Examples of Solving PDEs by Order Completion

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Abstract

So far, the order completion method for solving PDEs, introduced in 1990, can solve by far the most general linear and nonlinear systems of PDEs, with possible initial and/or boundary data. Examples of solving various PDEs with the order completion method are presented. Some of such PDEs do not have global solutions by any other known methods, or are even proved not to have such global solutions.
The presentation next aims to be as summary, and in fact, sketchy as possible, even if by that it may create some difficulty. However, nowadays, being subjected to an ever growing "information overload", that approach may turn out to be not the worst among two bad alternatives.
Details can be found in [1], while on the other hand, alternative longer "short presentations" are in [6-8].

0. Bare bones presentation

The most general systems of nonlinear PDEs and possibly associated initial and/or boundary problems have equations of the form

$$(0.1) \qquad T(x,D)U(x) = F(x, U(x), ..., D_x^p U(x), ...) = f(x), \quad x \in \Omega$$

where $\Omega \subseteq \mathbb{R}^n$ are open, possibly unbounded, $p \in \mathbb{N}^n$, $|p| \leq m$, F jointly continuous in all arguments, f continuous, except for certain possible discontinuities on closed, nowhere dense subsets of Ω.

Clearly, we have the associated mappings

$$(0.2) \qquad \mathcal{C}^m(\Omega) \ni U \stackrel{T(x,D)}{\longrightarrow} T(x,D)U \in \mathcal{C}^0(\Omega)$$

The *essential* ingredient in solving such general PDEs like in (0.1) by the *order completion* method is the following rather trivial and easy to prove classical analysis result on *local, one-sided approximative solutions* of the equations (0.1), namely

Lemma

Given $f \in \mathcal{C}^0(\Omega)$ which satisfies condition (2.4) in the section 2 below, then

$$(0.3) \qquad \begin{aligned} & \forall \ x_0 \in \Omega, \ \epsilon > 0 \ : \\ & \exists \ \delta > 0, \ P \text{ polynomial in } x \in \mathbb{R}^n \ : \\ & \forall \ x \in \Omega \ : \\ & \quad ||x - x_0|| \leq \delta \implies f(x) - \epsilon \leq T(x,D)P(x) \leq f(x) \end{aligned}$$

$\square$

Now it is easy to *patch up globally nearly on the whole of* Ω the local, one-sided approximative solutions obtained in the Lemma above, and obtain

Proposition

Given $f \in \mathcal{C}^0(\Omega)$ which satisfies condition (2.4) in the section 2 below, then

$$(0.4) \qquad \begin{aligned} & \forall \ \epsilon > 0 \ : \\ & \exists \ \Gamma_\epsilon \subseteq \Omega \text{ closed, nowhere dense, } U_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon) \ : \\ & \quad f - \epsilon \leq T(x,D)U_\epsilon \leq f \ \text{ on } \Omega \setminus \Gamma_\epsilon \end{aligned}$$

and one can assume that

$$(0.5) \qquad mes(\Gamma_\epsilon) = 0, \quad \epsilon > 0$$

$\square$

As consequence, one can *always* obtain the *existence of global solutions* U of equations (0.1), by suitable *Dedekind order completion* of appropriate extensions of the spaces of functions $\mathcal{C}^m(\Omega)$ and $\mathcal{C}^0(\Omega)$ in (0.2).

The remarkable fact - see Theorem in section 4 below - is that such global solutions U on the *whole* domain Ω *always* exist, and in addition, they can be assimilated with usual *measurable* functions on Ω. In fact, these global solutions U are even more regular, being *Hausdorff continuous*, [2,3,7,9].

Therefore, contrary to the general perception, including of "leading" specialists in PDEs, [5], the above Theorem - as well as its various developments in [1,10], offer for the first time in the literature :

- *type independent, general, blanket* global existence and regularity results for such unprecedented general nonlinear systems of PDEs and their possibly associated initial and/or boundary problems, as given by the PDEs in (0.1).

Remarks

1) An essential feature of the above order completion method in solving PDEs is that the construction of the Dedekind order completion does *not* involve algebra, and even less topology, but instead, *only* set theory and partial order relations.

2) A most important consequence is that the order completion method does *not* distinguish between linear and nonlinear PDEs.

