Integral Calculus in Ultrapower Fields

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Dedicated to Marie-Louise Nykamp

"Your Algebra and Geometry is your Destiny ..."

Anonymous

"A field of force represents the discrepancy between the natural geometry of a coordinate system and the abstract geometry arbitrarily ascribed to it."

Arthur Eddington

"Science is not done scientifically ..."

Anonymous

Abstract

Infinitely many *ultrapower* field extensions $\mathbb{F}_{\mathcal{U}}$ are constructed for the usual field $\mathbb{R}$ of real numbers by using only elementary algebra, thus allowing for the benefit of both infinitely small and infinitely large scalars, and doing so *without* the considerable usual technical difficulties involved in setting up the Transfer Principle in Nonstandard

Analysis. A natural Integral Calculus - which extends the usual one on the field $\mathbb{R}$ - is set up in these fields $\mathbb{F}_{\mathcal{U}}$. A separate paper presents the same for the Differential Calculus.

1. The Historically Unique Status of the Usual Field $\mathbb{R}$ of Real Numbers

The historically special, and in fact, unique, and thus natural, or canonical status of the usual field $\mathbb{R}$ of real numbers comes from the following

Theorem 1.1. [1, p. 30]

There is, up to order isomorphism, exactly one *complete ordered field*, namely, the usual field $\mathbb{R}$ of real numbers.

$\square$

This basic result, well known in mathematics, also settles the closely interrelated issues of *infinitesimals* and the *Archimedean* property, namely

Theorem 1.2. [1, p. 30]

Every complete ordered field $\mathbb{F}$ is Archimedean.

An ordered field $\mathbb{F}$ is Archimedean, if and only if it is order isomorphic with a subfield of $\mathbb{R}$.

$\square$

Now since obviously $\mathbb{R}$, and hence its subfields, do *not* have nontrivial infinitesimals, while on the other hand, all ultrapower fields $\mathbb{F}_{\mathcal{U}}$, see section 2 below, *do* have them, we obtain

Corollary 1.1.

Every ultrapower field $\mathbb{F}_{\mathcal{U}}$ is non-Archimedean.

$\square$

In fact, the stronger results holds

Corollary 1.2.

Every ordered field $\mathbb{F}$ with nontrivial infinitesimals is non-Archimedean.

$\square$

In this way, having *nontrivial infinitesimals*, and being *non-Archimedean* are equivalent in the case of ordered fields.

The definition of the concepts *ordered field, complete field, Archimedean field, order isomorphism, infinitesimals*, and eventually other ones needed, can be found in [17, pp. 26-30].

2. Ultrapower Fields

Detailes related to the sequel can be found [4-24], as well as in any 101 Algebra college course which deals with *groups, rings, fields*, and *quotient* constructions in groups and rings.

Let Λ be any given *infinite* set which will be used as *index set*, namely, with indices $\lambda \in \Lambda$. A usual particular case is when $\Lambda = \mathbb{N}$ and $\lambda = n \in \mathbb{N} = \Lambda$.

Now we denote by

$$(2.1) \qquad \mathbb{R}^\Lambda$$

the set of all real valued functions defined on Λ, that is, functions $x : \Lambda \longrightarrow \mathbb{R}$. Sometime it will be convenient to write such a function as the family of real numbers $(x_\lambda)_{\lambda \in \Lambda}$ indexed by $\lambda \in \Lambda$. Also, in certain cases we shall have $\Lambda = \mathbb{N}$, thus such a family becomes the usual infinite sequence $(x_n)_{n \in \mathbb{N}}$.

Clearly, with the usual operations of addition and multiplication of such functions, as well as the multiplication of such functions with scalars from $\mathbb{R}$, it follows that $\mathbb{R}^\Lambda$ is a commutative unital algebra on $\mathbb{R}$.

Now it is easy to see that $\mathbb{R}^\Lambda$ is *not* a field, since it has *zero divisors*. That is, exist elements $x, y \in \mathbb{R}^\Lambda, x \neq 0, y \neq 0$, such that nevertheless $xy = 0$. For instance, if we take $\nu, \mu \in \Lambda, \nu \neq \mu$, and take $x = (x_\lambda)_{\lambda \in \Lambda},\ y = (y_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}^\Lambda$, such that $x_\nu = 1$, while $x_\lambda = 0$, for $\lambda \in \Lambda, \lambda \neq \nu$, and similarly, $y_\mu = 1$, while $y_\lambda = 0$, for $\lambda \in \Lambda, \lambda \neq \mu$, then obviously $x \neq 0,\ y \neq 0$, and yet $xy = 0$.

