RELATIONS PROPERTIES COMPATIBILITY

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ABSTRACT. It was shown here that just transitive relation may be considered as closure and its presence is necessary and sufficient to order a set linearly, and it is not possible to do this by using other relation's property – neither reflexivity nor symmetry. By interaction, it occurs due to ambiguity of their definition – it was shown earlier (ref. [3]) they have various appearances. Among them just the only transitivity is determined uniquely. At the same time the last one doesn't exist separately from any others. Circumstances of their joint existence are clarifying in this article.

Contents

1. Something about "reflexivity" of tolerances

One may think that reflexivity and symmetry are necessary and sufficient conditions to appear transitivity. However, often *tolerances* are determined as reflexive and symmetric relations which transitivity is not required (e.g., according to [1]). Meanwhile, P. M. Cohn [2] mentioned[1] that similar relations are intransitive and even they are not reflexive. To understand who of them was right there may be applied general method that was developed in my earlier article [3].

First of all, talking about relations, it was meant that they are established at least on the binary Cartesian product – rather Cartesian square. But firstly, it needs to determine lower Cartesian powers[2]. So, the *first Cartesian power* is just a non-empty disordered set as it is commonly determined

$$A^1 = A = \{x : x \in A\} \neq \emptyset. \tag{1.1}$$

This is disjoint union of disordered "pairs"

$$\langle a \rangle = \{a, a\} = \{a\}. \tag{1.2}$$

[1] But he did not advance an argument to prove it. Perhaps it is based on empirical observation. Who knows?
[2] Obviously, another kind of Cartesian product simply doesn't exist – it will be proved a little bellow.

They are usual one-element sets[3] but written in terms of Cartesian products. As far as all of those "pairs" are exclusively diagonal, there wouldn't be another Cartesian product but a kind of power. *Nullary Cartesian power* just coincides with Boolean of empty set

$$A^0 = 2^{\emptyset} = \{\{\,\}\} = \{\emptyset\}. \tag{1.3}$$

That's why it is not empty.

Such definitions allow determining relations of lower *arity*. They differ from binary ones just by randomness. Since the first Cartesian power may be represented by set union, its subset – *unary relation* – may be represented by set intersection

$$(A^1 = A \cup A = A \cup I \supseteq A \cap I = I) \leftarrow\!\!\dashv (A \supseteq I). \tag{1.4}$$

Here, idempotent set union plays the role of Cartesian "square", and – of relation's one – its intersection with self-subset I. Due to commutativity of these actions, such relations are exclusively symmetrical. Any "pairs" compiling such relations are exclusively diagonal

$$\{i, i\} = \{i\} = \{i : i \in A \cap I\} = I.$$

"Diagonal" of such set is determined by the expression

$$\mathrm{id}_I = \{i : \{i, x\} \in I \wedge \{x, i\} \in I\}. \tag{1.5}$$

Such relations seem to be established reflexive, implying their following transitivity. But the left part of transitivity condition (inclusion (3.4.1)[4]), due to "pair's" randomness, yet, may be simplified by elements x

$$\{x : \{x, i\} \in I \wedge \{i, x\} \in I\} \supseteq \{i : i \in I\}. \tag{1.6}$$

But this inclusion has the opposite direction that occurs while one deals with transitive relation. To clarify the reason of the phenomenon, it may be noticed that formula (1.5) is not unique definition of this one. There exists another one that is not quite equivalent to the previous definition. Because a set A doesn't consists any elements but just diagonal's disordered "pairs", it totally coincides with diagonal and inclusion may be written as

$$A = \{x\} = \mathrm{id}_A \supseteq \mathrm{id}'_A = \mathrm{id}_I = \{i\} = I. \tag{1.7}$$

Such definition lets conclude that relation is contained inside diagonal is pseudo-reflexive. So, no one could state that such relations are equivalences – due to identity's definition ambiguity of such sets. Apparently, such *pseudo-equivalences* may be called *tolerances*. Inclusion direction in formula (1.6) doesn't correspond to conjunction's one – instead of this followings occur

[3] Initial letters of Latin alphabet (e.g. a, b, c) denote here fixed sets' elements, the final ones (e.g. x, y, z) -- any element of sets running all of the meanings. For example, they may be connected by formula $\{x\} = \{a, b, \dots, c\}$. So, disordered sets are perfectly defined by one of their elements. All equipotential disordered sets are isomorphic each other.

