

We observe that the constants inside the parenthesis of equation (2.3-4) form a “new” dimensionless constant. Thus, we denote the inverse of this new constant with $\alpha$ and, for reasons we shall discuss later, we shall call it the *atomic structure constant*

$$\alpha = \frac{e^2}{2\epsilon_0 h c} \tag{2.3-5}$$

Thus we have rediscovered the most famous and enigmatic constant of Nature: the *fine-structure constant,* $\alpha$ . According to Sommerfeld's theory, this constant determines the fine structure of the spectral lines of hydrogen.

This allows us to simplify equation (2.3-4) as follows

$$r_n = \left(n\sqrt{\frac{n^2}{\alpha^2}-1}\right)\frac{h}{2\pi\,m_0 c} \tag{2.3-6}$$

$$n = 1, 2, 3, 4 \ldots$$

Equation (2.3-6) gives the atomic radii of the hydrogen atom as a function of the principal quantum number $n$ (We have substituted $r$ with $r_n$ to indicate that the atomic radius is a function of $n$).
It is worthy to observe that the fine-structure constant, $\alpha$ , has appeared naturally in this equation without introducing any additional postulates and without multiplying and dividing by $c$. Only a quantum relativistic treatment of the atom is able to produce this result.

Now if we consider the Compton wavelength for the electron

$$\lambda_{Ce}=\frac{h}{m_0 c} \qquad (2.3\text{-}7)$$

we can express equation (2.3-6) in terms of this wavelength

$$r_n=\left(n\sqrt{\frac{n^2}{\alpha^2}-1}\right)\frac{\lambda_{Ce}}{2\pi} \qquad (2.3\text{-}8)$$

$n = 1, 2, 3, 4 \ldots$

Finally considering that

$$\bar{\lambda}_{Ce}=\frac{\lambda_{Ce}}{2\pi} \qquad (2.3\text{-}9)$$

we can write equation (2.3-8) in terms of $\bar{\lambda}_{Ce}$

$$r_n=\left(n\sqrt{\frac{n^2}{\alpha^2}-1}\right)\bar{\lambda}_{Ce} \qquad (2.3\text{-}10)$$

$n = 1, 2, 3, 4 \ldots$

In **Appendix 1** we have calculated the value of the atomic radius using formula (2.3-6)

## 2.4. Atomic Energy Levels

To find the expression for the energy levels for the hydrogen atom we use equations (2.1-7), (2.1-8) and (2.1-10)

$$E=\sqrt{(pc)^2+(m_0c^2)^2}-m_0c^2-\frac{ke^2}{r} \qquad (2.4\text{-}1)$$

We add the quantity $m_0c^2+\frac{ke^2}{r}$ to both sides to isolate the square root on the second side. Then we square both sides of the equation to get rid of the square root. This gives

$$\left(E+m_0c^2+\frac{ke^2}{r}\right)^2=(pc)^2+(m_0c^2)^2 \qquad (2.4\text{-}2)$$

Now we use equations (2.1-3) and (2.1-4a) to get

$$\left(E+m_0c^2+\frac{ke^2}{r}\right)^2=\frac{m_0^2v^2c^2}{\left(\sqrt{1-\beta^2}\right)^2}+(m_0c^2)^2 \qquad (2.4\text{-}3)$$

then we work on the second side of the equation and we get the following simplification

$$\left(E + m_0 c^2 + \frac{k e^2}{r}\right)^2 = \frac{m_0^2 c^4}{1 - \frac{v^2}{c^2}} \tag{2.4-4}$$

Now we use the equation (2.2-9) for the orbital velocity of the electron. This yields

$$\left(E + m_0 c^2 + \frac{k e^2}{r}\right)^2 = \frac{m_0^2 c^4}{1 - \left(\frac{1}{n} \frac{e^2}{2\epsilon_0 h}\right)^2 \frac{1}{c^2}} \tag{2.4-5}$$