Here we can recall from Algebra that given any *ideal* $\mathcal{I}$ in $\mathbb{R}^\Lambda$, we have

$$(2.2) \qquad \mathbb{R}^\Lambda / \mathcal{I} \text{ is a field} \quad \Longleftrightarrow \quad \mathcal{I} \text{ is a maximal ideal}$$

By the way, above, and in the sequel, by an *ideal* $\mathcal{I}$ in $\mathbb{R}^\Lambda$, we understand a *proper* ideal $\mathcal{I}$, namely, such that $\{0\} \subsetneqq \mathcal{I} \subsetneqq \mathbb{R}^\Lambda$.

The issue of identifying maximal ideals $\mathcal{I}$ in $\mathbb{R}^\Lambda$ needed in (2.2), can be reduced to *ultrafilters* on Λ. Namely, there is a simple bijection between *ideals* $\mathcal{I}$ in $\mathbb{R}^\Lambda$ and *filters* $\mathcal{F}$ on Λ, as follows

$$(2.3) \qquad \begin{array}{l} \mathcal{I} \ \longmapsto \ \mathcal{F}_\mathcal{I} = \{\ I \subseteq \Lambda \mid \exists\ x \in \mathcal{I}\ :\ Z(x) \subseteq I\ \} \\ \\ \mathcal{F} \ \longmapsto \ \mathcal{I}_\mathcal{F} = \{\ x \in \mathbb{R}^\Lambda \mid Z(x) \in \mathcal{F}\ \} \end{array}$$

in other words, we have

$$(2.4) \qquad \mathcal{I} \longmapsto \mathcal{F}_\mathcal{I} \longmapsto \mathcal{I}_{\mathcal{F}_\mathcal{I}} = \mathcal{I}, \qquad \mathcal{F} \longmapsto \mathcal{I}_\mathcal{F} \longmapsto \mathcal{F}_{\mathcal{I}_\mathcal{F}} = \mathcal{F}$$

where for $x \in \mathbb{R}^\Lambda$, we denoted $Z(x) = \{\lambda \in \Lambda \mid x(\lambda) = 0\}$, that is, the *zero set* of the function x.

Now it is easy to see that

$$(2.5) \qquad \begin{array}{l} \mathcal{U} \text{ ultrafilter on } \Lambda \quad \Longrightarrow \quad \mathcal{I}_\mathcal{U} \text{ maximal ideal in } \mathbb{R}^\Lambda \\ \\ \mathcal{I} \text{ maximal ideal in } \mathbb{R}^\Lambda \quad \Longrightarrow \quad \mathcal{F}_\mathcal{I} \text{ ultrafilter on } \Lambda \end{array}$$

Obviously, identifying maximal ideals $\mathcal{I}_\mathcal{U}$ in $\mathbb{R}^\Lambda$ is more difficult than

identifying ultrafilters $\mathcal{U}$ on Λ.
In view of that, we start with an ultrafilter $\mathcal{U}$ on Λ and associate with it the maximal ideal $\mathcal{I}_{\mathcal{U}}$. Now according to (2.2) we obtain the *ultrapower field*

(2.6) $\quad \mathbb{F}_{\mathcal{U}} = \mathbb{R}^\Lambda / \mathcal{I}_{\mathcal{U}}$

Here however, we have to be careful about the fact that there are *two* kind of ultrafilters with respect to the quotient fields (2.6). We shall only be interested in *free* or *non-principal ultrafilters* $\mathcal{U}$, namely, those which satisfy the condition

(2.7) $\quad \bigcap_{I \in \mathcal{U}} I = \phi$

The reason is that the alternative kind of ultrafilters $\mathcal{U}$, called *principal ultrafilters* or *ultrafilters fixed* at some given $\lambda_0 \in \Lambda$, namely, for which

(2.8) $\quad \bigcap_{I \in \mathcal{U}} I = \{\lambda_0\} \neq \phi$

are not of interest here, since it is easy to see that their corresponding quotient fileds (2.6) are isomorphic with $\mathbb{R}$.

Consequently, we shall from now on start with a *free ultrafilter* $\mathcal{U}$ on Λ. Then, two essential properties of the ultrapower field $\mathbb{F}_{\mathcal{U}}$ corresponding to $\mathcal{U}$ according to (2.6), are as follows.

First, we have the *embedding*, that is, *injective homomorphism* of fields

(2.9) $\quad \mathbb{R} \ni r \longmapsto {}^*r = u(r) + \mathcal{I}_{\mathcal{U}} \in \mathbb{F}_{\mathcal{U}}$

where $u(r) \in \mathbb{R}^\Lambda$ is defined by $u(r) : \Lambda \ni \lambda \longmapsto r \in \mathbb{R}$, thus $u(r)$ is the *constant* function with value r. When there is no confusion, we shall write ${}^*r = r$, for $r \in \mathbb{R}$, thus (2.9) will be considered as an *inclusion* of fields, hence, with $\mathbb{R}$ being a *strict subfield* of $\mathbb{F}_{\mathcal{U}}$, namely