[4] Below in all of strange numerical notations the 1st number will mean source in reference list. E.g., notation (3.4.1) means formula (4.1) from the source [3].

$$\left.\begin{array}{l}\{x:\{x,j\}\wedge\{k,x\}\}=\{x:\{j,x\}\wedge\{x,k\}\}\not\subseteq\{j,k\}\\ \{x:\{x,j\}\wedge\{k,x\}\}=\{x:\{j,x\}\wedge\{x,k\}\}\not\supseteq\{j,k\}\end{array}\right\}. \tag{1.8}$$

These formulae express intransitivity of tolerances. Among subsets of set A obligatorily exists those that equal to null. Choosing it as index set, there may be written for such tolerance

$$\left.\begin{array}{l}(B\cap I=\emptyset)\wedge(C\cap I=\emptyset)\nleftrightarrow(B\cap C=\emptyset)\\ (B\cap I\neq\emptyset)\wedge(C\cap I\neq\emptyset)\nleftrightarrow(B\cap C\neq\emptyset)\end{array}\right\}. \tag{1.9}$$

In addition, *arity* of tolerance is not so certain. On the one hand, relation arity is determined by multiplicands' quantity of Cartesian power – in this sense it is the unary relation. But, on the other hand, this is a correspondence between two sets. They are different, in spite of obligatory inclusion between them, so, this is binary.

The only relation established on empty set is equivalence anyway – its only improper subset coincides with its diagonal[5].

2. Dependence of reversibility on the other relation's properties

However, stated reasons are rather similar to some empirical statement of transitivity deficiency that, as a matter of fact, doesn't expose an origin of phenomenon than explains this. To do it there may be noticed following. As it has been shown earlier [3] relation definition in dependence on its kind of symmetry relation intersection with its reversal one plays the major role than relation of its own. It makes to doubt in the fact of transitivity, symmetry and reflexivity keeping while relations go through some activity[6]. E.g., the definition (3.4.9) contains such intersection but it is the part of diagonal and, anyway, it is pseudo-reflexive still remaining symmetrical. It's too hard to understand without knowledge of what anti-symmetric relation would be during reversion. But starting from asymmetrical relations there may be shown that their intersection with any other relation is asymmetrical too. According to formula (3.4.21), assuming intersection's commutativity and associativity, there may be written

$$(\mu\cap\nu)\cap(\mu\cap\nu)^{-1}=\mu\cap\mu^{-1}\cap\nu\cap\nu^{-1}. \tag{2.1}$$

But it confirms just relation's anti-reflexivity. Due to the relation $\mu^{-1}=\neg\mu$ is ambiguous; its intersection with another relation is empty. Therefore, there may be written

$$(\mu^{-1}=\neg\mu)\mapsto[(\mu\cap\nu)^{-1}=\emptyset]. \tag{2.2}$$

[5] Such certainty is something akin to that natural numbers' series (which are linearly ordered) generation may be obtained by sequential ordering of empty set. Perhaps, it may be a cause of their linear order. As it will be shown below, there is no such certainty of identity definition in the most other cases – in previous article [3] double-sided diagonal was called "almost" identity non-accidentally.

[6] Terms operations and actions aren't used here for a while because they have concrete algebraic meaning. But sets differ from algebraic structures just by operation's definition in last ones. As it widely known (e.g. from [2]) binary operation is ternary relation that established at a set transforming it into algebraic structure (group, module etc.). If it is transitive there might talk about algebraic closure. But this is the matter of another research.

