Perfect cuboid does not exist

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In memory of my wife Bendeberi O.I.,
initiator and inspirer of this research.

Abstract

A perfect cuboid, i.e., a rectangular parallelepiped having integer edges, integer face diagonals, and integer space diagonal, is proved to be non-existing.

Keywords
perfect cuboid, rational cuboid, integer brick.

The perfect cuboid problem [1] closely resembles that of Pythagoras, both in age and matter, but still has not been solved. Its possible formulation is as follows: does a cuboid exist in which the lengths of the edges and all diagonals are positive integer numbers? In other words, it is the problem of the existence of positive integer A, B, C, D, E, F, and G, satisfying the system of Diophantine equations

$$\begin{aligned} G^2 &= A^2 + B^2 + C^2, \\ D^2 &= A^2 + B^2, \\ E^2 &= A^2 + C^2, \\ F^2 &= B^2 + C^2. \end{aligned} \tag{1}$$

It follows from (1) that

$$\begin{aligned} G^2 &= A^2 + F^2, \\ G^2 &= B^2 + E^2, \\ G^2 &= C^2 + D^2. \end{aligned} \tag{2}$$

Assume that the problem has a solution. Without loss of generality, the cuboid may be thought of as a primitive one, with relatively prime A, B, C. One of these numbers (A) is odd whereas the other two are even. From the first equality (1) and equations (2) it follows that G has only prime divisors of the form $4T + 1$, and at least two of them must be different in order to provide three different representations of G^2 in terms of sums of squared integers.

In general, the Pythagorean triples in (2) are not primitive and therefore we represent

$$
\begin{aligned}
&A = (L^2 - l^2)G_1, \qquad F = 2LlG_1;\\
&E = (M^2 - m^2)G_2, \quad B = 2MmG_2;\\
&D = (N^2 - n^2)G_3, \qquad C = 2NnG_3;\\
&G = G_0G_1G_2G_3.
\end{aligned} \tag{3}
$$

Since A is odd, the pairs (L, l), (M, m) and (N, n) consist of mutually prime integers, one odd and the other even. Each G_i may be prime or compound, and one or two of them may be equal to unity but all $G_i > 1$ with $i > 0$ are relatively prime. Substituting (3) into (2), one obtains the system of identities whereas equations (1) are transformed into equivalent equalities, and anyone of them is necessary and sufficient for the existence of a perfect cuboid.

Let us represent G_i as sums of squares of two integers, $\bar{Q}_i > 0$ and $\bar{q}_i$, as well as in terms of positive integer Q_i and rational q_i:

$$
\begin{aligned}
&G_0 = \bar{Q}_0^2 + \bar{q}_0^2 = Q_0^2(1 + q_0^2),\\
&G_1 = \bar{Q}_1^2 + \bar{q}_1^2 = Q_1^2(1 + q_1^2),\\
&G_2 = \bar{Q}_2^2 + \bar{q}_2^2 = Q_2^2(1 + q_2^2),\\
&G_3 = \bar{Q}_3^2 + \bar{q}_3^2 = Q_3^2(1 + q_3^2).
\end{aligned} \tag{4}
$$

If $G_i = 1$ then $\bar{Q}_i = Q_i = 1$ and $\bar{q}_i = q_i = 0$, whereas if $G_i > 1$ then $\bar{Q}_i$ and $\bar{q}_i$ are mutually prime integers, one odd and the other even, and we can take either $Q_i = \bar{Q}_i$, $q_i = \bar{q}_i/\bar{Q}_i$, or $Q_i = |\bar{q}_i|$, $q_i = \bar{Q}_i/\bar{q}_i$. Therefore, the equations obtained remain valid when replacing any $q_i \neq 0$ by $1/q_i$ (remind that at least two of q_i are nonzero). In particular, one can choose q_i with even numerators and odd denominators.

As a necessary and sufficient condition for the existence of a perfect cuboid we choose the fourth equation (1):

$$
Q_1^4(1 + q_1^2)^2L^2l^2 = Q_2^4(1 + q_2^2)^2M^2m^2 + Q_3^4(1 + q_3^2)^2N^2n^2. \tag{5}
$$

Using (3) and (4) we can express L, l, M, m, N and n in terms of Q_i and q_i,

$$
\begin{aligned}
&\frac{L^2 + l^2}{Q_0^2Q_2^2Q_3^2} = (1 + q_0^2)(1 + q_2^2)(1 + q_3^2) = (1 - q_0q_2 - q_0q_3 - q_3q_2)^2 + (q_0 + q_2 + q_3 - q_0q_2q_3)^2,\\
&\frac{M^2 + m^2}{Q_0^2Q_1^2Q_3^2} = (1 + q_0^2)(1 + q_1^2)(1 + q_3^2) = (1 - q_0q_1 - q_0q_3 - q_3q_1)^2 + (q_0 + q_1 + q_3 - q_0q_1q_3)^2,\\
&\frac{N^2 + n^2}{Q_0^2Q_2^2Q_1^2} = (1 + q_0^2)(1 + q_2^2)(1 + q_1^2) = (1 - q_0q_2 - q_0q_1 - q_1q_2)^2 + (q_0 + q_2 + q_1 - q_0q_2q_1)^2,
\end{aligned} \tag{6}
$$