(2.10) $\quad \mathbb{R} \subsetneqq \mathbb{F}_{\mathcal{U}}$

The second property is that the ultrapower field $\mathbb{F}_\mathcal{U}$ has the following *linear order* that extends the usual linear order on $\mathbb{R}$. Given $\xi = (x_\lambda)_{\lambda \in \Lambda} + \mathcal{I}_\mathcal{U},\ \eta = (y_\lambda)_{\lambda \in \Lambda} + \mathcal{I}_\mathcal{U} \in \mathbb{F}_\mathcal{U}$, then

$$(2.11) \qquad \xi \ \leq_\mathcal{U} \ \eta$$

is defined by the condition

$$(2.12) \qquad \{ \ \lambda \in \Lambda \mid x_\lambda \leq y_\lambda \ \} \in \mathcal{U}$$

It follows that

$$(2.13) \qquad r \leq s \iff {}^*r \ \leq_\mathcal{U} \ {}^*s, \qquad r, s \in \mathbb{R}$$

also, for $\xi, \eta, \chi \in \mathbb{F}_\mathcal{U}$, we have

$$(2.14) \qquad \begin{array}{l} \xi \ \leq_\mathcal{U} \ \eta \implies \xi + \chi \ \leq_\mathcal{U} \ \eta + \chi \\ \\ \xi \ \leq_\mathcal{U} \ \eta, \ \ \chi \ \geq_\mathcal{U} \ 0 \implies \chi \xi \ \leq_\mathcal{U} \ \chi \eta \end{array}$$

And now, to the far more rich structure of the ultrapower fields $\mathbb{F}_\mathcal{U}$, when compared with the usual field $\mathbb{R}$ of real numbers, see [4-24]. This rich structure is due to the presence of infinitesimals, or infinitely small elements, and then, opposite to them, the presence of infinitely larger elements. And all that rich structure is contributing to the non-Archimedean structure of $\mathbb{F}_\mathcal{U}$.

Let us first extend the *absolute value* to the ultrapower fields $\mathbb{F}_\mathcal{U}$ as follows. Given $\xi = (x_\lambda)_{\lambda \in \Lambda} + \mathcal{I}_\mathcal{U} \in \mathbb{F}_\mathcal{U}$, then we define

$$(2.15) \qquad |\xi| = (|x_\lambda|)_{\lambda \in \Lambda} + \mathcal{I}_\mathcal{U} \in \mathbb{F}_\mathcal{U}$$

Now, each ultrapower field $\mathbb{F}_\mathcal{U}$ has the following three kind of subsets. The subset of *infinitely small*, or equivalently, *infinitesimal* elements is denoted by $Monad_\mathcal{U}(0)$. The subset of *finite elements* is denoted by $Galaxy_\mathcal{U}(0)$. And at last, the subset of *infinitely large* elements is $\mathbb{F}_\mathcal{U} \setminus Galaxy_\mathcal{U}(0)$.

We make now explicit the respective properties which define these three kind of elements, namely, infinitesimal, finite, and infinite.

Given $\xi = (x_\lambda)_{\lambda\in\Lambda} + \mathcal{I}_\mathcal{U} \in \mathbb{F}_\mathcal{U}$, then $\xi \in Monad_\mathcal{U}(0)$, if and only if

$$(2.16) \qquad \forall\, r \in \mathbb{R},\ r > 0 \ : \ |\xi| \ \leq_\mathcal{U} \ {}^*r$$

further, $\xi \in Galaxy_\mathcal{U}(0)$, if and only if

$$(2.17) \qquad \exists\, r \in \mathbb{R},\ r > 0 \ : \ |\xi| \ \leq_\mathcal{U} \ {}^*r$$

Obviously

$$(2.18) \qquad Monad_\mathcal{U}(0) \subsetneqq Galaxy_\mathcal{U}(0)$$

Finally, ξ is infinitely large, that is, $\xi \in \mathbb{F}_\mathcal{U} \setminus Galaxy_\mathcal{U}(0)$, if and only if

$$(2.19) \qquad \forall\, r \in \mathbb{R},\ r > 0 \ : \ |\xi| \ \geq_\mathcal{U} \ {}^*r$$

It is easy now to see that

Proposition 2.1.

None of the ultrapower fields $\mathbb{F}_\mathcal{U}$ is complete, thus each of them is non-Archimedean..

Proof.

