Fermat prime criterion related to Landau's fourth problem

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Abstract

In this paper we consider the infinity of primes represented by quadratic polynomial with $4(M_p–2)^2+1$, basing on a hypothesis as sufficient condition in which Fermat primes are criterion for the infinity of such primes, where M_p is Mersenne prime, and give an elementary argument for existence of infinitely many primes of the form x^2+1 because of such primes to be a subset of primes of the form x^2+1. As an addition, an elementary argument on the infinity of Mersenne primes is also given.

Keywords Mersenne prime · Fermat prime criterion · primes represented by quadratic polynomial with $4(M_p–2)^2+1$ · primes of the form x^2+1

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1. Introduction

Are there infinitely many primes of the form x^2+1? It has been an unsolved problem in mathematics. Landau listed it as the last one of four basic problems about primes at the 1912 ICM. H. Iwaniec showed that there are infinitely many numbers of the form x^2+1 with at most two prime factors in 1978[1,2]. A theorem proved in 1997 by J. Friedlander and H. Iwaniec shows that there are infinitely many primes of the form x^2+y^4[3]. However, such two results do not imply that there is an infinite number of primes of the form x^2+1. In our previous work[4] we posed a conjecture that there is at least one prime factor $(k^{1/2}\, 2^{a/2})^2+1$ of Fermat number for $F_n–1 \le a < F_{n+1}–1$ (n = 0,1,2,3,…), where $k^{1/2}$ is odd positive integer and a is even positive integer (F_n is Fermat number). Some of known evidences support the conjecture to hold till $a < F_{4+1} –1 = 4294967296$. Since such prime factors of Fermat numbers are a subset of primes of the form x^2+1, if the conjecture is proven then primes of the form x^2+1 are infinite. Similarly, in this paper we will consider the infinity of primes to be a subset of primes of the form x^2+1, which are represented by quadratic polynomial with $4(M_p–2)^2+1$, and if corresponding Fermat prime criterion is acceptable as a hypothesis then an elementary argument on the infinity of primes of the form x^2+1 will be given.

2. Primes represented by quadratic polynomials of Mersenne primes

Definition 2.1 If M_p is Mersenne prime then $Q_p=a(M_p)^2+bM_p+c$ is called a quadratic polynomial with $a(M_p)^2+bM_p+c$.

Definition 2.2 If number represented by quadratic polynomial with $a(M_p)^2+bM_p+c$ is a prime then the number is called prime represented by quadratic polynomial with $a(M_p)^2+bM_p+c$.

Definition 2.3 Exponents of all Mersenne primes $M_p=2^p-1$ are called common basic sequence of number for primes represented by any quadratic polynomial of Mersenne primes i.e. p=2, 3, 5, 7, 13, 17, 19, 31,….

Definition 2.4 For numbers represented by a given quadratic polynomial of Mersenne primes, if the first few continuous terms (at lest two terms) are prime then corresponding p-values are called original continuous prime number sequence of primes represented by the quadratic polynomial of Mersenne primes.

Hypothesis 2.1 For primes represented by a given quadratic polynomial of Mersenne primes, these primes are infinite if both the sum of corresponding original continuous prime number sequence and the first such prime are Fermat primes.

3. Primes represented by quadratic polynomial with $2(M_p)^2-1$

We consider primes represented by quadratic polynomial with $2(M_p)^2-1$ as the first example for studying the infinity of primes represented by quadratic polynomial of

Mersenne primes.

Definition 3.1 If M_p is Mersenne prime then $Q_p=2(M_p)^2-1$ is called quadratic polynomial with $2(M_p)^2-1$.

Definition 3.2 If number represented by $Q_p=2(M_p)^2-1$ is a prime then Q_p is called a prime represented by the quadratic polynomial with $2(M_p)^2-1$.

An elementary argument on the infinity of primes represented by quadratic polynomial with $2(M_p)^2-1$ should be given. The first few terms represented by quadratic polynomial with $2(M_p)^2-1$ are as follows

$Q_2=2(M_2)^2-1=2\cdot 3^2-1=17$, prime;

$Q_3=2(M_3)^2-1=2\cdot 7^2-1=97$, prime;

$Q_5=2(M_5)^2-1=2\cdot 31^2-1=1921$, composite;

$Q_7=2(M_7)^2-1=2\cdot 127^2-1=32257$, prime;

…

By Definition 2.3 basic sequence of number of primes represented by quadratic polynomial with $2(M_p)^2-1$ is p=2, 3, 5, 7, 13, 17, 19, 31,…, thus we have the following lemma.

Lemma 3.1 The original continuous prime number sequence for primes represented by quadratic polynomial with $2(M_p)^2-1$ is p=2, 3.

Proof Since the first two continuous terms Q_2=17, Q_3=97 are all prime but the third term Q_5=1921 is composite, by Definition 2.4 the original continuous prime number sequence of primes represented by quadratic polynomial with $2(M_p)^2–1$ is p=2, 3.

Proposition 3.1 Primes represented by quadratic polynomial with $2(M_p)^2–1$ are infinite.

Proof Since the sum of original continuous prime number sequence of primes represented by quadratic polynomial with $2(M_p)^2–1$ i.e. 2+3=5 is a Fermat prime i.e. F_1 and the first prime represented by quadratic polynomial with $2(M_p)^2–1$ i.e. $Q_2=2(M_2)^2–1=17$ is also a Fermat prime i.e. F_2, by Hypothesis 2.1 primes represented by quadratic polynomial with $2(M_p)^2–1$ are infinite.

Proposition 3.2 Primes of the form $2x^2–1$ are infinite.