And already it proves the statement about asymmetry of such intersection. But union with any relation may be not asymmetric – even when relation μ^{-1} is empty, there may be written

$$(\mu \cup \nu) \cap (\mu \cup \nu)^{-1} = (\mu \cap \nu^{-1}) \cup (\mu^{-1} \cap \nu).$$

Anyway, the 2nd operand in the right part is empty, but there may not say the same about the 1st one. The 1st one is empty when it is satisfied to implication

$$(\mu \cap \nu^{-1} = \emptyset) \mapsto [(\mu \cup \nu) \cap (\mu \cup \nu)^{-1} = \emptyset]. \tag{2.3}$$

Notably, sets μ and ν^{-1} are disjoint independently on set ν^{-1} is empty or not. In other cases it is not asymmetric due to it is not anti-reflexive.

There may be done similar discourses on the union and intersection of anti-symmetric relations. But firstly, it needs to be done preliminary notes. It's easy to show that union of diagonal or its fragment with anti-symmetric relation is reflexive or pseudo-reflexive with keeping of anti-symmetry of a result. Because union and intersection are distributive there may be written formula

$$(\alpha \cup \mathrm{id}) \cap (\alpha \cup \mathrm{id})^{-1} = (\alpha \cap \alpha^{-1}) \cup \mathrm{id}. \tag{2.4}$$

It coincides with one of the definitions (3.4.9) confirming by this anti-symmetry of the result. Coincidence with equality (3.4.13) confirms its reflexivity. Apparently, it occurs for union with non-empty diagonal's fragment $\tilde{\alpha}$ too, that may be written as

$$\left.\begin{array}{c} \tilde{\alpha} \cap \tilde{\alpha}^{-1} \subset (\alpha \cap \alpha^{-1}) \cup \mathrm{id} = \mathrm{id} \\ \tilde{\alpha} \cap \mathrm{id} \neq \emptyset \\ \tilde{\alpha} \not\subseteq \mathrm{id} \\ \tilde{\alpha} \not\supseteq \mathrm{id} \end{array}\right\}. \tag{2.5}$$

This inclusion coincides with one of the definitions (3.4.9) confirming pseudo-reflexivity of such union. Using pseudo-reflexivity of initial and final relations, there may be written equalities

$$(\alpha \cap \xi) \cap (\alpha \cap \xi)^{-1} = (\alpha \cap \alpha^{-1}) \cap (\xi \cap \xi^{-1}) \subseteq \mathrm{id} \cap \mathrm{id} = \mathrm{id}. \tag{2.6}$$

As it was in asymmetry case, anti-symmetry may not be keeping at a union. Reason by analogy leads to expression

$$(\alpha \cup \xi) \cap (\alpha \cup \xi)^{-1} = (\alpha \cap \alpha^{-1}) \cup (\xi \cap \xi^{-1}) = (\alpha \cap \xi^{-1}) \cup (\xi \cap \alpha^{-1}). \tag{2.7}$$

If relation ξ is asymmetric, then it is the case described by equality (2.3), but if it is anti-symmetric, then it depends on reversion features of pseudo-reflexive relation.

Talking about reversibility in dependence on kind appearance of reflexivity there may be noticed that, indeed, intersection of pseudo-reflexive anti-symmetric relation with its reversal one is irreversible. In fact, reversing intersection $\tilde{\alpha} \cap \tilde{\alpha}^{-1} \subset \mathrm{id}$, it will lead to the one, that result should be reflexive

$$(\tilde{\alpha} \cap \tilde{\alpha}^{-1})^{-1} = \alpha^{-1} \cap \alpha \supset \mathrm{id}.$$

However, it would contradict to statement of intersection commutativity. Reversing here is provided by pairs' elements permutation and in logical terms[7] it looks like

$$\left.\begin{array}{c}\langle x,j\rangle\in\tilde{\alpha}\wedge\langle x,k\rangle\in\tilde{\alpha}^{-1}\nRightarrow\langle j,x\rangle\in\tilde{\alpha}^{-1}\wedge\langle k,x\rangle\in\tilde{\alpha}\\ \tilde{\alpha}\cap\tilde{\alpha}^{-1}\nsubseteq(\tilde{\alpha}\cap\tilde{\alpha}^{-1})^{-1}\end{array}\right\}. \tag{2.8}$$

Because pseudo-reflexive anti-symmetric relation is represented by direct sum with diagonal's fragment, then this object is irreversible.