Obviously the subset $Monad_\mathcal{U}(0) \subsetneqq \mathbb{F}_\mathcal{U}$ is bounded both from below and above, since, for instance, we have $Monad_\mathcal{U}(0) \subsetneqq [-1, 1]$. However, $Monad_\mathcal{U}(0)$ does *not* have either an infimum, or a supremum in $\mathbb{F}_\mathcal{U}$. Assume, on the contrary, that $\xi \in \mathbb{F}_\mathcal{U}$ is a supremum of $Monad_\mathcal{U}(0)$. Let us take any $\eta \in Monad_\mathcal{U}(0)$, such that $0 \ <_\mathcal{U} \ \eta \ <_\mathcal{U} \ \xi$. Now, if $\xi \in Monad_\mathcal{U}(0)$, then obviously $\xi + \eta \in Monad_\mathcal{U}(0)$, thus ξ is not the supremum of $Monad_\mathcal{U}(0)$, since $\xi \ <_\mathcal{U} \ \xi + \eta \in Monad_\mathcal{U}(0)$. On the other hand, if $\xi \notin Monad_\mathcal{U}(0)$, then clearly $\xi - \eta \notin Monad_\mathcal{U}(0)$, thus again, ξ is not the supremum of $Monad_\mathcal{U}(0)$, as $\chi \ <_\mathcal{U} \ \xi - \eta \ <_\mathcal{U} \ \xi$, for all $\chi \in Monad_\mathcal{U}(0)$.
The rest results from Corollary 1.2.

3. Infinite Sums in Ultrapower Fields

In pursuing our project mentioned in the Abstract, namely, to set up a Differential and Integral Calculus in ultrapower extensions $\mathbb{F}_{\mathcal{U}}$ of the usual field $\mathbb{R}$ of real numbers, and do so *without* the use of the Transfer Principle, or a topological type structure on $\mathbb{F}_{\mathcal{U}}$, in this paper our ultimate aim is to define Riemann type integrals for suitable functions

$$(3.1) \qquad f : [\alpha, \beta]_{\mathcal{U}} \longrightarrow \mathbb{F}_{\mathcal{U}}$$

where $[\alpha, \beta]_{\mathcal{U}} \subseteq \mathbb{F}_{\mathcal{U}}$ is an *interval* in the sense of (2.11), namely, $[\alpha, \beta]_{\mathcal{U}} = \{ \xi \in \mathbb{F}_{\mathcal{U}} \mid \alpha \leq_{\mathcal{U}} \xi \leq_{\mathcal{U}} \beta \}$ and $\alpha \neq \beta$. The fact that the intervals $[\alpha, \beta]_{\mathcal{U}}$ can have infinitely large end points α or β, may lead to Riemann sums with *infinitely* many terms

$$(3.2) \qquad \sum_{i \in I} (\delta_i - \gamma_i) f(\xi_i)$$

where $\alpha \leq_{\mathcal{U}} \gamma_i \leq_{\mathcal{U}} \xi_i \leq_{\mathcal{U}} \delta_i \leq_{\mathcal{U}} \beta$, and $\gamma_i \neq \delta_i$, while I is an infinite index set.

It is however convenient to pursue the issue of summation in (3.2) more generally, namely, to consider sums with *infinitely* many terms of the form

$$(3.3) \qquad S = \sum_{i \in I} a_i$$

thus where I may be an infinite index set, while $a_i \in \mathbb{F}_{\mathcal{U}}$, where the a_i themselves may possibly be *infinitely large*, in the sense of (2.19).

Obviously, in case the summation index set I in (3.3) is finite, then $S \in \mathbb{F}_{\mathcal{U}}$ is always well defined, regardless of the terms a_i being infinitely large, or not. Indeed, in this case S is simply the finite sum of certain terms in the field $\mathbb{F}_{\mathcal{U}}$, and as such it is always well defined, simply by the commutative group structure of $\mathbb{F}_{\mathcal{U}}$.

Thus the problem of summation in (3.3) only arises when the summa-

tion index set I in (3.3) is *infinite*.
However, we shall as well refer to the sums in (3.3) in general, that is, also in the trivial case when the index sets I are finite.

Let us now start with the particular case of (3.3), when the index set I is infinite, while we have

(3.4) $\quad a_i = a, \quad i \in I$

for a certain given

(3.5) $\quad a \in \mathbb{F}_{\mathcal{U}}$

Then intuitively, we may expect that in this particular case we would have S in (3.3) given by some kind of "product"

(3.6) $\quad S = (car I) \times a$

where $car I$ denotes the *cardinal* number of the *infinite* set I.

Obviously, the problem here is that the "product" in (3.6) is in general between two rather different entities, since a need not be a cardinal number, while on the other hand, $car I$ need not be an element of $\mathbb{F}_{\mathcal{U}}$. Thus this "product" does not make sense, unless each of its two factors is made to correspond in a suitable way to the same kind of entity, an entity which, furthermore, has to posses an operation of multiplication. And in our case $\mathbb{F}_{\mathcal{U}}$ seems to be a natural candidate in this regard, in view of (3.4), (3.5).

Further, we can note that even in the yet more particular case when we would take

(3.7) $\quad a = 1_{\mathbb{F}_{\mathcal{U}}} \in \mathbb{F}_{\mathcal{U}}$

in (3.6), thus seemingly eliminating the mentioned problem of "multiplication", we would still remain with the problem that in the resulting equality

$$(3.8) \qquad S = car I$$

the entity $S \in \mathbb{F}_{\mathcal{U}}$ given by (3.3), is in general supposed to have a different nature, than that of a cardinal number.

Remark 3.1.

