# A Proof of the Beal's Conjecture

Zhang Tianshu

Zhanjiang city, Guangdong province, China

Email: chinazhangtianshu@126.com

**Introduction:** The Beal's Conjecture was discovered by Andrew Beal in 1993. Later the conjecture was announced in December 1997 issue of the Notices of the American Mathematical Society. Yet, it is still both unproved and un-negated a conjecture hitherto.

## Abstract

First we classify A, B and C according to their respective odevity, and ret rid of two kinds from $A^X+B^Y=C^Z$. Then affirm $A^X+B^Y=C^Z$ such being the case A, B and C have a common prime factor by examples. After that, prove $A^X+B^Y\neq C^Z$ under these circumstances that A, B and C have not any common prime factor by mathematical analyses with the aid of the symmetric law of odd numbers. Finally we have proven that the Beal's conjecture holds water after the comparison between $A^X+B^Y=C^Z$ and $A^X+B^Y\neq C^Z$ under the given requirements.

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# A Proof of the Conjecture

The Beal's Conjecture states that if $A^X+B^Y=C^Z$, where A, B, C, X, Y and Z are positive integers, and X, Y and Z are all greater than 2, then A, B and C must have a common prime factor.

We consider limits of values of above-mentioned A, B, C, X, Y and Z as given requirements, including their all or some parts in any inequality or in any equality, thereinafter.

First, we classify A, B and C according to their respective odevity, and remove following two kinds from $A^X+B^Y=C^Z$.

**1.** If A, B and C all are positive odd numbers, then $A^X+B^Y$ is an even number, yet $C^Z$ is an odd number, evidently there is only $A^X+B^Y\neq C^Z$ according to an odd number ≠ an even number.

**2.** If any two in A, B and C are positive even numbers, and another is a positive odd number, then when $A^X+B^Y$ is an even number, $C^Z$ is an odd number, yet when $A^X+B^Y$ is an odd number, $C^Z$ is an even number, so there is only $A^X+B^Y\neq C^Z$ according to an odd number ≠ an even number.

Thus we continue to have merely two kinds of $A^X+B^Y=C^Z$ under the given requirements as listed below.

**1.** A, B and C all are positive even numbers.

**2.** A, B and C are two positive odd numbers and a positive even number.

For indefinite equation $A^X+B^Y=C^Z$ under the given requirements plus aforementioned either qualification, in fact, it has many sets of solutions of positive integers. Let us instance four concrete equations to prove such a viewpoint, as the follows listed.

When A, B and C all are positive even numbers, if let A=B=C=2, X=Y=3, and Z=4, then indefinite equation $A^X+B^Y=C^Z$ is exactly equality $2^3+2^3=2^4$. Evidently $A^X+B^Y=C^Z$ has a set of solutions of positive integers (2, 2, 2) here, and A, B and C have common even prime factor 2.

In addition, if let A=B=162, C=54, X=Y=3, and Z=4, then indefinite equation $A^X+B^Y=C^Z$ is exactly equality $162^3+162^3=54^4$. Evidently $A^X+B^Y=C^Z$ has a set of solutions of positive integers (162, 162, 54) here, and A, B and C have two common prime factors, i.e. even 2 and odd 3.

When A, B and C are two positive odd numbers and a positive even number, if let A=C=3, B=6, X=Y=3, and Z=5, then indefinite equation $A^X+B^Y=C^Z$ is exactly equality $3^3+6^3=3^5$. Manifestly $A^X+B^Y=C^Z$ has a set of solutions of positive integers (3, 6, 3) here, and A, B and C have common prime factor 3.

In addition, if let A=B=7, C=98, X=6, Y=7, and Z=3, then indefinite equation $A^X+B^Y=C^Z$ is exactly equality $7^6+7^7=98^3$. Manifestly $A^X+B^Y=C^Z$ has a set of solutions of positive integers (7, 7, 98) here, and A, B and C have common prime factor 7.

Thus it can seen, indefinite equation $A^X+B^Y=C^Z$ under the given requirements plus aforementioned either qualification can hold water according to above-mentioned four concrete examples, but A, B and C must have at least one common prime factor >1.

If we can prove that there is only $A^X+B^Y\neq C^Z$ under the given requirements plus the qualification that A, B and C have not any common prime factor, then we precisely proved completely that there is only $A^X+B^Y=C^Z$ under the given requirements plus the qualification that A, B and C must have a common prime factor >1.

Since A, B and C have common prime factor 2 where A, B and C all are positive even numbers, so these circumstances that A, B and C have not any common prime factor can only occur under the prerequisite that A, B and C are two positive odd numbers and a positive even number.

If A, B and C have not any common prime factor, then any two of them have not any common prime factor either. Because on the supposition that any two of them have a common prime factor, namely $A^X+B^Y$ or $C^Z-A^X$ or $C^Z-B^Y$ have a common prime factor, yet another has not it, then from this it would lead to $A^X+B^Y\neq C^Z$ or $C^Z-A^X\neq B^Y$ or $C^Z-B^Y\neq A^X$ according to the unique factorization theorem of natural number.

Since it is so, if we can prove that there is only inequality $A^X+B^Y\neq C^Z$ under the given requirements plus the qualification that A, B and C have not any common prime factor, then the Beal's conjecture is surely tenable, otherwise it will be negated.

Unquestionably, let following two inequalities add together, are able to replace completely $A^X+B^Y\neq C^Z$ under the given requirements plus the qualification that A, B and C have not any common prime factor.