Returning to the matter of relation transitivity in dependence on its kind of symmetry appearance, there may be written for asymmetric relation

$$\left.\begin{array}{c}\mu\cap\mu^{-1}=\emptyset\\ \mu\cap\mu=\mu\end{array}\right\}.$$

Also, these expressions may be written as something like that

$$\left.\begin{array}{c}\langle x,j\rangle\in\mu\wedge\langle x,k\rangle\notin\mu^{-1}\Rightarrow(j\neq k)\\ \langle x,j\rangle\in\mu\wedge\langle x,k\rangle\in\mu\Rightarrow(j=k)\end{array}\right\}.$$

Elements' permutation in 2nd operand of 1st of these expressions leads to transfer from commutative intersection to non-commutative multiplication

$$\left.\begin{array}{c}\langle x,j\rangle\in\mu\wedge\langle j,x\rangle\notin\mu^{-1}\Rightarrow\langle x,x\rangle\notin\mathrm{id}\\ \mu\circ\mu^{-1}=\mu\circ\neg\mu=\emptyset\end{array}\right\}.$$

Changing of intersection by multiplication in the idem-potency condition leads to formulae

$$\left.\begin{array}{c}\langle x,j\rangle\in\mu\wedge\langle k,x\rangle\in\mu\nRightarrow\langle j,k\rangle\in\mu\\ \mu\circ\mu\nsubseteq\mu\end{array}\right\}. \tag{2.9}$$

It points to intransitivity of asymmetric relation – just due to elements' permutation in pairs of asymmetric relations is impossible, the left part of this is non-commutative.

There may be written something like that for anti-symmetric relations

$$\left.\begin{array}{c}\emptyset\subseteq\alpha\cap\alpha^{-1}\subseteq\mathrm{id}\\ \alpha\cap\alpha=\alpha\end{array}\right\}.$$

Elements' permutation in 2nd pair in the left part of the 1st expression (2.8) may be written as

$$\langle x,j\rangle\in\alpha\wedge\langle k,x\rangle\in\alpha.$$

If such relation is reversible, there may be written sequence

$$\langle x,j\rangle\in\alpha\wedge\langle k,x\rangle\in\alpha\Rightarrow\langle k,x\rangle\in\alpha^{-1}\wedge\langle x,j\rangle\in\alpha^{-1}\Rightarrow\langle k,j\rangle\in\alpha^{-1}.$$

And it is already plain to see that only anti-reflexive anti-symmetric relation or its direct sum with diagonal is transitive. Summarizing, there may be obtained criterion for anti-symmetric relation to be transitive

[7] It means the usage of conjunction and disjunction instead of signs cap (∩) and cup (∪).

$$\left.\begin{array}{l}(\alpha \cap \alpha^{-1} = \emptyset) \mapsto (\alpha^2 \subset \alpha) \\ (\alpha \cap \alpha^{-1} = \mathrm{id}) \mapsto (\alpha^2 = \alpha)\end{array}\right\}. \tag{2.10}$$

As a simple example the case of proper inclusion may be observed. Transitivity condition here is written as

$$(A \subset B) \wedge (B \subset C) \Rightarrow (A \subset C). \tag{2.11}$$

Composition of relations is described here, but it is not their intersection, which is empty due to their anti-reflexivity.

3. Equivalences' negative powering features

Natural powering was defined, at least, for relations. But it's too hard to say definitely about negative powering and its coincidence with their reversing even for relations, in spite of symbol similarity for these activities. General property (3.3.16) is not fulfilled in most cases – due to inequality (3.3.12) occurs

$$\beta^{n+1} \circ \beta^{-1} \neq \mathrm{id}_A \circ \beta^n. \tag{3.1}$$

But there is exclusion from this rule for multiplicative idempotent. Only in this case one may talk about such powering and its coincidence with reversing. Nevertheless, regardless non-commutativity of anti-symmetric relations' multiplication, composition $\alpha \circ \alpha^{-1}$ is equivalence – it may be powered in natural degree and general property (3.4.2) is inherent to it. Anyway for both of it and any of its own natural power there may be written sequence of transitivity conditions

$$\mathrm{id} \subseteq \varepsilon^{n+1} \subseteq \varepsilon^n \subseteq \varepsilon^{n-1} \subseteq \cdots \subseteq \varepsilon \subseteq A^2. \tag{3.2}$$