Which can also be written as

$$\left(E + m_0 c^2 + \frac{k e^2}{r}\right)^2 = \frac{m_0^2 c^4}{1 - \frac{1}{n^2}\left(\frac{e^2}{2\epsilon_0 h c}\right)^2} \tag{2.4-6}$$

Now we introduce the fine-structure constant given by equation (2.3-5) into the above equation. This simplifies the equation as follows

$$\left(E + m_0 c^2 + \frac{k e^2}{r}\right)^2 = \frac{m_0^2 c^4}{1 - \frac{\alpha^2}{n^2}} \tag{2.4-7}$$

Taking the square root on both sides and re-arranging yields

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{\alpha^2}{n^2}}} - m_0 c^2 - \frac{k e^2}{r} \tag{2.4-8}$$

Now we substitute the atomic radius, *r*, with the second side of equation (2.3-8). This yields

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{\alpha^2}{n^2}}} - m_0 c^2 - \frac{e^2}{2\epsilon_0} \frac{1}{n\sqrt{\frac{n^2}{\alpha^2} - 1}\ \lambda_{Ce}} \tag{2.4-9}$$

Now we perform the following series of mathematical steps

$$E = \left(\frac{1}{\sqrt{1 - \frac{\alpha^2}{n^2}}} - 1\right) m_0 c^2 - \left(\frac{1}{n\sqrt{\frac{n^2}{\alpha^2} - 1}}\right)\left(\frac{e^2}{2\epsilon_0 h c}\right) m_0 c^2 \tag{2.4-10}$$

$$E=\left(\frac{1}{\sqrt{1-\frac{\alpha^2}{n^2}}}-1-\frac{\alpha}{n\sqrt{\frac{n^2}{\alpha^2}-1}}\right)m_0c^2 \tag{2.4-11}$$

$$E=\left(\frac{1}{\sqrt{1-\frac{\alpha^2}{n^2}}}-\frac{\alpha}{n\sqrt{\frac{n^2}{\alpha^2}-1}}-1\right)m_0c^2 \tag{2.4-12}$$

$$E=\left[\frac{n\sqrt{\frac{n^2}{\alpha^2}-1}-\alpha\sqrt{1-\frac{\alpha^2}{n^2}}}{n\sqrt{\left(1-\frac{\alpha^2}{n^2}\right)\left(\frac{n^2}{\alpha^2}-1\right)}}-1\right]m_0c^2 \tag{2.4-13}$$

$$E=\left[\frac{n\sqrt{\frac{n^2}{\alpha^2}-1}-\alpha\sqrt{1-\frac{\alpha^2}{n^2}}}{\frac{n\left(n^2-\alpha^2\right)}{n\alpha}}-1\right]m_0c^2 \tag{2.4-14}$$

$$E=\left[\frac{n\sqrt{\frac{n^2-\alpha^2}{\alpha^2}}-\alpha\sqrt{\frac{n^2-\alpha^2}{n^2}}}{\frac{n^2-\alpha^2}{\alpha}}-1\right]m_0c^2 \tag{2.4-15}$$

$$E=\left[\frac{\alpha\sqrt{n^2-\alpha^2}\left(\frac{n}{\alpha}-\frac{\alpha}{n}\right)}{n^2-\alpha^2}-1\right]m_0c^2 \tag{2.4-16}$$

$$E=\left[\frac{\sqrt{n^2-\alpha^2}\left(\frac{n\alpha}{\alpha}-\frac{\alpha^2}{n}\right)}{n^2-\alpha^2}-1\right]m_0c^2 \tag{2.4-17}$$

$$E=\left[\frac{\sqrt{n^2-\alpha^2}\left(n^2-\alpha^2\right)}{n\left(n^2-\alpha^2\right)}-1\right]m_0c^2 \tag{2.4-18}$$

$$E=\left(\frac{1}{n}\sqrt{n^2-\alpha^2}-1\right)m_0c^2 \tag{2.4-19}$$

$$E=\left(\sqrt{\frac{n^2-\alpha^2}{n^2}}-1\right)m_0c^2 \tag{2.4-20}$$

Now we substitute *E* with $E_n$ to indicate that the energy levels depend on the quantum number *n*. This yields

$$E_n = \left( \sqrt{1 - \frac{\alpha^2}{n^2}} - 1 \right) m_0 c^2 \qquad (2.4\text{-}21)$$

n= 1, 2, 3, 4, ...