As noted, the problem in ultrapower fields $\mathbb{F}_{\mathcal{U}}$ with sums (3.3) is obviously as follows. If the index set I is finite, then the respective sum - independently of its terms $a_i \in \mathbb{F}_{\mathcal{U}}$ - is always well defined, simply according to the group structure of $\mathbb{F}_{\mathcal{U}}$.
Therefore, the only problem with such sums is in the case of *infinite* index sets I.
Now, so far, there have only been two ways to deal with that situation in the literature.
One is to introduce some convergence, or in general, a topological type structure on $\mathbb{F}_{\mathcal{U}}$, and based on that, to define what it means when an infinite sum (3.3) *converges*.
The other way, used in Nonstandard Analysis, is to supplement the simple, elementary, purely algebraic, or more precisely, field structure of $\mathbb{F}_{\mathcal{U}}$, with a *Transfer Principle* which will automatically extend a variety of properties from finite to infinite structures, among them properties useful in defining Differential and Integral Calculus on $\mathbb{F}_{\mathcal{U}}$.

Here in this paper, however, we suggest a *third way* to define infinite sums (3.3), and do so *without* either topology, or Transfer Principle, and instead, we use a simple and natural way to overcome the mentioned difficulty inherent in such infinite sums.

$\square$

We note that the unit element in the ultrapower field $\mathbb{F}_{\mathcal{U}}$ is given by

$$(3.9) \qquad 1_{\mathbb{F}_{\mathcal{U}}} = (u_\lambda)_{\lambda \in \Lambda} + \mathcal{I}_{\mathcal{U}} \in \mathbb{F}_{\mathcal{U}}$$

where $u_\lambda = 1 \in \mathbb{R}$, for $\lambda \in \Lambda$. Let now

$$(3.10) \qquad \mathbb{Z}_{\mathbb{F}_{\mathcal{U}}} \subseteq \mathbb{F}_{\mathcal{U}}$$

be the *subring* in $\mathbb{F}_{\mathcal{U}}$ which consists from all the elements of the form

$$(3.11) \qquad \nu = (n_\lambda)_{\lambda \in \Lambda} + \mathcal{I}_{\mathcal{U}} \in \mathbb{F}_{\mathcal{U}}$$

where

$$(3.12) \qquad n_\lambda \in \mathbb{Z}, \quad \lambda \in \Lambda$$

In view of (2.11), we obviously have the property, see (2.15)

$$(3.13) \qquad \forall \; \alpha \in \mathbb{F}_{\mathcal{U}} \; : \; \exists \; \nu \in \mathbb{Z}_{\mathbb{F}_{\mathcal{U}}} \; : \; |\alpha| \; \leq_{\mathcal{U}} \; \nu$$

Let now - as one of the main ideas in the sequel - consider the index set I in (3.3) to be an *interval* of the form

$$(3.14) \qquad I = [\, i_{min}, i_{max} \,]_{\mathcal{U}} =$$

$$= \{ \; \nu \in \mathbb{Z}_{\mathbb{F}_{\mathcal{U}}} \mid i_{min} \; \leq_{\mathcal{U}} \; \nu \; \leq_{\mathcal{U}} \; i_{max} \; \} \subseteq \mathbb{Z}_{\mathbb{F}_{\mathcal{U}}}$$

for suitably given $i_{min}, i_{max} \in \mathbb{Z}_{\mathbb{F}_{\mathcal{U}}}, \; i_{min} \; \leq_{\mathcal{U}} \; i_{max}$, in the sense of the partial order $\leq_{\mathcal{U}}$ in (2.11) when restricted to $\mathbb{Z}_{\mathbb{F}_{\mathcal{U}}}$. Choosing in (3.3) again the particular case of (3.4), (3.5), (3.7), we obtain, see (3.9)

$$(3.15) \qquad S_I = \textstyle\sum_{i \in I} 1_{\mathbb{F}_{\mathcal{U}}}$$

Then, regarding (3.14), (3.15), it is natural to accept the following :

Transfer Principle Replacing Convention

$$(3.16) \qquad S_I = \textstyle\sum_{i \in I} 1_{\mathbb{F}_{\mathcal{U}}} = i_{max} - i_{min} + 1_{\mathbb{F}_{\mathcal{U}}} \in \mathbb{Z}_{\mathbb{F}_{\mathcal{U}}}$$

$\square$

We show in the sequel how the seemingly particular assumption (3.16) can nevertheless have convenient consequences. Among them, it offers the possibility of effective summation for a large variety of sums (3.3) with infinite index sets I. Thus (3.16) *compensates* for the fact that, neither any Transfer Principle, nor any convergence, or in general, topological type structures, are considered in this paper on the ultrapower fields $\mathbb{F}_{\mathcal{U}}$.

We shall need two further conventions as well.