3) One can - if one wishes - set aside the whole of the usual theories of weak, generalized, distributional, Sobolev type, and other similar solutions.

4) Needless to say, the usual solution theories for PDEs, focused on their many highly specialized types of equations, can deliver results with additional properties for solutions, and therefore, they are useful.

5) The fact, however, remains that, so far, the order completion method for solving PDEs, as introduced in [1], is the *only* such method in the known literature which has the above properties 1), 2) and 3).

And now, some details ...

1. What does it mean to solve an equation ?

$$(1.1) \qquad T(x) = y$$

$$(1.2) \qquad T : X \longrightarrow Y, \quad y \in Y \text{ given}, \quad x \in X, \quad x = ?$$

Three cases :

$$(1.3.1) \qquad T \text{ bijective} \quad \Longrightarrow \quad \forall\, y \in Y \;:\; \exists!\, x \in X \;:\; T(x) = y$$

$$(1.3.2) \qquad T \text{ surjective} \quad \Longrightarrow \quad \forall\, y \in Y \;:\; \exists\, x \in X \;:\; T(x) = y$$

and the *difficult case* :

$$(1.3.3) \qquad T \text{ not surjective} \quad \Longrightarrow \quad \exists\, y \in Y \;:\; \forall\, x \in X \;:\; T(x) \neq y$$

The following two most simple examples appear to exhaust all the kind of situation that may appear when trying to solve equations by the Dedekind order completion method :

Pythagoras

$$(1.4) \qquad x^2 = 2$$

$$(1.5) \qquad X = \mathbb{Q} \ni x \xrightarrow{T} T(x) = x^2 \in Y = \mathbb{Q}, \quad y = 2 \in \mathbb{Q} = Y$$

In this case, a solution of equation (1.4) is given by the *Dedekind cut*

$x = \{r \in \mathbb{Q} \mid r^2 \leq 2\} \in \mathbb{R}$, thus this solution $x \in \mathbb{R}$ can be perfectly *approximated on the left side* by rational numbers r in the mentioned Dedekind cut.

Complex Numbers

$$(1.6) \qquad x^2 + 1 = 0, \quad x^2 = -1$$

$$(1.7) \qquad X = \mathbb{R} \ni x \xrightarrow{T} T(x) = x^2 \in Y = \mathbb{R}, \quad y = -1 \in \mathbb{R} = Y$$

Here, there is *no* way to set up a partial order on $\mathbb{R}$, so as to obtain a Dedekind cut in $\mathbb{R}$ which would approximate well enough the solution $x = i \in \mathbb{C}$ of equation (1.6).

Fortunately, nonlinear systems of PDEs even of such generality as those in (0.1) prove to belong to the first above case, that is, the case of the equation (1.4) of Pythagoras.

2. The generality of PDE systems solved by order completion

$$(2.1) \qquad T(x, D)U(x) = F(x, U(x), ..., D_x^p U(x), ...) = f(x), \quad x \in \Omega$$

$\Omega \subseteq \mathbb{R}^n$ open, possibly unbounded, $p \in \mathbb{N}^n$, $|p| \leq m$, F jointly continuous in all arguments, f continuous, except for certain possible discontinuities on closed, nowhere dense subsets of Ω.

Let

$$(2.2) \qquad R_x = \left\{ F(x, \xi_0, ..., \xi_p, ...) \left| \begin{array}{l} p \in \mathbb{N}^n, \ |p| \leq m \\ \xi_p \in \mathbb{R} \end{array} \right. \right\}, \quad x \in \Omega$$

and the corresponding conditions :

$$(2.3) \qquad f(x) \in R_x, \quad x \in \Omega$$

$$(2.4) \qquad f(x) \in int(R_x), \quad x \in \Omega$$

Obviously

$$(2.5) \qquad (\ R_x = \mathbb{R}, \quad x \in \Omega \ ) \quad \Longrightarrow \quad (2.4) \quad \Longrightarrow \quad (2.3)$$

We shall assume (2.4).