This is the formula for the energy of the “allowed” stationary levels (also known as allowed stationary states) of hydrogen. According to the conventional interpretation proposed by Bohr, the stationary levels are those atomic energy levels in which the electron does not radiate energy.

Equation (2.4-21) gives the atomic energy as a function of the quantum number *n*. Thus electrons in this model (as in Bohr's model) can orbit the nucleus in only certain discrete energy levels. This means that the energy of the electrons is quantized. The lowest energy level of the hydrogen atom is known as the ground state while the states with higher energies are known as excited states. Since the energy levels are negative values, the less negative the value of the level, the higher its energy.

It is worthy to observe that the fine-structure constant appears naturally in formula (2.4-21). This is due to three laws of physics:

a) the Coulomb law
b) the de Broglie law, and
c) The Einstein's relativistic mass law

Each of these laws “contribute” with one or more constants to make the fine-structure constant. Thus the Coulomb law contributes with two constants: the elementary charge ( $e$ ) and the permittivity constant ( $\epsilon_0$ ). The de Broglie law contributes with the Plank's constant ( $h$ ) and Einstein's relativistic mass law contributes with the speed of light ( $c$ ). But which formula contributes with the number (1/2)? Probably the Coulomb law.

The energy of the fundamental level (ground level) can be calculated through equation (2.4-21) taking *n* =1. This gives

$$E_1 = \left( \sqrt{1 - \frac{\alpha^2}{1^2}} - 1 \right) m_0 c^2 = -2.179\,901\,193 \times 10^{-18} J$$

and considering the conversion factor, $F_{JeV}$ , between Joules and electron-volts

$$F_{JeV} = 1.602\,176\,564 \times 10^{-19} \frac{J}{eV}$$

we can express the energy of the ground state in electron-volts

$$E_1 = \frac{-2.179\,901\,193 \times 10^{-18} J}{F_{JeV}} \approx -13.605\,873\,66\, eV$$

The measured value according to CODATA 2010 is

$$E_{measured} \approx -13.598\,434\,005\,136\, eV$$

The absolute error, $\Delta E$ , is

$$\Delta E = E_1 - E_{measured} \approx -13.605\,873\,66\, eV - (-13.598\,434\,005\,136\ eV) \approx$$
$$\Delta E = \approx -0.0074\, eV$$

As expected, the error is very small.

## 2.5. Frequencies of the Spectral Lines

A hydrogen atom can absorb and emit electromagnetic radiation. Electrons "orbiting" the proton can a) make a transition to a higher energy level by absorbing electromagnetic radiation (photons). This process is responsible for the emission spectrum, or b) make a transition to a lower energy level by emitting electromagnetic radiation (photons). This process produces the emission spectrum. Since the ground state corresponds to a state of minimum energy, electrons in this state cannot emit electromagnetic radiation. The frequency of the photon emitted by a hydrogen atom can be calculated using the Einstein's relationship

$$hf = E_i - E_j \qquad (2.5\text{-}1)$$

where

$E_i =$ upper energy level (also known as initial energy level)
$E_j =$ lower energy level (also known as $E_f$ or final energy level)

$$f = \frac{1}{h}(E_i - E_j) \qquad (2.5\text{-}2)$$

Equation (2.4-21) allows us to write the energy of the levels *i* and *j*

$$E_i = \left(\sqrt{1 - \frac{\alpha^2}{i^2}} - 1\right) m_0 c^2 \qquad (2.5\text{-}3a)$$

$$E_j = \left(\sqrt{1 - \frac{\alpha^2}{j^2}} - 1\right) m_0 c^2 \qquad (2.5\text{-}3b)$$