Proof By Proposition 3.1 primes represented by quadratic polynomial with $2(M_p)^2–1$ are infinite and primes represented by quadratic polynomial with $2(M_p)^2–1$ are a subset of primes of the form $2x^2–1$, hence primes of the form $2x^2–1$ are infinite.

In fact, there is a very great probability for appearing of primes of the form $2x^2–1$, because there are 44 x-values less 100 i.e. x=2, 3, 4, 6, 7, 8, 10, 11, 13, 15, 17, 18, 21, 22, 24, 25, 28, 34, 36, 38, 39, 41, 42, 45, 46, 49, 50, 52, 56, 59, 62, 63, 64, 69, 73, 76, 80, 81, 85, 87, 91, 92, 95, 98 to generate primes of the form $2x^2–1$. More such primes can be viewed in the on-line Encyclopedia of Integer Sequences. It should be conjectured that the primes with such great probability will be infinite, however, there has not been any argument or proof about the infinity of such primes. By Hypothesis 2.1 we give an elementary argument on the infinity of primes represented by quadratic polynomial with $2(M_p)^2–1$, which makes it become possible that we can consider the infinity of primes of the form $2x^2–1$ because such primes are a subset of primes of the form $2x^2–1$. This is a suitable example for studying the infinity of primes represented by quadratic polynomials of Mersenne primes by Hypothesis 2.1 to imply the next example, in which we will directly consider the infinity of primes of the form x^2+1 by means of a suitable quadratic polynomial of Mersenne primes, not to be an isolated case.

4. Primes represented by quadratic polynomial with $4(M_p–2)^2+1$

Definition 4.1 If M_p is Mersenne prime then $Q_p=4(M_p–2)^2+1$ is called quadratic polynomial with $4(M_p–2)^2+1=4(M_p)^2–16M_p+17$.

Definition 4.2 If number represented by $Q_p= 4(M_p–2)^2+1$ is a prime then Q_p is called a prime represented by quadratic polynomial with $4(M_p–2)^2+1$.

An elementary argument on the infinity of primes represented by quadratic polynomial with $4(M_p–2)^2+1$ should be given. The first few terms represented by quadratic polynomial with $4(M_p–2)^2+1$ are as follows

$Q_2=4(M_2–2)^2+1=2^2+1=5$, prime;

$Q_3=4(M_3–2)^2+1=10^2+1=101$, prime;

$Q_5= 4(M_5–2)^2+1=58^2+1=3365$, composite;

$Q_7=4(M_7–2)^2+1=250^2+1=62501$, prime;

…

By Definition 2.3 basic sequence of number of primes represented by quadratic polynomial with $4(M_p–2)^2+1$ is p=2, 3, 5, 7, 13, 17, 19, 31,…, thus we have the following lemma.

Lemma 4.1 The original continuous prime number sequence of primes represented by quadratic polynomial with $4(M_p–2)^2+1$ is p=2, 3.

Proof Since the first two continuous terms $Q_2=5$, $Q_3=101$ are all prime but the third term $Q_5=3365$ is composite, by Definition 2.4 the original continuous prime number sequence of primes represented by quadratic polynomial with $4(M_p–2)^2+1$ is p=2, 3.

Proposition 4.1 Primes represented by quadratic polynomial with $4(M_p–2)^2+1$ are infinite.

Proof Since the sum of original continuous prime number sequence of primes represented by quadratic polynomial with $4(M_p–2)^2+1$ i.e. 2+3=5 is a Fermat prime i.e. F_1 and the first prime represented by quadratic polynomial with $4(M_p–2)^2+1$ i.e. $Q_2=4(M_2–2)^2+1=5$ is also a Fermat prime i. e. F_1, by Hypothesis 2.1 primes represented by quadratic polynomial with $4(M_p–2)^2+1$ are infinite.

Proposition 4.2 Primes of the form x^2+1 are infinite.

Proof When $4(M_p–2)^2+1$ is written in $(2(M_p–2))^2+1$, obviously primes represented by quadratic polynomial with $(2(M_p–2))^2+1$ are a subset of primes of the form x^2+1. By Proposition 4.1 primes represented by quadratic polynomial with $4(M_p–2)^2+1=(2(M_p–2))^2+1$ are infinite, hence primes of the form x^2+1 are infinite.

In addition, the infinity of primes represented by quadratic polynomial with $4(M_p–2)^2+1$ will lead to an elementary argument for the infinity of Mersenne primes.

Proposition 4.3 Mersenne primes M_p are infinite.

Proof By Proposition 4.1 primes represented by quadratic polynomial with $4(M_p–2)^2+1$ are infinite, hence p-values of such primes are infinite. Since p-values of primes represented by quadratic polynomial with $4(M_p–2)^2+1$ are a subset of common basic sequence of number for primes represented by any quadratic polynomial of Mersenne primes as Definition 2.3 sates, the common basic sequence of number for primes represented by any quadratic polynomial of Mersenne primes are infinite. In the common basic sequence of number all p-values correspond to distinct Mersenne primes, hence Mersenne primes are infinite.

5. Conclusion

Gauss-Wantzel theorem shows existence of connections between Fermat primes and constructible regular polygons in which Fermat primes are criterion for the constructibility of regular polygons (not including the constructibility of regular 2^k-sided polygons). Similarly, in this paper Hypothesis 2.1 is a sufficient condition to be proven, in which Fermat primes are criterion for the infinity of primes represented by quadratic polynomials of Mersenne primes. If Hypothesis 2.1 is acceptable then Landau's fourth problem seems to be able to get an elementary argument, even the infinity of Mersenne primes may get an elementary argument as an addition.

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