First, let any sum (3.3) be given with an infinite index set I as in (3.14), while the terms a_i in (3.3), satisfy the boundedness condition

$$(3.17) \qquad \exists\, a', a'' \in \mathbb{F}_{\mathcal{U}} \;:\; a' \;\leq_{\mathcal{U}}\; a_i \;\leq_{\mathcal{U}}\; a'', \quad i \in I$$

Let us then accept as well the following :

Summation Convention 1

$$(3.18) \qquad a'(\textstyle\sum_{i\in I} 1_{\mathbb{F}_{\mathcal{U}}}) = a' S_I \;\leq_{\mathcal{U}}\; S = \sum_{i\in I} a_i \;\leq_{\mathcal{U}}$$

$$\leq_{\mathcal{U}}\; a'' S_I = a''(\textstyle\sum_{i\in I} 1_{\mathbb{F}_{\mathcal{U}}})$$

Remark 3.2.

The relations (3.16) and (3.18) do *not* necessarily mean that the entity $S = \sum_{i\in I} a_i$ is defined, and even less, that it is defined as an element of $\mathbb{F}_{\mathcal{U}}$.

In (3.18), the entity $S_I = \sum_{i\in I} 1_{\mathbb{F}_{\mathcal{U}}} \in \mathbb{F}_{\mathcal{U}}$ is defined in view of (3.16), and therefore, are also defined the entities $a'(\sum_{i\in I} 1_{\mathbb{F}_{\mathcal{U}}}) = a' S_I,\ a'' S_I = a''(\sum_{i\in I} 1_{\mathbb{F}_{\mathcal{U}}}) \in \mathbb{F}_{\mathcal{U}}$.

The meaning of entities such as $S = \sum_{i\in I} a_i$ in (3.3), (3.16) or (3.18) will be clarified in section 5 in the sequel.

□

Now based on the above, let us denote by

$$(3.19) \qquad \mathcal{S}_{\mathcal{U}}$$

the set of *finite* or *infinite* sums (3.3) for which condition (3.14) holds.

Clearly, in view of Remark 3.2., the set $\mathcal{S}_{\mathcal{U}}$ need not necessarily be a subset of $\mathbb{F}_{\mathcal{U}}$, since it may contain undefined sums like those in (3.3) and which have infinite index sets I.

Further, for every such sum $S = \sum_{i \in I} a_i \in \mathcal{S}_{\mathcal{U}}$ with finite or infinite index set I, let us denote by

$$(3.20) \qquad \mathcal{S}_{\mathcal{U}}(S)$$

the set of corresponding *finite* families $\Sigma = (S(I_1) = \sum_{i_1 \in I_1} a_{i_1}, \ldots, S(I_m) = \sum_{i_m \in I_m} a_{i_m})$, of sums $S(I_k) = \sum_{i_k \in I_k} a_{i_k}$, with $m \in \mathbb{N}$ and $1 \leq k \leq m$, such that

(3.21) each of $I_1, \ldots, I_m$ is an interval of the form (3.14), and they constitute together a *partition* of I

Obviously, each I_k, with $1 \leq k \leq m$, can be finite or infinite, and we also have

$$(3.22) \qquad S(I_k) = \sum_{i_k \in I_k} a_{i_k} \in \mathcal{S}_{\mathcal{U}}, \quad 1 \leq k \leq m$$

Given now a sum $S = \sum_{i \in I} a_i \in \mathcal{S}_{\mathcal{U}}$ with an infinite index set I, and a corresponding family of sums $\Sigma = (S(I_1) = \sum_{i_1 \in I_1} a_{i_1}, \ldots, S(I_m) = \sum_{i_m \in I_m} a_{i_m}) \in \mathcal{S}_{\mathcal{U}}(S)$, then obviously at least one of the index sets $I_k \subseteq \mathbb{Z}_{\mathbb{A}_{\mathcal{U}}}$ is infinite, thus the corresponding sum $S(I_k)$ need not necessarily be defined as an element of $\mathbb{F}_{\mathcal{U}}$.

Let us, therefore, accept also the following :

Summation Convention 2

$$(3.23) \qquad S = S(I_1) + \ldots + S(I_m)$$

Remark 3.3.

As above in (3.18), the relation (3.23) does *not* necessarily mean that the entity $S = S(I_1) + \ldots + S(I_m)$ is defined, and even less, that it is defined as an element of $\mathbb{F}_{\mathcal{U}}$. On the other hand, those terms $S(I_k)$ in S in (3.23) which correspond to finite index sets I_k, are well defined as elements of $\mathbb{F}_{\mathcal{U}}$, according to the structure of Σ in the definition (3.20).