From equations (2.5-2), (2.5-3a) and (2.5-3b) we obtain the frequency $f_{ij}$ of the spectral lines

$$f_{ij} = \left(\sqrt{1 - \frac{\alpha^2}{i^2}} - 1\right) \frac{m_0 c^2}{h} - \left(\sqrt{1 - \frac{\alpha^2}{j^2}} - 1\right) \frac{m_0 c^2}{h} \qquad (2.5\text{-}4)$$

After a short mathematical work we get to the final formula

$$f_{ij}=\left(\sqrt{1-\frac{\alpha^2}{i^2}}-\sqrt{1-\frac{\alpha^2}{j^2}}\right)\frac{m_0c^2}{h} \qquad (2.5\text{-}5)$$

This is the formula for the frequencies of the spectral lines of the hydrogen atom. It is worthy to remark that the Rydberg constant does not appear in this formula.

## 2.6. Wavelengths of the Spectral Lines

The formula for the wavelengths, $\lambda_{ij}$ , of the spectral lines of hydrogen can be derived substituting in equation (2.1-11) the frequency, $f$, with the second side of equation (2.5-5). This gives

$$\lambda_{ij}=\left(\frac{1}{\sqrt{1-\frac{\alpha^2}{i^2}}-\sqrt{1-\frac{\alpha^2}{j^2}}}\right)\frac{h}{m_0c} \qquad (2.6\text{-}1)$$

The spectral lines of all atoms are the result of the transitions of electrons between different energy levels. It is worthy to remark that the Rydberg constant does not appear in this formula either.

Sometimes it is convenient to express the wavelength of the spectral lines in Angstroms. This can be done by multiplying the previous formula by $10^{10}$

$$\lambda_{ij}(A)=\left(\frac{10^{10}}{\sqrt{1-\frac{\alpha^2}{i^2}}-\sqrt{1-\frac{\alpha^2}{j^2}}}\right)\frac{h}{m_0c} \qquad (2.6\text{-}2)$$

where $\lambda_{ij}(A)$ is the wavelength in Angstroms.

To illustrate this theory we shall calculate the wavelengths corresponding to the Balmer series with formula (2.6-2) by substituting the quantum number for the lower energy level $j$ with 2. Thus, formula (2.6-2) becomes

$$\lambda_{ij}(A)=\left(\frac{10^{10}}{\sqrt{1-\frac{\alpha^2}{i^2}}-\sqrt{1-\frac{\alpha^2}{2^2}}}\right)\frac{h}{m_0c} \qquad \text{Equivalent to Balmer formula} \qquad (2.6\text{-}3)$$

$i$ = 3,4,5, ...

It is worthy to remark that formula (2.6-3) yields the value of the wavelengths directly in Angstroms. Table 1, table 2 and table 3 show some selected spectral lines for the Lyman ($j$ =1), Balmer ($j$ =2) and Paschen ($j$ =3) spectral series, respectively.

## Lynman Series (ultraviolet)

| $j$<br>(Quantum number corresponding to the lower energy level) | $i$<br>(Quantum number corresponding to the higher energy level) | $\lambda$<br>(Theoretical wavelength in Angstroms) | $\lambda_{experimental\ air}$<br>(Hyperphysics) |
|---|---|---|---|
| 1 | 2 | 1215.002 514 | 1215.66 |
| 1 | 3 | 1025.160 266 | 1025.83 |
| 1 | 4 | 972.004 473 3 | 972.54 |
| 1 | 5 | 949.223 367 6 | 949.76 |
| 1 | 6 | 937.290 426 3 | 937.82 |
| 1 | ∞ | 911.254 918 2 | 911.8 |

**Table 1**: Theoretical wavelengths for selected lines of the Lynman series.

## Balmer Series (near UV and visible)

The following table shows some selected wavelengths for the Balmer series ($j$ =2 and $i$ = 3,4,5,...)