3. Local one-sided solutions

Lemma

Given $f \in \mathcal{C}^0(\Omega)$, then

$$(3.1) \qquad \begin{aligned} & \forall \ x_0 \in \Omega, \ \epsilon > 0 \ : \\ & \exists \ \delta > 0, \ P \text{ polynomial in } x \in \mathbb{R}^n \ : \\ & \forall \ x \in \Omega \ : \\ & \quad ||x - x_0|| \leq \delta \quad \Longrightarrow \quad f(x) - \epsilon \leq T(x, D)P(x) \leq f(x) \end{aligned}$$

Proposition

Given $f \in \mathcal{C}^0(\Omega)$, then

$$\forall \ \epsilon > 0 \ :$$

$$(3.2) \qquad \exists \ \Gamma_\epsilon \subseteq \Omega \text{ closed, nowhere dense, } U_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon) \ :$$

$$f - \epsilon \leq T(x, D)U_\epsilon \leq f \quad \text{on } \Omega \setminus \Gamma_\epsilon$$

and one can ssume that

$$(3.3) \qquad mes(\Gamma_\epsilon) = 0, \quad \epsilon > 0$$

Corollary

Given $f \in \mathcal{C}^0(\Omega)$, then

$$
\begin{array}{ll}
 & \forall \; x_0 \in \Omega \;\; : \\[1em]
 & \exists \; \delta > 0 \;\; : \\[1em]
(3.4) & \forall \; A \subseteq B(x, \delta), \;\; A \text{ finite } \;\; : \\[1em]
 & \exists \; U \in \mathcal{C}^\infty(\Omega) \;\; : \\[1em]
 & \quad T(y, D) U(y) = f(y), \quad y \in A
\end{array}
$$

and furthermore

$$
\begin{array}{ll}
 & \forall \; A \subseteq \Omega, \;\; A \text{ discrete } \;\; : \\[1em]
(3.5) & \exists \; U \in \mathcal{C}^\infty(\Omega) \;\; : \\[1em]
 & \quad T(x, D) U(x) = f(x), \quad x \in A
\end{array}
$$

4. Global solutions

For $0 \leq l \leq \infty$, let

$$
(4.1) \qquad \mathcal{C}^l_{nd}(\Omega) = \left\{ \; u \; \left| \; \begin{array}{l} \exists \; \Gamma \subseteq \Omega, \;\; \Gamma \text{ closed, nowhere dense } \;\; : \\[1em] \quad u \in \mathcal{C}^l(\Omega \setminus \Gamma) \end{array} \right. \right\}
$$

Then

$$
(4.2) \qquad \mathcal{C}^m(\Omega) \subset \mathcal{C}^m_{nd}(\Omega) \stackrel{T(x,D)}{\longrightarrow} \mathcal{C}^0_{nd}(\Omega) \subset \hat{\mathcal{M}}^0_{nd}(\Omega) \stackrel{id}{\longrightarrow} \mathcal{P}(\mathcal{C}^0_{nd}(\Omega))
$$

and thus we have the *injective* mapping

$$
(4.3) \qquad \hat{\mathcal{M}}^m_T(\Omega) \stackrel{\hat{T}}{\longrightarrow} \hat{\mathcal{M}}^0_{nd}(\Omega) \stackrel{id}{\longrightarrow} \mathcal{P}(\mathcal{C}^0_{nd}(\Omega))
$$

where $\hat{X}$ denotes the *Dedekind order completion* of the poset $(X, \leq)$, according to MacNeille, 1937, [1].

Also, we have the *commutative* diagram

$$(4.4)\qquad \begin{array}{ccc} \mathcal{C}^m_{nd}(\Omega) \ni U & \stackrel{T(x,D)}{\longrightarrow} & T(x,D)U \in \mathcal{C}^0_{nd}(\Omega) \\ \downarrow & & \downarrow \\ \hat{\mathcal{M}}^m_T(\Omega) \ni < U] & \stackrel{\hat{T}}{\longrightarrow} & \hat{T}(< U]) \in \hat{\mathcal{M}}^0_{nd}(\Omega) \end{array}$$

where for $U, V \in \mathcal{C}^l_{nd}(\Omega)$, we have

$$(4.5)\qquad U \leq V \quad \Longleftrightarrow \quad U(x) \leq V(x), \quad x \in \Omega \setminus \Gamma$$

with $U, V \in \mathcal{C}^l(\Omega \setminus \Gamma)$, $\Gamma \subset \Omega$, Γ closed, nowhere dense, and

$$(4.6)\qquad < U] = \{V \in \mathcal{C}^l_{nd}(\Omega) \mid V \leq U\}$$

Theorem

We have

$$(4.7)\qquad \hat{T}(\hat{\mathcal{M}}^m_T(\Omega)) = \hat{\mathcal{M}}^0(\Omega)$$