Therefore, one might say that equivalences may have proper subsets – they are equivalences too. But it could be powered only as integration – "associativity" here is inappropriate, because "degree" α^{-n} is undetermined

$$[(\alpha^{-1})^n \neq \alpha^{-n}] \mapsto [\varepsilon^n = (\alpha \circ \alpha^{-1})^n \neq \alpha^n \circ \alpha^{-n}]. \tag{3.3}$$

It doesn't coincide with reversal relation powering. Even when both multiplicands are equivalences it is not always possible due to absence of commutativity

$$\left.\begin{array}{l}\alpha^{n+1} \circ \alpha^{-1} = \alpha^n \circ \alpha \circ \alpha^{-1} \supset \alpha^n \circ \mathrm{id} = \alpha^n \\ \alpha^{-1} \circ \alpha^{n+1} = \alpha^{-1} \circ \alpha \circ \alpha^n = \mathrm{id} \circ \alpha^n = \alpha^n\end{array}\right\}. \tag{3.4}$$

All the more, it is not possible when one of compositions is tolerant

$$\left.\begin{array}{l}\alpha^{n+1} \circ \alpha^{-1} = \alpha^n \circ \alpha \circ \alpha^{-1} = \alpha^n \circ \mathrm{id} = \alpha^n \\ \alpha^{-1} \circ \alpha^{n+1} = \alpha^{-1} \circ \alpha \circ \alpha^n \subset \mathrm{id} \circ \alpha^n = \alpha^n\end{array}\right\}. \tag{3.5}$$

Anyway there may be written inclusions' sequence

$$\tau \cap \tau^{-1} \subseteq \mathrm{id} \subseteq \tau \cup \tau^{-1} \subseteq A^2. \tag{3.6}$$

It determines *partial* order relation. But among varieties of the last of these inclusions equality is possible for some relation $\hat{\tau}$

$$\tau \cup \tau^{-1} = A^2. \tag{3.7}$$

In this case set is appeared to be *fully* indicated and ordered – this order is *linear*. Thus, any relation is subset of this relation

$$\rho \subseteq \hat{\tau}. \tag{3.8}$$

It may be called set's *transitive closure*. Asymmetric relation is not observed as a candidate to play the role. There may be shown that this role can play only reflexive anti-symmetric and transitive relations – they are called *order* relations.

Talking about Cartesian product of its own as binary relation, one may say that this is surely equivalence. Anyway there may be written for this

$$\left.\begin{array}{r} A^2 \supseteq \mathrm{id} \\ A^2 = (A^2)^{-1} \\ A^2 \circ A^2 = A^2 \end{array}\right\}. \tag{3.9}$$

The same as in formula (3.2.9), defining diagonal, here transitivity condition may be interpreted like idempotent intersection of sets A^2, but as opposed to diagonal this activity is not distributive with "intersection"[8] and that's why it is not associative

$$(A \times A) \circ (A \times A) \neq A \times A \cap A \times A. \tag{3.10}$$

Square of Cartesian square as a binary relation surely differs from 4th Cartesian power (obviously, the last one is *quaternary* relation) by using composition forming activity "∘" but not by Cartesian square "×" between these multiplicands. Although, by analogy with intersection, idem-potency of such composition lets write expression that is similar to formula (3.3.14)

$$\rho \circ A^2 = A^2 \circ \rho = \rho. \tag{3.11}$$

Therefore, as a matter of fact, diagonal is not unique identity that may be established on a set. It explains the reason of usage adverb "almost[9]". Any equivalence differs from diagonal usually is called *non-trivial*, and one differs from Cartesian square – is *proper*.

References

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[8] As any intersection is not surely distributive with relations' multiplication. But this statement is not supplied by proof here.

[9] If there could say so, anyway, in this situation.