| $j$<br>(Quantum number corresponding to the lower energy level) | $i$<br>(Quantum number corresponding to the higher energy level) | $\lambda$<br>(Theoretical wavelength in Angstroms) | $\lambda_{experimental\ air}$<br>(Measured wavelength in Angstroms CODATA 2010. NIST Atomic Spectra Database Lines Data [2] – minimum value) |
|---|---|---|---|
| 2 | 3 | 6561.091 217 | 6562.22 (red) |
| 2 | 4 | 4860.070 714 | 4861.29671127 (blue green) |
| 2 | 5 | 4339.350 151 | 4340.472 (blue) |
| 2 | 6 | 4100.686 560 | 4101.70746200 (violet) |
| 2 | 7 | 3969.059 974 | 3970.075 (near UV) |
| 2 | 8 | 3888.058 997 | 3889.02486539 (near UV) |
| 2 | 9 | 3834.409 000 | 3835.397 (near UV) |
| 2 | ∞ | 3645.056 068 | 3656.65 (near UV) |

**Table 2**: Theoretical wavelengths for selected lines of the Balmer series.

### Paschen Series (infrared)

| $j$ (Quantum number corresponding to the lower energy level) | $i$ (Quantum number corresponding to the higher energy level) | $\lambda$ (Theoretical wavelength in Angstroms) | $\lambda_{experimental\ air}$ (Hyperphysics) |
|---|---|---|---|
| 3 | 4 | 18746.021 7 | 18750.1 |
| 3 | 5 | 12814.667 11 | 12818.1 |
| 3 | 6 | 10935.184 36 | 10938.0 |
| 3 | 7 | 10046.701 64 | 10049.8 |
| 3 | 8 | 9543.435 185 | 9546.20 |
| 3 | ∞ | 8201.391 317 | 8220 |

**Table 3**: Theoretical wavelengths for selected lines of the Paschen series.

## 3. A New Meaning of the Fine-Structure Constant

Let us consider the equation of the atomic radius we derived in section 2.3

$$r = \frac{h}{2\pi\, m_0\, c}\left(n\sqrt{\frac{n^2}{\alpha^2}-1}\right) \qquad (3\text{-}1 = 2.3\text{-}6)$$

This equation tells us the new meaning of the fine-structure constant. For the hydrogen atom to exist the square root of equation (3-1) must be positive. This means that

$$\frac{n^2}{\alpha^2}-1>0 \qquad (3\text{-}2)$$

and this leads to the following result

$$\alpha<n \qquad (3\text{-}3)$$

The lowest value of the second side of inequation (3-3) occurs when $n = 1$. Thus we can rewrite the condition for the existence of the hydrogen atom as

$$\alpha<1 \qquad (3\text{-}4)$$

or

$$c > \frac{e^2}{2\epsilon_0 h} \tag{3-5}$$

Now we define the constant $c_H$ as

$$c_H = \frac{e^2}{2\epsilon_0 h} \tag{3-6}$$

$$c_H = 2\,187691.266 \frac{m}{S}$$

$c_H$ is the speed of the electron in its fundamental or lowest energy level. This velocity is also known as $v_1$. However, since this constant is a threshold for the speed of light (in vacuum) to ensure the existence of the hydrogen atom, we shall use a new nomenclature, $c_H$, to indicate that this constant is a new constant of Nature.

Now inequation (3-5) can be rewritten as

$$c > c_H \tag{3-7}$$

This inequation reveals one of the most important conditions for the existence of hydrogen in the Universe

*Condition for the Existence of Hydrogen*

*A necessary condition for the existence of hydrogen atoms is that the speed of light in vacuum to be greater than* $c_H$ *:* $c > c_H$ *; or in other words, that the fine-structure constant to be less than 1:* $\alpha < 1$ .

*Since* $\alpha$ *= 0.007 297 352 57, this condition is satisfied.*

Now we are one step forward in finding out why the fine-structure constant takes on the value it does. A point to observe is that a more complete relativistic atomic theory which took into account all the atomic quantum numbers (such as the Dirac theory) will, probably, produce a slightly different value of the threshold $c_H$ .