1. $A^X+B^Y\neq 2^Z G^Z$ under the given requirements plus the qualification that A, B and 2G have not any common prime factor, where 2G=C.

We divide all positive odd numbers into two kinds of A and B, namely the form of A is 1+4n, and the form of B is 3+4n, where n≥0. Odd numbers of A plus B from small to great are respectively arranged below.

A: 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61…1+4n …

B: 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63…3+4n …

Since a sum of two odd numbers of A divided by 2 is an odd number, i.e. $(1+4n_1)+(1+4n_2)=2+4(n_1+n_2)$, $[2+4(n_1+n_2)]/2=1+2(n_1+n_2)$; also a sum of two odd numbers of B divided by 2 is an odd number too, i.e. $(3+4n_1)+(3+4n_2)=6+4(n_1+n_2)$, $[6+4(n_1+n_2)]/2=3+2(n_1+n_2)$, where $n_1\in n$, $n_2\in n$.

While $2^Z G^Z$ divided by 2 are an even numbers $2^{Z-1}G^Z$, consequently $A^X$ and $B^Y$ within $A^X+B^Y\neq 2^Z G^Z$ can only belong to two of A or two of B. In this case, there is only $A+B=2^Z G^Z$.

2. $A^X+2^Y D^Y\neq C^Z$ under the given requirements plus the qualification that A, 2D and C have not any common prime factor, where 2D=B.

We again divide all odd numbers of A into two kinds, i.e. $A_1$ and $A_2$, and again divide all odd numbers of B into two kinds, i.e. $B_1$ and $B_2$. Or rather, the form of $A_1$ is 1+8n; the form of $B_1$ is 3+8n; the form of $A_2$ is 5+8n; and the form of $B_2$ is 7+8n, where n≥0. The four kinds of odd numbers are all positive odd numbers. They are arranged as follows respectively.

$A_1$: 1, 9, 17, 25, 33, 41, 49, 57, 65, 73, 81, 89, 97, 105…1+8n …

$B_1$: 3, 11, 19, 27, 35, 43, 51, 59, 67, 75, 83, 91, 99, 107…3+8n …

$A_2$: 5, 13, 21, 29, 37, 45, 53, 61, 69, 77, 85, 93, 101, 109…5+8n …

$B_2$: 7, 15, 23, 31, 39, 47, 55, 63, 71, 79, 87, 95, 103, 111…7+8n …

Since a difference between an odd number of $A_1$ and an odd number of $A_2$ divided by 4 is an odd number, and a difference between an odd number of $B_1$ and an odd number of $B_2$ divided by 4 is an odd number, yet $2^YD^Y$ divided by 4 is even number $2^{Y-2}D^Y$, consequently $A^X$ and $C^Z$ within $A^X+2^YD^Y\neq C^Z$ can only either belong to one of $A_1$ and one of $A_2$, or belong to one of $B_1$ and one of $B_2$. In this case, there is only $A+2^YD^Y=B$ or $B+2^YD^Y=A$. Since B and A on the rights of two equalities can be substituted by C, and B and A on the lefts of two equalities can substitute from each other, so employ $A+2^YD^Y=C$ to express $A+2^YD^Y=B$ and $B+2^YD^Y=A$, thereinafter.

We shall go a step further to prove that A and B within $A+B=2^ZG^Z$ are not two odd numbers of greater exponents, and C and A within $A+2^YD^Y=C$ are not two odd numbers of greater exponents either, where greater exponents ≥3, similarly hereinafter.

In other words, this needs us to prove $A^X+B^Y\neq 2^ZG^Z$ and $A^X+2^YD^Y\neq C^Z$ under the given requirements plus the qualification that A, B and C have not any common prime factor >1, where 2G=C, 2D=B.

We again divide $A^X+B^Y\neq 2^ZG^Z$ into two kinds, i.e. (1) $A^X+B^Y\neq 2^Z$, when

$G=1$, and (2) $A^X+B^Y \neq 2^Z G^Z$, where G has at least an odd prime factor >1. Likewise divide $A^X+2^Y D^Y \neq C^Z$ into two kinds, i.e. (3) $A^X+2^Y \neq C^Z$, when $D=1$, and (4) $A^X+2^Y D^Y \neq C^Z$, where D has at least an odd prime factor >1. We list from small to great seriate positive odd numbers and label a belongingness of each of them, well then you would discover that permutations of four kinds of odd numbers are possessed of a certain law.

$1^k$, $A_1$; 3, $B_1$; 5, $A_2$; 7, $B_2$; $(2^3)$;9, $A_1$; 11, $B_1$; 13, $A_2$; 15, $B_2$; $(2^4)$;

17, $A_1$; 19, $B_1$; 21, $A_2$; 23, $B_2$; 25, $A_1$; $3^3$, $B_1$; 29, $A_2$; 31, $B_2$; $(2^5)$;

33, $A_1$; 35, $B_1$; 37, $A_2$; 39, $B_2$; 41, $A_1$; 43, $B_1$; 45, $A_2$; 47, $B_2$;

49, $A_1$; 51, $B_1$; 53, $A_2$; 55, $B_2$; 57, $A_1$; 59, $B_1$; 61, $A_2$; 63, $B_2$; $(2^6)$;

65, $A_1$; 67, $B_1$; 69, $A_2$; 71, $B_2$; 73, $A_1$; 75, $B_1$; 77, $A_2$; 79, $B_2$;

$3^4$, $A_1$; 83, $B_1$; 85, $A_2$; 87, $B_2$; 89, $A_1$; 91, $B_1$; 93, $A_2$; 95, $B_2$;