In different equations (6), q_i with the same index are equal in absolute value but their signs may be selected independently for each equation. With this in mind, we write

$$L = Q_0Q_2Q_3[1 - q_2q_3 - q_0(q_2 + q_3)],\ l = Q_0Q_2Q_3[q_0(1 - q_2q_3) + q_2 + q_3],$$

$$M = Q_0Q_1Q_3[1 - q_1q_3i_3 - q_0i_0(q_1 + q_3i_3)],\ m = Q_0Q_1Q_3[q_0i_0(1 - q_1q_3i_3) + q_1 + q_3i_3], \tag{7}$$

$$N = Q_0Q_1Q_2[1 - q_1q_2j_1j_2 - q_0j_0(q_1j_1 + q_2j_2)],\ n = Q_0Q_1Q_2[q_0j_0(1 - q_1q_2j_1j_2) + q_1j_1 + q_2j_2],$$

where each q_i has the sign selected when it appears in (7) for the first time, and possible changes of signs in subsequent formulas (7) are taken into account by sign factors i_0, i_3 and j_0, j_1, j_2, equal to unity in absolute value.

Substituting (7) into (5) yields the equation

$$\begin{aligned}&(1 + q_1^2)^2[1 - q_2q_3 - q_0(q_2 + q_3)]^2[q_0(1 - q_2q_3) + q_2 + q_3]^2 = \\ &= (1 + q_2^2)^2[1 - q_1q_3i_3 - q_0i_0(q_1 + q_3i_3)]^2[q_0i_0(1 - q_1q_3i_3) + q_1 + q_3i_3]^2 \\ &+ (1 + q_3^2)^2[1 - q_1q_2j_1j_2 - q_0j_0(q_1j_1 + q_2j_2)]^2[q_0j_0(1 - q_1q_2j_1j_2) + q_1j_1 + q_2j_2]^2,\end{aligned} \tag{8}$$

which is a fourth-degree polynomial with respect to each q_i. Let $q_0 \neq 0$. Replacing q_0 by $1/q_0$ in (8) and multiplying the resulting equation by q_0^4 one obtains a new equation (hereafter referred to as (8a)) which is a fourth-degree polynomial of q_0 differing from (8) by interchanging the factors at q_0^n and q_0^{4-n}, and is fulfilled at the same q_0, q_1, q_2, q_3 as (8). Now let us subtract (8a) from (8). After a simple algebra, the factor at $(1 + q_1^2)^2$ in (8) takes the form

$$\begin{aligned}&(1 + q_0^4)(1 - q_2q_3)^2(q_2 + q_3)^2 + q_0^2[(1 - q_2q_3)^4 + (q_2 + q_3)^4 - 4(1 - q_2q_3)^2(q_2 + q_3)^2] \\ &+2q_0(1 - q_0^2)(1 - q_2q_3)(q_2 + q_3)[(1 - q_2^2)(1 - q_3^2) - 4q_2q_3].\end{aligned}$$

A similar factor in (8a) will differ only in the signs of odd degrees of q_0 so that the difference of the left-hand sides of (8) and (8a) is

$$4q_0(1 - q_0^2)(1 + q_1^2)^2(1 - q_2q_3)(q_2 + q_3)[(1 - q_2^2)(1 - q_3^2) - 4q_2q_3].$$

Calculating in the same way the difference of their right-hand sides and taking into account that, in accordance with (4), $q_0^2 \neq 1$ we obtain the equation

$$\begin{aligned}&(1 + q_1^2)^2(1 - q_2q_3)(q_2 + q_3)[(1 - q_2^2)(1 - q_3^2) - 4q_2q_3] \\ &-(1 + q_2^2)^2i_0(1 - q_1q_3i_3)(q_1 + q_3i_3)[(1 - q_1^2)(1 - q_3^2) - 4q_1q_3i_3] \\ &-(1 + q_3^2)^2j_0(1 - q_1q_2j_1j_2)(q_1j_1 + q_2j_2)[(1 - q_1^2)(1 - q_2^2) - 4q_1q_2j_1j_2] = 0.\end{aligned} \tag{9}$$