If we express the fine-structure constant as a function of $c_H$ we get the first new meaning of this constant

*The fine-structure constant in terms of* $c_H$ *and c*

*The fine-structure constant is the ratio between the minimum value of the speed of light necessary for the hydrogen atom to exist and the speed of light*

$$\alpha = \frac{c_H}{c} \tag{3-8}$$

Note: in the literature you may find that $\alpha$ is defined by the ratio $\alpha = v_1 / c$ . Despite the fact that $v_1$ and $c_H$ are numerically identical, conceptually they are totally different.

in other words

$$c = \frac{1}{\alpha} c_H \tag{3-9}$$

$$c \approx 137.035\,999\,074\, c_H$$

This result shows that the speed of light in vacuum satisfies condition (3-7).

## 4. Summary and Comparison

The following table compares the formulas of the Bohr model with the corresponding formulas of this formulation.

| **Variable** | **Bohr Model**<br>($n$ = 1,2,3,4, ...) | **Relativistic Model**<br>($n$ = 1,2,3,4, …) |
|---|---|---|
| Orbital velocity of the electron | $v_n = \frac{1}{n} \frac{e^2}{2\epsilon_0 h}$ | $v_n = \frac{1}{n} \frac{e^2}{2\epsilon_0 h}$ |
| Atomic radii | $r_n = n^2 \frac{h^2 \epsilon_0}{\pi m_0 e^2}$ | $r_n = \left( n \sqrt{\frac{n^2}{\alpha^2} - 1} \right) \frac{h}{2\pi m_0 c}$ |
| Atomic energy levels | $E_n = -\frac{1}{n^2} \frac{m_0 e^4}{8 \epsilon_0^2 h^2}$ | $E_n = \left( \sqrt{1 - \frac{\alpha^2}{n^2}} - 1 \right) m_0 c^2$ |
| Frequency of the atomic spectral lines | $f_{ji} = \left( \frac{1}{j^2} - \frac{1}{i^2} \right) \left( \frac{m_0 e^4}{8 \epsilon_0^2 h^3} \right)$ | $f_{ij} = \left( \sqrt{1 - \frac{\alpha^2}{i^2}} - \sqrt{1 - \frac{\alpha^2}{j^2}} \right) \frac{m_0 c^2}{h}$ |
| Wavelength of the atomic spectral lines | $\lambda_{ji} = \frac{1}{\left( \frac{1}{j^2} - \frac{1}{i^2} \right) \left( \frac{m_0 e^4}{8 \epsilon_0^2 h^3 c} \right)}$ | $\lambda_{ij} = \frac{1}{\left( \sqrt{1 - \frac{\alpha^2}{i^2}} - \sqrt{1 - \frac{\alpha^2}{j^2}} \right)} \frac{h}{m_0 c}$ |
| Important constant of the theory | Rydberg's constant<br>$R_\infty = \frac{m_0 e^4}{8 \epsilon_0^2 h^3 c}$ | Fine-structure constant<br>(atomic structure constant)<br>$\alpha = \frac{e^2}{2 \epsilon_0 h c}$ |

**Table 4:** Formula comparison. The last row shows the salient constant of both theories.

## 5. Conclusions

One of the most amazing things about the Bohr model is that it was remarkably successful in explaining the empirical Rydberg formula and other aspects of the hydrogen atom from a set of hybrid physical laws.

However Nature had a hidden surprise for us. The present relativistic formulation of the atom shows that the Rydberg constant does not appear anywhere in the formula for the spectral lines. But then we can ask: Where are *e* and $\epsilon_0$ in this formulation? The answer is: Nature "incorporated" these two constants, along with *h* and *c*, into the fine-structure constant while the rest mass of the electron appears separately in the formulas for the atomic radius, the energy levels and the wavelength of the spectral lines (see table 4).