97, $A_1$; 99, $B_1$; 101, $A_2$; 103, $B_2$; 105, $A_1$; 107, $B_1$; 109, $A_2$; 111, $B_2$;

113, $A_1$; 115, $B_1$; 117, $A_2$; 119, $B_2$; 121, $A_1$; 123, $B_1$; $5^3$, $A_2$; 127, $B_2$; $(2^7)$;

129, $A_1$; 131, $B_1$; 133, $A_2$; 135, $B_2$; 137, $A_1$; 139, $B_1$; 141, $A_2$; 143, $B_2$;

145, $A_1$; 147, $B_1$; 149, $A_2$; 151, $B_2$; 153, $A_1$; 155, $B_1$; 157, $A_2$; 159, $B_2$;

161, $A_1$; 163, $B_1$; 165, $A_2$; 167, $B_2$; 169, $A_1$; 171, $B_1$; 173, $A_2$; 175, $B_2$;

177, $A_1$; 179, $B_1$; 181, $A_2$; 183, $B_2$; 185, $A_1$; 187, $B_1$; 189, $A_2$; 191, $B_2$;

193, $A_1$; 195, $B_1$; 197, $A_2$; 199, $B_2$; 201, $A_1$; 203, $B_1$; 205, $A_2$; 207, $B_2$;

209, $A_1$; 211, $B_1$; 213, $A_2$; 215, $B_2$; 217, $A_1$; 219, $B_1$; 221, $A_2$; 223, $B_2$;

225, $A_1$; 227, $B_1$; 229, $A_2$; 231, $B_2$; 233, $A_1$; 235, $B_1$; 237, $A_2$; 239, $B_2$;

241,$A_1$;$3^5$,$B_1$;245,$A_2$;247,$B_2$; 249, $A_1$;251, $B_1$;253, $A_2$;255, $B_2$;$(2^8)$;

$257, A_1; 259, B_1; 261, A_2; 263, B_2$; $265$, $A_1; 267$, $B_1; 269$, $A_2; 271$, $B_2$; …→

Thus it can seen, permutations of seriate positive odd numbers of from small to great are infinitely many cycles of $A_1B_1A_2B_2$.

To wit: $A_1B_1A_2B_2A_1B_1A_2B_2A_1B_1A_2B_2A_1B_1A_2B_2$ $A_1B_1A_2B_2A_1B_1A_2B_2$…→

We shall set to prove aforementioned four inequalities orderly from the sequence of positive odd numbers to proceed, thereinafter.

**Firstly,** Prove $A^X+B^Y\neq 2^Z$, where A and B are positive odd numbers without any common prime factor >1, and X, Y and Z are integers ≥3.

We add $2^{Z-1}$ and $2^Z$ into the above-listed sequence of odd numbers. If we regard $2^{Z-1}$ as a symmetric center, then $2^{Z-1}-1\in B_2$ with $2^{Z-1}+1\in A_1$, $2^{Z-1}-3 \in A_2$ with $2^{Z-1}+3\in B_1$, $2^{Z-1}-5\in B_1$ with $2^{Z-1}+5\in A_2$, $2^{Z-1}-7\in A_1$ with $2^{Z-1}+7\in B_2$ etc are one-to-one bilateral symmetries respectively.

We consider such permutations of odd numbers as a symmetric law of odd numbers whose symmetric center is $2^{Z-1}$, as follows listed.

$A_1B_1A_2B_2$…$A_1B_1A_2B_2A_1B_1A_2B_2$ $(2^{Z-1})$ $A_1B_1A_2B_2A_1B_1A_2B_2$…$A_1B_1A_2B_2$→

After regard $2^{Z-1}$ as a symmetric center, if leave from $2^{Z-1}$, then there are finite cycles of $B_2A_2B_1A_1$ leftwards until $7(B_2)5(A_2)3(B_1)1(A_1)$, and there are infinitely many cycles of $A_1B_1A_2B_2$ rightwards up to infinite.

Under the symmetric law of odd numbers, two distances from $2^{Z-1}$ to two symmetric odd numbers are each other's equivalent, i.e. $B_2+(1+8n)=2^{Z-1}$ and $A_1-(1+8n)=2^{Z-1}$, $A_2+(3+8n)=2^{Z-1}$ and $B_1-(3+8n)= 2^{Z-1}$, $B_1+(5+8n)=2^{Z-1}$

and $A_2$-(5+8n)=$2^{Z-1}$, $A_1$+(7+8n)=$2^{Z-1}$ and $B_2$-(7+8n)=$2^{Z-1}$ at each other's-symmetric places on two sides of $2^{Z-1}$, where n≥0.

Consequently, on the one hand, a sum of every two symmetric odd numbers is equal to $2^Z$, i.e. $A_1+B_2 =2^Z$ where $A_1>B_2$; $B_1+A_2=2^Z$ where $B_1>A_2$; $A_2+B_1=2^Z$ where $A_2>B_1$; and $B_2+A_1=2^Z$ where $B_2 >A_1$. On the other hand, a sum of any two non-symmetric odd numbers is unequal to $2^Z$, thus we consider not, even need not to prove these inequalities.

Moreover all odd numbers on an identical distance which departs from $2^{Z-1}$ on the either side of $2^{Z-1}$ belong within a kind and the same, no matter which values of Z-1.

Since either $A^X$ or $B^Y$ within $A^X+B^Y=2^Z$ belongs within A, and another belongs within B according to the preceding inference, so A+B=$2^Z$, also A and B are bilateral symmetry whereby $2^{Z-1}$ to act as the center of the symmetry, then either A or B is greater than $2^{Z-1}$, yet another is smaller than $2^{Z-1}$.