$\square$

Now suppose given any $S = \sum_{i \in I} a_i \in \mathcal{S}_{\mathcal{U}}$ and $\Sigma = (S(I_1) = \sum_{i_1 \in I_1} a_{i_1}, \ldots, S(I_m) = \sum_{i_m \in I_m} a_{i_m}) \in \mathcal{S}_{\mathcal{U}}(S)$. Then for each $1 \leq k \leq m$, the sum $S(I_k)$ can be associated with its corresponding particular version in (3.15), namely

$$(3.24) \qquad S_{I_k} = \textstyle\sum_{i_k \in I_k} 1_{\mathbb{F}_{\mathcal{U}}}$$

Thus assuming that, see (3.14)

$$(3.25) \qquad I_k = \{ \ \nu \in \mathbb{Z}_{\mathbb{F}_{\mathcal{U}}} \mid i_{k,min} \ \leq_{\mathcal{U}} \ \nu \ \leq_{\mathcal{U}} \ i_{k,max} \ \} \subseteq \mathbb{Z}_{\mathbb{F}_{\mathcal{U}}}$$

then (3.16) implies

$$(3.26) \qquad S_{I_k} = i_{k,max} - i_{k,min} + 1_{\mathbb{F}_{\mathcal{U}}} \in \mathbb{F}_{\mathcal{U}}$$

and in view of (3.16), (3.23) we obtain the equality of usual two *finite* sums computed in $\mathbb{F}_{\mathcal{U}}$, namely

$$(3.27) \qquad i_{max} - i_{min} + 1_{\mathbb{F}_{\mathcal{U}}} = \textstyle\sum_{1 \leq k \leq m} (i_{k,max} - i_{k,min} + 1_{\mathbb{F}_{\mathcal{U}}})$$

which is correct, due to (3.14), (3.21).

In particular, from (3.14), (3.21), we obviously obtain

$$(3.28) \qquad i_{k,max} + 1_{\mathbb{F}_{\mathcal{U}}} = i_{k+1,min}, \qquad 1 \leq k \leq m-1$$

Now, similar with (3.17), let us assume for each $1 \leq k \leq m$ that

$$(3.29) \qquad \exists \ a'_k, a''_k \in \mathbb{F}_{\mathcal{U}} \ : \ a'_k \ \leq_{\mathcal{U}} \ a_{i_k} \ \leq_{\mathcal{U}} \ a''_k, \quad i_k \in I_k$$

In view of (3.18), (3.20), we obtain for each $S(I_k) = \sum_{i_k \in I_k} a_{i_k} \in \mathcal{S}_{\mathcal{U}}$, with $1 \leq k \leq m$, that

$$(3.30) \qquad a'_k S_{I_k} \;\leq_{\mathcal{U}}\; S(I_k) \;\leq_{\mathcal{U}}\; a''_k S_{I_k}, \quad 1 \leq k \leq m$$

hence (3.18), (3.23) give

$$(3.31) \qquad \sum_{1 \leq k \leq m} a'_k S_{I_k} \;\leq_{\mathcal{U}}\; S =$$

$$= S(I_1) + \ldots + S(I_m) \;\leq_{\mathcal{U}}\; \sum_{1 \leq k \leq m} a''_k S_{I_k}$$

and here, both sums at the left and right extremes, respectively, are well defined elements of $\mathbb{F}_{\mathcal{U}}$, in view of (3.16).

Let us now consider for every given sum $S = \sum_{i \in I} a_i \in \mathcal{S}_{\mathcal{U}}$, the corresponding sets

$$(3.32) \qquad \begin{aligned} L_{\mathbb{F}_{\mathcal{U}}}(S) &= \left\{ s \in \mathbb{F}_{\mathcal{U}} \;\middle|\; \begin{array}{l} \exists\ s' = \sum_{1 \leq k \leq m} a'_k S_{I_k} \text{ in } (3.19) - (3.31) : \\ \\ s \leq_{\mathcal{U}} s' \end{array} \right\} \\ R_{\mathbb{F}_{\mathcal{U}}}(S) &= \left\{ s \in \mathbb{F}_{\mathcal{U}} \;\middle|\; \begin{array}{l} \exists\ s'' = \sum_{1 \leq k \leq m} a''_k S_{I_k} \text{ in } (3.19) - (3.31) : \\ \\ s'' \leq_{\mathcal{U}} s \end{array} \right\} \end{aligned}$$

called, respectively, the *lower* and *upper associated approaximations* with the sum $S = \sum_{i \in I} a_i \in \mathcal{S}_{\mathcal{U}}$.