The square roots in the formulas I presented in this paper (see table 4) indicate the relativistic nature of this formulation. Despite the fact that the Bohr model and the model presented here produce similar numerical results there is a profound conceptual difference between these two models. The Bohr model ignores relativity and therefore it ignores one of the most important laws of nature. The result, this is the price we have to pay, is the absence of the fine-structure constant in the formulas of the atomic radius, in the energy levels and in the frequencies of that theory (see table 4). However the Bohr formula for the wavelength of the spectral lines

*Bohr formula for the wavelengths of spectral lines*
*(in terms of the Rydberg constant)*

$$\lambda_{ji} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right)\left(\frac{m_0 e^4}{8 \epsilon_0^2 h^3 c}\right)} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right) R_\infty}$$

can show the fine-structure constant if we rewrite it as follows

*Bohr formula for the wavelengths of spectral lines*
*(in terms of the fine-structure constant)*

$$\lambda_{ji} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right)\left(\frac{m_0 e^2}{4 \epsilon_0 h^2}\right)\left(\frac{e^2}{2 \epsilon_0 h c}\right)} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right)\left(\frac{m_0 e^2}{4 \epsilon_0 h^2}\right)\alpha} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right)\left(\frac{m_0}{2 h}\right) c_H \alpha}$$

As far as I know, no-one, at the time Bohr published his theory (1913), realized that the above formula (*Bohr formula for the wavelengths of spectral lines)* contained the most enigmatic constant of Nature: the fine-structure constant. The world would have to wait until 1916 for Sommerfeld to discover this constant through his relativistic theory of the hydrogen atom with elliptical orbits [2].

In the light of the relativistic theory presented in this paper it seems that the Rydberg constant, as far as the spectral lines is concerned, is just the result of the non-relativistic approach used by Bohr and not a true indication of the dependence of the spectral lines on this constant. This is the only example, I know of, in the history of physics where an approximate empirical formula – the Rydberg formula - was confirmed by an approximate and incomplete theory (all atomic theories are approximations). In other words the Bohr model is not a theoretical confirmation of a numerical formula but a pure coincidence due to a non-relativistic approach. However, quantitatively speaking, the Bohr model was a magnificent triumph for science because it was the first atomic model ever developed. Qualitatively it was also a brilliant peace of work since it unveiled the quantum nature of the atom. If Bohr would have used the relativistic approach I have used in the present formulation, the Rydberg formula would have never been theoretically confirmed. I believe that Bohr also developed a relativistic formulation but, in the end, he decided to publish the non-relativistic theory simply because it matched Rydberg's empirical work.

Having said that we have to clarify that the Rydberg constant still "appears", implicitly, in this formulation. To obtain the Rydberg constant we need to divide the classical kinetic energy of the electron (*we must use the electron rest mass*) in its lowest orbit by *hc (the Planck's constant times the speed of light in vacuum).* Mathematically

$$\frac{\frac{1}{2} m_o v_1^2}{hc} = \frac{1}{2} \frac{m_o c \alpha^2}{h} = \frac{m_0 e^4}{8 \epsilon_0^2 h^3 c} = R_\infty$$

Another prediction of this theory concerns with the existence of hydrogen. We have seen that one of the necessary conditions for the existence of hydrogen, and therefore for the existence of life, is that the fine-structure constant to be smaller than 1. Mathematically

$$\alpha < 1$$

Nature as we all know, "has passed this test" with flying colors.

One important point to observe is that we have defined the new constant of nature $c_H$ as

*A new constant of Nature*

$$c_H = \frac{e^2}{2 \epsilon_0 h}$$

If we observe the first row of table 4, which corresponds to the orbital velocity of the electron, we find that both theories yield exactly the same equation for the velocity of the electron. This suggests that the velocity $c_H$ should be an exceptionally important physical parameter. The evidence is perhaps the fact that $c_H$ is part of the fine-structure constant as shown by the following expression

*The fine-structure constant in terms of the new constant of Nature*

$$\alpha = \frac{c_H}{c}$$

This in turn suggests that there is another even more fundamental constant of Nature that I shall call it the *energy-distance constant*. Let $u_d$ denote this constant and I let us define it with the following ratio

*The energy-distance constant*

$$u_d = \frac{e^2}{2\epsilon_0}$$

Now the fine-structure constant can be re-written in terms of $u_d$

$$\alpha = \frac{u_d}{hc}$$

Since the denominator, *hc,* has the same dimensions as $u_d$ I shall define the quantum energy-distance constant, $u_q$ , as follows