By now, we just list odd numbers which have a common base number, and label a belongingness of each of them, below.

| | | | |
|---|---|---|---|
| $1^1$, $A_1$; | $3^1$=3, $B_1$; | $5^1$=5, $A_2$; | $7^1$=7, $B_2$; ($2^3$); |
| $1^2$, $A_1$; | $3^2$=9, $A_1$; | $5^2$=25, $A_1$; | $7^2$=49, $A_1$; |
| $1^3$, $A_1$; | $3^3$=27, $B_1$; | $5^3$=125, $A_2$; | $7^3$=343, $B_2$; |
| $1^4$, $A_1$; | $3^4$=81, $A_1$; | $5^4$=625, $A_1$; | $7^4$=2481, $A_1$; |
| $1^5$, $A_1$; | $3^5$=243, $B_1$; | $5^5$=3125, $A_2$; | $7^5$=16807, $B_2$; |

$1^6$, $A_1$; $3^6=729$, $A_1$; $5^6=15625$, $A_1$; $7^6=117609$, $A_1$;

… … … …

$9^1=9$, $A_1$; $11^1=11$, $B_1$; $13^1=13$, $A_2$; $15^1=15$, $B_2$; $(2^4)$;

$9^2=81$, $A_1$; $11^2=121$, $A_1$; $13^2=169$, $A_1$; $15^2=225$, $A_1$;

$9^3=729$, $A_1$; $11^3=1331$, $B_1$; $13^3=2197$, $A_2$; $15^3=3375$, $B_2$;

$9^4=6561$, $A_1$; $11^4=14641$, $A_1$; $13^4=28561$, $A_1$; $15^4=50625$, $A_1$;

$9^5=59049$, $A_1$; $11^5=161051$, $B_1$; $13^5=371293$, $A_2$; $15^5=759375$, $B_2$;

$9^6=531441$, $A_1$; $11^6=1771561$, $A_1$; $13^6=4826809$, $A_1$; $15^6=11390625$, $A_1$;

… … … …

$17^1=17$, $A_1$; $19^1=19$, $B_1$; $21^1=21$, $A_2$; $23^1=23$; $B_2$…

$17^2=289$, $A_1$; $19^2=361$, $A_1$; $21^2=441$, $A_1$; $23^2=529$; $A_1$…

$17^3=4193$, $A_1$; $19^3=6859$, $B_1$; $21^3=9261$, $A_2$; $23^3=12167$; $B_2$…

$17^4=83521$, $A_1$; $19^4=130321$, $A_1$; $21^4=194481$, $A_1$; $23^4=279841$; $A_1$…

$17^5=1419857$, $A_1$; $19^5=2476099$, $B_1$; $21^5=4084101$, $A_2$; $23^5=6436343$, $B_2$…

$17^6=24137569$,$A_1$;$19^6=47045881$,$A_1$;$21^6=85766121$,$A_1$;$23^6=148035889$,$A_1$..

… … … …

From above listed odd numbers, we are not difficult to see, on the one hand, all odd numbers whereby $A_1$ to act as a base number belong still within $A_1$; all odd numbers whereby $B_1$ to act as a base number belong within $B_1$ plus $A_1$, and one $B_1$ alternates with one $A_1$; all odd numbers whereby $A_2$ to act as a base number belong within $A_2$ plus $A_1$, and one $A_2$ alternates with one $A_1$; and all odd numbers whereby $B_2$ to act as a base number belong within $B_2$ plus $A_1$, and one $B_2$ alternates with one $A_1$.

On the other hand, we classify them into set four kinds according to their respective belongingness, well then all odd numbers of even exponents and odd numbers 1+8n of odd exponents belong within $A_1$; odd numbers 3+8n of odd exponents belong within $B_1$; odd numbers 5+8n of odd exponents belong within $A_2$; and odd numbers 7+8n of odd exponents belong within $B_2$, where $n \geq 0$.

Excepting odd number 1, two adjacent odd numbers which have a common base number are an even number ≥6 apart, but also such even numbers are getting greater and greater along which exponents of these odd numbers are getting greater and greater.

At all events, whether odd numbers of odd exponents or odd numbers of even exponents, all of them are included and dispersed within aforementioned four kinds of odd numbers, thus they entirely conform to the symmetric law of odd numbers.

Thereinafter we shall prove $A^X+B^Y\neq 2^Z$ by mathematical induction under these circumstances that $A^X\in B_2$ with $B^Y\in A_1$, $A_1+B_2=2^Z$; $A^X\in A_1$ with $B^Y\in B_2$, $B_2+A_1=2^Z$; $A^X\in A_2$ with $B^Y\in B_1$, $A_2+B_1=2^Z$; and $A^X\in B_1$ with $B^Y\in A_2$, $B_1+A_2=2^Z$, where $A_1$, $B_2$, $A_2$, and $B_1$ under their respective definiendum are one another's- different positive odd numbers.

**(1)***When Z-1=3, odd numbers on two sides of $2^3$ are listed below.

$1^3$, 3, 5, 7, $(2^3)$, 9, 11, 13, 15…→

To wit: $A_1B_1A_2\ B_2\ (2^3)\ A_1\ B_1\ A_2\ B_2\ \ldots\rightarrow$

It is clear at a glance, that there are not two odd numbers of greater exponents on two places of every bilateral symmetry whereby $2^3$ to act as the center of the symmetry.

When Z-1=4, odd numbers on two sides of $2^4$ are listed below.