Obviously, in view of (3.31), we have

$$(3.33) \qquad \forall\ s' \in L_{\mathbb{F}_{\mathcal{U}}}(S),\ s'' \in R_{\mathbb{F}_{\mathcal{U}}}(S) \ :\ s' \leq_{\mathcal{U}} s''$$

Then, naturally, we can give the following

Definition 3.1. Riemann Summation

A sum $S = \sum_{i \in I} a_i$ in (3.3), which belongs to $\mathcal{S}_{\mathcal{U}}$, is called *Riemann sumable*, if and only if there exist a *unique* $\widetilde{S} \in \mathbb{F}_{\mathcal{U}}$, such that

$$(3.34) \qquad \forall\ s' \in L_{\mathbb{F}_{\mathcal{U}}}(S),\ s'' \in R_{\mathbb{F}_{\mathcal{U}}}(S) \ :\ s' \leq_{\mathcal{U}} \widetilde{S} \leq_{\mathcal{U}} s''$$

in which case $\widetilde{S} \in \mathbb{F}_{\mathcal{U}}$ is called the *Riemann sum* of the sum $S = \sum_{i \in I} a_i \in \mathcal{S}_{\mathcal{U}}$, and one writes

$$(3.35) \qquad (R)S = (R) \sum_{i \in I} a_i = \widetilde{S}$$

Now let us turn - based on the above treatment of general sums (3.3) - to the definition of the *Riemann Integral* itself, in the case of suitable functions f in (3.1). For that purpose, we have to specify both the appropriate functions f in (3.1), and the corresponding families of sums (3.2), with the latter being obviously particular cases of sums (3.3), namely

Definition 3.2. Riemann Type Sums

A sum (3.2) is called *Riemann type sum*, if and only if

$$(3.36) \qquad [\,\alpha, \beta\,]_{\mathcal{U}} = \bigcup_{i \in I} [\,\gamma_i, \delta_i\,]_{\mathcal{U}}$$

and for $i, j \in I$, $i \neq j$, we have that

$$(3.37) \qquad [\,\gamma_i, \delta_i\,]_{\mathcal{U}} \cap [\,\gamma_j, \delta_j\,]_{\mathcal{U}} \text{ is a set of at most one point}$$

$\square$

Obviously, if $[\,\gamma_i, \delta_i\,]_{\mathcal{U}} \cap [\,\gamma_j, \delta_j\,]_{\mathcal{U}}$ is a set of one point, then it is either the one point set $\{\,\delta_j\,\} = \{\,\gamma_i\,\}$, or the one point set $\{\,\delta_i\,\} = \{\,\gamma_j\,\}$.

Let us now consider any function $f : [\,\alpha, \beta\,]_{\mathcal{U}} \longrightarrow \mathbb{F}_{\mathcal{U}}$ in (3.1). With the help of (3.32), we consider the following associated *lower*, respectively, *upper* sums

$$(3.38) \qquad \begin{aligned} L_{\mathbb{F}_{\mathcal{U}}}(f) &= \bigcup L_{\mathbb{F}_{\mathcal{U}}}(S) \\ R_{\mathbb{F}_{\mathcal{U}}}(f) &= \bigcup R_{\mathbb{F}_{\mathcal{U}}}(S) \end{aligned}$$

where each of the two unions is taken over all *Reimann type sums* S associated with the function f according to (3.2).

Now of course, we can give

Definition 3.3. Riemann Integral

A function f in (3.1) is *Riemann integrable*, if and only if there exist a *unique* $\widetilde{S} \in \mathbb{F}_{\mathcal{U}}$, such that

$$\forall\ s' \in L_{\mathbb{F}_{\mathcal{U}}}(f),\ s'' \in R_{\mathbb{F}_{\mathcal{U}}}(f)\ :\ s' \leq_{\mathcal{U}} \widetilde{S} \leq_{\mathcal{U}} s'' \tag{3.39}$$

in which case $\widetilde{S} \in \mathbb{F}_{\mathcal{U}}$ is called the *Riemann integral* of the function f, and one writes

$$(R) \int_{[\,\alpha,\beta\,]_{\mathcal{U}}} f(x)dx = \widetilde{S} \tag{3.40}$$

Remark 3.4.

The above, obviously, has to be supplemented with a clarification of the status of the three conventions, namely, the one in (3.16) replacing the Transfer Principle, and the two summation conventions in (3.18) and (3.23).

4. On the Compatibility of the New Axioms

As mentioned in Remark 3.4., one has to clarify the situation with what amounts to no less than *three axioms* introduced in (3.16), (3.18) and (3.23), and used in an essential manner in the definition of the Riemann Integral given in Definition 3.3.

One easy way in this regard, although an indirect one, is as follows. One can try to show to what extent the respective relations (3.16), (3.18) and (3.23) may indeed hold in usual Nonstandard Analysis, [1,2], or in the Internal Set Theory, [3]. And in case they do hold in either of the mentioned two theories, then clearly, the relations (3.16), (3.18) and (3.23), seen as axioms, are *compatible* we the Zermelo-Fraenkel Set Theory.

Alternatively, there is the possibility that the relations (3.16), (3.18) and (3.23) are not compatible with the Zermelo-Fraenkel Set Theory.

In such a case however, it is important to note that there is *no* need, and even *less* any obligation to discard them altogether from the very beginning.
Indeed, all these three relations (3.16), (3.18) and (3.23) appear to be *natural*, thus quite likely, *not* leading too soon to contradictions. And thus, it makes sense to develop the theory of Riemann Integration introduced in section 3, and do so according to recent developments in Inconsistent Mathematics, [25].

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