*The quantum energy-distance constant*

$$u_q = hc$$

Thus we arrive to the second new meaning of fine-structure constant

*The fine-structure constant in terms of the energy-distance constants*

*The fine-structure constant is the ratio between the energy-distance constant and the quantum energy-distance constant*

$$\alpha = \frac{u_d}{u_q}$$

This relationship indicates that there is a deep connection between electromagnetism (through $u_d$ ) and the Planck scale (through $u_q$ ). Therefore this equation seems to represent one of the most fundamental meanings of the fine-structure constant. Any "theory of everything" should be able to explain this relationship from even more fundamental principles. However this term is very unfortunate since no theory will ever explain everything. For example a true theory of everything should explain what you and I are thinking right now, why we are not intelligent enough so that we wouldn't need a theory of everything to understand the universe we live in, what cities looked like 3000 years ago (unless time travel into the past were possible), and that seems to be impossible.

Another point to observe is that even if the fine-structure constant were ten times larger than the value we observe today, Nature would still satisfy the above condition ( $\alpha < 1$ ) and therefore hydrogen atoms would still have formed (we are not considering any other constant tuning in this analysis), and life would still have emerged. This implies that, as far as the hydrogen atom is concerned, Nature has a relatively wide "built-in" margin to ensure the emergence of life. This suggests that, in this context and only in this context, there is no fine tuning as far as $\alpha$ is concerned. Other investigations indicate that if the fine-structure constant were larger than 0.1, stellar fusion (see Appendix 2) would no longer occur. This means that if $\alpha$ were to increase by a factor of 13.7, approximately, fusion would no longer occur in stars. Because this is a relatively large factor these investigations also suggest that there is no fine tuning as far as $\alpha$ is concerned.

In summary, we have shown that the emergence of the fine-structure constant in this theory is the result of three different laws of physics: the Coulomb law, the de-Broglie law and the Einstein's relativistic mass law.

Furthermore, since $\alpha$ appears in the expressions of the atomic radii, the atomic energy levels and the wavelengths of the atomic spectral lines, this analysis suggests that a more general and suitable name for this constant could have been the *atomic structure constant*.

# Appendix 1
# Atomic Radius

The atomic radius of the hydrogen atom can be calculated from equation (2.2-6) for $n=1$

$$r_1 = \frac{h}{2\pi\, m_0 c}\left(1\sqrt{\frac{1^2}{\alpha^2}-1}\right)$$

The standard or usual nomenclature for the atomic radius is $a_0$ therefore the previous equation becomes

$$a_0 = \frac{h}{2\pi\, m_0 c}\left(\sqrt{\frac{1}{\alpha^2}-1}\right)$$

the value this equation yields is

$a_0 = 5.291\ 631\ 206 \times 10^{-11}\, m$
$a_0 = 0.529\ 163\ 120\ 6\ A$

and the diameter of the hydrogen atom is

$d_0 = 1.058\ 326\ 241 \times 10^{-10}\, m$
$d_0 = 1.058\ 326\ 241\, A$

The atomic radius given by CODATA 2010 is

$a_{0\_\mathrm{CODATA}} = 0.529\ 177\ 210\ 92(17) \times 10^{-10}\, m$
$a_{0\_\mathrm{CODATA}} = 0.529\ 177\ 210\ 92(17)\, A$

# Appendix 2
# Nuclear Fusion

Fusion is a nuclear process by which a number of particles combine together to produce a heavier particle whose mass is less than the addition of the masses of the particles that fused together in the process. The mass difference is converted into energy. Stellar fusion is nuclear fusion that occurs in stars including our Sun. As a result, stars emit electromagnetic radiation, including visible light. Thus stars are bright because of nuclear fusion.

## REFERENCES

[1] NIST, *Atomic Spectra Database Lines Data*, retrieved October 4, 2014, from http://physics.nist.gov/asd

[2] C. Churchill, *The structure of hydrogen*, Chapter 9, Draft Version, (2010)