$1^4$, 3, 5, 7, 9, 11, 13, 15, ($2^4$) 17, 19, 21, 23, 25, 27, 29, 31…→

To wit: $A_1B_1A_2\ B_2\ A_1B_1A_2\ B_2\ (2^4)\ A_1B_1A_2\ B_2\ A_1B_1A_2\ B_2\ldots\rightarrow$

Evidently there are not two odd numbers of greater exponents on two places of every bilateral symmetry whereby $2^4$ to act as the center of the symmetry.

When Z-1=5 and Z-1=6, odd numbers on two sides of $2^6$ including $2^5$ are listed below.

$1^6$, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, $3^3$, 29, 31, ($2^5$), 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, ($2^6$), 65, 67, 69, 71, 73, 75, 77, 79, $3^4$, 83, 85, 87, 89, 91, 93, 95, 97, 99, 101, 103, 105, 107, 109, 111, 113, 115, 117, 119, 121, 123, $5^3$, 127…→

To wit: $A_1B_1A_2B_2A_1B_1A_2B_2A_1B_1A_2B_2A_1B_1A_2B_2\ (2^5)\ A_1B_1A_2B_2A_1B_1A_2B_2\ A_1B_1A_2B_2A_1B_1A_2B_2\ (2^6)\ A_1B_1A_2B_2A_1B_1A_2B_2A_1B_1A_2B_2A_1B_1A_2B_2A_1B_1A_2\ B_2\ A_1B_1A_2B_2A_1B_1A_2B_2A_1B_1A_2B_2\ldots\rightarrow$

Likewise there are not two odd numbers of greater exponents on two places of every bilateral symmetry whereby $2^6$ including $2^5$ to act as a center of either symmetry.

From above several cases, we can get $A^X+B^Y\neq 2^4$, $A^X+B^Y\neq 2^5$, $A^X+B^Y\neq 2^6$ and $A^X+B^Y\neq 2^7$ such being the case $X\geq 3$ and $Y\geq 3$.

**(2)***Suppose that when Z-1=K and K≥6, there are not two odd numbers of greater exponents on two places of every bilateral symmetry whereby $2^K$ to act as the center of the symmetry. Namely there is only $A^X+B^Y\neq 2^{K+1}$ such being the case $X\geq 3$ and $Y\geq 3$.

**(3)***Prove that when Z-1=K+1, there are not two odd numbers of greater exponents either on two places of every bilateral symmetry whereby $2^{K+1}$ to act as the center of the symmetry. That is to say, this needs us to prove $A^X+B^Y\neq 2^{K+2}$ such being the case $X\geq 3$ and $Y\geq 3$.

**Proof *** We know that permutations of odd numbers on two sides of $2^{Z-1}$ conform to the symmetric law of odd numbers, including odd numbers on two sides of $2^K$ and of $2^{K+1}$, where K≥6. Please, see symmetric permutations of odd numbers on two sides of $2^K$ and of $2^{K+1}$ below.

$A_1B_1A_2B_2\ldots B_1A_2B_2A_1B_1A_2B_2\ (2^K)\ A_1B_1A_2B_2A_1B_1A_2\ldots A_1B_1A_2B_2 \rightarrow$

$A_1B_1A_2B_2\ldots B_1A_2B_2A_1B_1A_2B_2\ (2^{K+1})\ A_1B_1A_2B_2A_1B_1A_2\ldots A_1B_1A_2B_2 \rightarrow$

Since either A or B $>2^K$ and another $< 2^K$ within $A+B=2^{K+1}$, so let $B>2^K$ and $A< 2^K$, then each of $A_1B_1A_2B_2\ldots B_1A_2B_2A_1B_1A_2B_2$ on the left of $2^K$ expresses A, and each of symmetry with A on the right of $2^K$ expresses B. Since all odd numbers on the left of $2^{K+1}$ are exactly all odd numbers of bilateral symmetry whereby $2^K$ to act as the center of the symmetry, thus each of one-to-one symmetric odd numbers whereby $2^K$ to act as a symmetric center expresses A after their symmetric center is changed into $2^{K+1}$, and each of symmetry with A on the right of $2^{K+1}$ expresses B. Besides we divide all odd numbers of bilateral symmetry whereby $2^{K+1}$ to act as the center of the symmetry into four equivalent segments per $2^{K-1}$ odd numbers by $2^K$, $2^{K+1}$ and $3\times2^K$. And number the ordinal of each segment of from left to right as №1, №2, №3 and №4. Then odd numbers at №1 segment and odd numbers at №4 segment are one-to-one bilateral symmetry whereby $2^{K+1}$ to act as the center of the symmetry; also odd numbers at №2 segment and odd numbers at №3 segment as well.

When $Z-1\leq K$, there are not two odd numbers of greater exponents on two places of every bilateral symmetry whereby $2^{Z-1}$ to act as the center of the symmetry. Of course, there are four kinds of symmetric odd numbers always, i.e. $A_1$ and $B_2$ where $A_1>B_2$; $B_1$ and $A_2$ where $B_1>A_2$; $A_2$ and $B_1$ where $A_2>B_1$; and $B_2$ and $A_1$ where $B_2>A_1$.

$A_1$ and $B_2$ away from $2^K$ is respectively $1+8n$ where $A_1>B_2$, and $n \geq 0$; $B_1$ and $A_2$ away from $2^K$ is respectively $3+8n$ where $B_1>A_2$, and $n \geq 0$; $A_2$ and $B_1$ away from $2^K$ is respectively $5+8n$ where $A_2>B_1$, and $n \geq 0$; $B_2$ and $A_1$ away from $2^K$ is respectively $7+8n$ where $B_2>A_1$, and $n \geq 0$.