Next, assuming that $q_1 \neq 0$ we replace in (9) q_1 by $1/q_1$, multiply the result by q_1^4 and subtract the equation obtained from (9). Since $q_1^2 \neq 1$, the resulting equation is

$$i_0(1 + q_2^2)^2[(1 + q_3^2)^2 - 8q_3^2] + j_0j_1(1 + q_3^2)^2[(1 + q_2^2)^2 - 8q_2^2] = 0. \tag{10}$$

If i_0 and j_0j_1 have the same signs, we obtain the equality

$$(1 + q_2^2)^2(1 + q_3^2)^2 = 4q_2^2(1 + q_3^2)^2 + 4q_3^2(1 + q_2^2)^2$$

that can not be satisfied by q_2 and q_3 with even numerators and odd denominators. If i_0 and j_0j_1 have opposite signs,

$$0 = q_2^2(1+q_3^2)^2 - q_3^2(1+q_2^2)^2 = (q_2^2 - q_3^2)(1 - q_2^2q_3^2),$$

and the only solution is $q_2^2 = q_3^2$. In view of (4) this means that $q_2 = q_3 = 0$ and $G_2 = G_3 = 1$.

Substituting $q_2 = q_3 = 0$ into (8) provides (remind that $i_0 + j_0j_1 = 0$)

$$2q_1^2 + q_0^2q_1^4 + q_0^2 + 2q_0^4q_1^2 - 10q_0^2q_1^2 = 0. \tag{11}$$

Let P be the greatest common divisor for even numerators of q_0 and q_1 so that $q_0 = Pp_0$ and $q_1 = Pp_1$. Then

$$(p_0^2p_1^4 + 2p_0^4p_1^2)P^4 - 10p_0^2p_1^2P^2 + 2p_1^2 + p_0^2 = 0. \tag{12}$$

In (12), the numerator of the left-hand side is odd (if the numerator of p_0 is odd) or becomes odd when divided by 2 (if the numerator of p_1 is odd). Hence, the equation (11) has no solutions (other than $q_0 = q_1 = 0$ that contradicts the initial assumptions).

Now, let $q_0 = 0$. From (8) we find

$$\begin{aligned}(1+q_1^2)^2(1-q_2q_3)^2(q_2+q_3)^2 - (1+q_2^2)^2(1-q_1q_3i_3)^2(q_1+q_3i_3)^2 \\ -(1+q_3^2)^2(1-q_1q_2j_1j_2)^2(q_1j_1+q_2j_2)^2 = 0.\end{aligned} \tag{13}$$

If $q_1 = 0$ as well, non-zero q_2 and q_3 should obey the equation

$$(1-q_2^2)(1-q_3^2) = 4q_2q_3,$$

but this is impossible for q_2 and q_3 with even numerators and odd denominators. If, however, $q_1 \neq 0$, we replace in (13) q_1 by $1/q_1$, multiply the result by q_1^4, and subtract the equation obtained from (13). Since $q_1^2 \neq 1$, we have the equation

$$\begin{aligned}0 &= (1+q_2^2)^2(1-q_3^2)\, q_3i_3 + (1+q_3^2)^2(1-q_2^2)\, q_2j_1j_2 \\ &= (|q_2| \pm |q_3|)(1 \mp |q_2||q_3|)[(1 \pm |q_2||q_3|)^2 - (|q_2| \mp |q_3|)^2],\end{aligned}$$

with the upper signs taken for the equal signs of q_3i_3 and $q_2j_1j_2$, and the lower signs for the opposite ones. This equation has no solution with even numerators and odd denominators.

Finally, when $q_0 \neq 0$ and $q_1 = 0$ the equation (9) takes the form

$$(1-q_2q_3)(q_2+q_3)[(1-q_2q_3)^2-(q_2+q_3)^2] = (1+q_2^2)^2(1-q_3^2)q_3i_0i_3+(1+q_3^2)^2(1-q_2^2)q_2j_0j_2. \tag{14}$$

If $q_2 \neq 0$, we replace q_2 by $1/q_2$, multiply the result by q_2^4, subtract the equation obtained from (14) and get

$$2q_2(1-q_2^2)[(1+q_3^2)^2(1-j_0j_2) - 8q_3^2] = 0.$$

If $j_0j_2 = -1$, there is no solution with an even numerator and odd denominator, and if $j_0j_2 = 1$ the only solution is $q_3 = 0$. Substituting it into (8) provides $(1+q_2^2)^2q_0^2 = 0$ which contradicts the assumption $q_0 \neq 0$. Likewise, we find that when $q_3 \neq 0$ there are no solutions as well.

Thus, it is proved that **a perfect cuboid does not exist**.

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References

[1] Guy, R.K. *Unsolved Problems in Number Theory.* New York: Springer-Verlag, 1994.