When $Z-1\leq K$, there are not two odd numbers of greater exponents on two places of every bilateral symmetry whereby $2^{Z-1}$ to act as the center of the symmetry, i.e. there is only $A^X+B^Y\neq2^Z$ such being the case $X\geq3$ and $Y\geq3$.

When $Z-1=K+1$, likewise, there are symmetric permutations of four kinds of odd numbers. In addition, all odd numbers of bilateral symmetries whereby $2^K$ to act as the center of symmetry are turned into all odd numbers on the left of $2^{K+1}$, yet on the right of $2^{K+1}$, odd numbers of symmetries with left odd numbers are formed from $2^{K+1}$ plus each and every odd number of bilateral symmetry whereby $2^K$ to act as the center of the symmetry.

Thus for odd numbers of bilateral symmetries whereby $2^{K+1}$ to act as the center of symmetry, a half of them retained still original places, and the half lies on the left of $2^{K+1}$, yet another half is formed from $2^{K+1}$ plus each and every odd number of bilateral symmetry whereby $2^K$ to act as the center of symmetry.

On the supposition that $A^X$ and $B^Y$ on the two sides of $2^K$ are any pair of bilateral symmetric odd numbers whereby $2^K$ to act as the center of the symmetry, then $B^Y$ plus $2^{K+1}$ is $B^Y+2^{K+1}$, but also $A^X$ and $B^Y+2^{K+1}$ are bilateral symmetry whereby $2^{K+1}$ to act as the center of the symmetry, thus there is $A^X+ (B^Y+2^{K+1}) =2^{K+2}$. After regard $2^{K+1}$ as the center of the symmetry, $A^X$ and $(B^Y+2^{K+1})$ are the very $A^X$ and $B^Y$, so get $A^X+B^Y=2^{K+2}$. Besides, 0 and $2^{K+2}$ are bilateral symmetry for symmetric center $2^{K+1}$ too, thus there is $B^Y+2^{K+1}=2^{K+2}-A^X$, and from this to get $A^X+B^Y=2^{K+1}$.

Like that, $A^X$ plus $2^{K+1}$ is $A^X+2^{K+1}$, but also $B^Y$ and $A^X+2^{K+1}$ are bilateral symmetry whereby $2^{K+1}$ to act as the center of the symmetry, thus there is $B^Y+(A^X+2^{K+1})=2^{K+2}$. After regard $2^{K+1}$ as the center of the symmetry, $B^Y$

and $(A^X+2^{K+1})$ are the very $A^X$ and $B^Y$, so get $A^X+B^Y=2^{K+2}$.

As well, 0 and $2^{K+2}$ are bilateral symmetry for symmetric center $2^{K+1}$, thus there is $A^X+2^{K+1}=2^{K+2}-B^Y$, and from this to get $A^X+B^Y=2^{K+1}$.

For above–mentioned two cases, please, see a simple illustration at the number axis to express them as follows.

$A^X+2^{K+1}$ $\quad$ $B^Y+2^{K+1}$

1, 3... $\quad A^X \quad 2^K \quad B^Y \quad 2^{K+1} \quad 2^{K+2}-B^Y \quad 3 \times 2^K \quad 2^{K+2}-A^X \quad 2^{K+2}$

If the suppositional pair of odd numbers $A^X$ and $B^Y$ on the two sides of $2^K$ is an odd number of greater exponent and an odd number of smaller exponent, then there is only $A^X+B^Y \neq 2^{K+1}$ such being the case $X \geq 3$ and $Y \geq 3$ in line with second step of the preceding supposition, where the smaller exponent is only 1 or 2, similarly hereinafter. So we deduce $B^Y+2^{K+1} \neq 2^{K+2}-A^X$ and $A^X+2^{K+1} \neq 2^{K+2}-B^Y$ from $A^X+B^Y \neq 2^{K+1}$.

Now that exist only to $B^Y+2^{K+1} \neq 2^{K+2}-A^X$ such being the case $X \geq 3$ and $Y \geq 3$, if let $B^Y+2^{K+1}=2^{K+2}-A^X$ by all means, then precisely speak that at least one in "$B^Y+2^{K+1}$" and "$2^{K+2}-A^X$" is unable to be turned into an odd numbers of greater exponent according to the successive inference.

Since $B^Y+2^{K+1}$ and $2^{K+2}-A^X$ share a place and the same, so both of them are an identical odd number in reality. Consequently $B^Y+2^{K+1}$ i.e. $2^{K+2}-A^X$ is unable to be turned into an odd numbers of greater exponent, yet it can only be an odd number of smaller exponent.

Like that, conclude that $A^X+2^{K+1}$ i.e. $2^{K+2}-B^Y$ can only be an odd number of smaller exponent too.

Taken one with another, if either $A^X$ or $B^Y$ in $A^X+B^Y=2^{K+2}$ is an odd number of greater exponent, then another is an odd number of smaller exponent inevitably, therefore there is only $A^X+B^Y\neq 2^{K+2}$ such being the case $X\geq 3$ and $Y\geq 3$.

If the suppositional pair of odd numbers $A^X$ and $B^Y$ on the two sides of $2^K$ is two odd numbers of smaller exponent, then there is $A^X+B^Y=2^{K+1}$ such being the case $X\leq 3$ and $Y\leq 3$. So we deduce $B^Y+2^{K+1}=2^{K+2}-A^X$ and $A^X+2^{K+1}=2^{K+2}-B^Y$ from $A^X+B^Y=2^{K+1}$. This explains that $B^Y+2^{K+1}$ and $2^{K+2}-A^X$ are an identical odd number on a place and the same; additionally $A^X+2^{K+1}$ and $2^{K+2}-B^Y$ as well. Judging from this, $B^Y+2^{K+1}$ is probably either an odd number of greater exponent or an odd number of smaller exponent; additionally $A^X+2^{K+1}$ as well.

Convert $B^Y+2^{K+1}$ and $A^X+2^{K+1}$ into $B^Y$ and convert $B^Y$ into $A^X$ according to the preceding stipulate for odd numbers on the two sides of $2^{K+1}$, well then $A^X$ and $B^Y$ are bilateral symmetry for symmetric center $2^{K+1}$, and there is $A^X+B^Y=2^{K+2}$ due to $A^X$ is an odd number of smaller exponent.

But, after $A^X$ is turned into an odd numbers of greater exponent, if $B^Y$ is an odd numbers of greater exponent too, then there is $A^X+B^Y\neq 2^{K+2}$ such being the case $X\geq 3$ and $Y\geq 3$; if turn $B^Y$ into an odd number of greater exponent though it is originally an odd number of smaller exponent, then there is $A^X+B^Y\neq 2^{K+2}$ such being the case $X\geq 3$ and $Y\geq 3$ as well.

Consequently when Z-1=K+1, there are not two odd numbers of greater exponents on two places of every bilateral symmetry whereby $2^{K+1}$ to act as the center of the symmetry. In other words, there is only $A^X+B^Y\neq 2^{K+2}$ such being the case $X\geq 3$ and $Y\geq 3$.

Apply the above-mentioned way of doing, we can continue to prove that when Z-1=K+2, Z-1=K+3…up to Z-1=every integer≥3, there are always $A^X+B^Y\neq 2^{K+3}$, $A^X+B^Y\neq 2^{K+4}$… $A^X+B^Y\neq 2^Z$, where A and B are positive odd numbers without any common prime factor >1, and X, Y and Z are integers ≥3.

**Secondly,** let us successively prove $A^X+B^Y\neq 2^Z G^Z$ under the given requirements plus the qualifications that A and B are two positive odd numbers, and G has at least an odd prime factor >1, and A, B and 2G have not any common prime factor >1.

We are necessary to use substitutive inequality $E^P+F^V\neq 2^M$ according to proven $A^X+B^Y\neq 2^Z$, where E and F are two positive odd numbers without any common prime factor >1, and P, V and M are integers ≥3.

To begin with, multiply each term of $E^P+F^V\neq 2^M$ by $G^M$, then we obtain $E^PG^M+F^VG^M\neq 2^MG^M$, where G has at least an odd prime factor >1.

For any positive even number, either it is able to be expressed as $A^X+B^Y$, or it is unable. Undoubtedly $E^PG^M+F^VG^M$ is a positive even number.

If $E^PG^M+F^VG^M$ is able to be expressed as $A^X+B^Y$, then there is

$A^X+B^Y \neq 2^M G^M$.

If $E^P G^M+F^V G^M$ is unable to be expressed as $A^X+B^Y$, then the even number has nothing to do with proving $A^X+B^Y \neq 2^M G^M$.

No matter how, there are $E^P G^M+F^V G^M \neq A^X+B^Y$ and $E^P G^M+F^V G^M \neq 2^M G^M$ under these circumstances. So let $E^P G^M+F^V G^M=A^X+B^Y+2b$ or $A^X+B^Y-2b$, where b is a positive integer. Also use sign "±" to denote signs "+" and "-" hereinafter, then obtain $E^P G^M+F^V G^M=A^X+B^Y \pm 2b$, so exists $A^X+B^Y \pm 2b \neq 2^M G^M$ due to $E^P G^M+F^V G^M \neq 2^M G^M$, i.e. $A^X+B^Y \neq 2^M G^M \pm 2b$.

Since 2b can express every positive even number, then $2^M G^M \pm 2b$ can express all positive even numbers except for $2^M G^M$.

For a positive even number, either it is able to be expressed as $2^K N^K$, or it is unable, where K is an integer >2, and N is a positive integer which has at least an odd prime factor >1.

On the one hand, there is $A^X+B^Y \neq 2^K N^K$ where $2^M G^M \pm 2b=2^K N^K$. On the other hand, $2^M G^M \pm 2b$ have nothing to do with proving $A^X+B^Y \neq 2^K N^K$ where $2^M G^M \pm 2b \neq 2^K N^K$.

That is to say, for $E^P G^M+F^V G^M \neq 2^M G^M$, if $E^P G^M+F^V G^M$ is unable to be expressed as $A^X+B^Y$, we can deduce $A^X+B^Y \neq 2^K N^K$ elsewhere too.

Hereto, we have proven $A^X+B^Y \neq 2^M G^M$ or $A^X+B^Y \neq 2^K N^K$ on the existence.

Since either M or K is to express an integer >2, also either G or N is a positive integer which has at least an odd prime factor >1, therefore $A^X+B^Y \neq 2^M G^M$ and $A^X+B^Y \neq 2^K N^K$ are of the same meaning. Thus let Z expresses M and K, and G expresses N, we get $A^X+B^Y \neq 2^Z G^Z$.

**Thirdly,** we proceed to prove $A^X+2^Y \neq C^Z$ under the given requirements plus the qualification that A and C are two positive odd numbers without any common prime factor >1.

Like that, we use substitutive inequality $E^P+F^V \neq 2^M$ once again, also supposed $F^V > E^P$, and let $F^V = C^Z$, then there is $E^P+C^Z \neq 2^M$.

Moreover, let $2^M > 2^3$, then there is $2^M = 2^{M-1} + 2^{M-1}$, and $2^{M-1} \geq 3$.

So there is $E^P + C^Z > 2^{M-1}+2^{M-1}$ or $E^P+C^Z < 2^{M-1}+2^{M-1}$.

Namely there is $C^Z-2^{M-1} > 2^{M-1}-E^P$ or $C^Z-2^{M-1} < 2^{M-1}-E^P$.

In addition, there is $A^X+E^P \neq 2^{M-1}$ according to similar $E^P+F^V \neq 2^M$.

Then we deduce $2^{M-1}-E^P > A^X$ or $2^{M-1}-E^P < A^X$ from $A^X+E^P \neq 2^{M-1}$.

Therefore there is $C^Z-2^{M-1} > 2^{M-1}-E^P > A^X$ or $C^Z-2^{M-1} < 2^{M-1}-E^P < A^X$.

Consequently there is $C^Z-2^{M-1} > A^X$ or $C^Z-2^{M-1} < A^X$.

In a word, there is $C^Z-2^{M-1} \neq A^X$, i.e. $A^X+2^{M-1} \neq C^Z$.

For $A^X+2^{M-1} \neq C^Z$, let $2^{M-1} = 2^Y$, we obtain $A^X+2^Y \neq C^Z$.

**Fourthly,** let us last prove $A^X+2^Y D^Y \neq C^Z$ under the given requirements plus the qualifications that A and C are two positive odd numbers, and D has at least an odd prime factor >1, and A, 2D and C have not any common prime factor >1.

Let us use substitutive inequality $H^U+2^Y \neq K^T$ according to proven $A^X+2^Y \neq C^Z$, where H and K are two positive odd numbers without any common prime factor >1, and U, Y and T are integers >2.

Via moving terms of $H^U+2^Y \neq K^T$, we obtain $K^T-H^U \neq 2^Y$. Like as the above way of Secondly section, first multiply each term of $K^T-H^U \neq 2^Y$ by $D^Y$ to obtain $K^TD^Y-H^UD^Y \neq 2^YD^Y$.

For any positive even number, either it is able to be expressed as $C^Z-A^X$, or it is unable. Unquestionably $K^TD^Y-H^UD^Y$ is a positive even number.

If $K^TD^Y-H^UD^Y$ is able to be expressed as $C^Z-A^X$, then there is $C^Z-A^X \neq 2^YD^Y$, i.e. $A^X+2^YD^Y \neq C^Z$.

If $K^TD^Y-H^UD^Y$ is unable to be expressed as $C^Z-A^X$, then the even number has nothing to do with proving $A^X+2^YD^Y \neq C^Z$.

No matter how, there are $K^TD^Y-H^UD^Y \neq C^Z-A^X$ and $K^TD^Y-H^UD^Y \neq 2^YD^Y$ under these circumstances.

Let $K^TD^Y-H^UD^Y=C^Z-A^X \pm 2d$, where d is a positive integer, well then there is $C^Z-A^X \pm 2d \neq 2^YD^Y$ due to $K^TD^Y-H^UD^Y \neq 2^YD^Y$, i.e. $C^Z-A^X \neq 2^YD^Y \pm 2d$.

Since 2d can express every positive even number, then $2^YD^Y \pm 2d$ can express all positive even numbers except for $2^YD^Y$.

For a positive even number, either it is able to be expressed as $2^SR^S$, or it is unable, where S is an integer >2, and R is a positive integer which has at least an odd prime factor >1.

On the one hand, there is $C^Z-A^X \neq 2^SR^S$ where $2^YD^Y \pm 2d=2^SR^S$, i.e. $A^X+2^SR^S \neq C^Z$. On the other hand, $2^YD^Y \pm 2d$ have nothing to do with proving $A^X+2^SR^S \neq C^Z$ where $2^YD^Y \pm 2d \neq 2^SR^S$.

That is to say, for $K^TD^Y-H^UD^Y \neq 2^YD^Y$, if $K^TD^Y-H^UD^Y$ is unable to be expressed as $C^Z-A^X$, we can deduce $A^X+2^SR^S \neq C^Z$ elsewhere too.

Thus far, we have proven $A^X+2^YD^Y \neq C^Z$ or $A^X+2^SR^S \neq C^Z$ on the existence.

Since either Y or S is to express an integer >2, also either D or R is a positive integer which has at least an odd prime factor >1, therefore $A^X+2^YD^Y \neq C^Z$ and $A^X+2^SR^S \neq C^Z$ are of the same meaning. So let D expresses R, and Y expresses S, we get $A^X+2^YD^Y \neq C^Z$.

To sun up, we have proven every kind of $A^X+B^Y \neq C^Z$ under the given requirements plus the qualification that A, B and C have not any common prime factor >1.

Previous, we have proven that $A^X+B^Y=C^Z$ has certain sets of solutions of positive integers under the given requirements plus the qualification that A, B and C have at least a common prime factor >1.

After the comprehensive comparison between $A^X+B^Y=C^Z$ and $A^X+B^Y \neq C^Z$ under the given requirements, we have reached such a conclusion inevitably, namely an indispensable prerequisite of the existence of $A^X+B^Y=C^Z$ under the given requirements is that A, B and C must have a common prime factor >1.

The proof was thus brought to a close. As a consequence, the Beal conjecture holds water.