In other words

$$(4.8)\qquad \begin{array}{l} \forall\ A \in \hat{\mathcal{C}}^0_{nd}(\Omega)\ : \\ \exists!\ F \in \hat{\mathcal{M}}^m_T(\Omega)\ : \\ \quad \hat{T}(F) = A \end{array}$$

Remark

Let $U \in \mathcal{C}^m(\Omega \setminus \Gamma)$, where $\Gamma \subset \Omega$, Γ closed, nowhere dense, be such that

$$(4.9)\qquad T(x,D)U(x) = f(x), \quad x \in \Omega \setminus \Gamma$$

then $F = < U]$ is a solution of (4.8).

5. Examples (see [1], pp. 65-73)

Questions :

1) What is the nature of the generalized solutions obtained in (4.8) ?

2) How are these generalized solutions connected to the earlier known classical, weak, distributional, Sobolev space, and other such generalized solutions ?

3) What is the meaning of the uniqueness property of solutions in (4.8) ?

Answer to questions 1) and 2) : The generalized solutions in (4.8) can be assimilated to usual *measurable* functions on Ω. Furthermore, due to very important results obtained by Roumen Anguelov, they can be assimilated with significantly more regular functions, namely with *Hausdorff continuous* ones, [2,3,5-8]. And still more, due to similarly important result obtained by Jan-Harm van der Walt, such generalized solutions can be even more regular,[10].
Consequently, the whole earlier theory of generalized functions and solutions is no longer necessary.
Not to mention that that earlier theory cannot come anywhere near to the solution of such general systems of nonlinear PDEs and the possibly associated initial and/or boundary problems.

The answer to question 3) is as follows. The unique generalized solution in (4.8) simply contains *all* the possible solutions of the respective PDE. This fact can also be seen in the examples next.

Example 1

Let $\Omega \subset \mathbb{R}^2$ an open bounded set, and let $f \in \mathcal{C}^0(\mathbb{R})$ be a *non-differentiable* function on a *dense* subset $S \subset \mathbb{R}$. We consider the PDE, more precisely, the ODE

(5.1) $\qquad D_t U(t,y) = f(y), \quad x = (t,y) \in \Omega$

Then the PDE (5.1) does *not* have any solution $U \in \mathcal{C}^1_{nd}(\Omega)$.

However, the unique generalized solution $F \in \hat{\mathcal{M}}^m_T(\Omega)$ in (4.8) contains all the functions $U \in \mathcal{C}^1(\Omega \setminus \Gamma)$, where $\Gamma \subset \Omega$ closed, nowhere dense, and

$$(5.2) \qquad D_t U(t,y) \leq f(y), \quad x = (t,y) \in \Omega \setminus \Gamma$$

Example 2

Let

$$(5.3) \qquad D_t U(t) = 0, \quad t \in \Omega = (-1,1) \subset \mathbb{R}$$

whose classical solutions are

$$(5.4) \qquad U(t) = c, \quad t \in \Omega = (-1,1)$$

where $c \in \mathbb{R}$ are given arbitrary. We shall now follow the way the unique generalized solution $F \in \hat{\mathcal{M}}^m_T(\Omega)$ in (4.8) are constructed. It is easy to see that F contains all the functions of the form

$$(5.5) \qquad U(t) = \epsilon_\nu t + c_\nu, \quad t \in \Omega_\nu$$

where for $\nu \in \mathbb{N}$, we have $\epsilon_\nu, c_\nu \in \mathbb{R}$, $\epsilon_\nu \leq 0$, $\Omega_\nu \subseteq (-1,1)$, open, pair-wise disjoint, and such that $\bigcup_{\nu \in \mathbb{N}} \Omega_\nu$ dense in $(-1,1)$. Thus

$$(5.6) \qquad D_t U(t) \leq 0, \quad t \in \bigcup_{\nu \in \mathbb{N}} \Omega_\nu \subset (-1,1)$$

Some further details

Here we try do convey the essence of the order completion method, and do so without getting in all the details. However, the following minimal details may be useful. We start, as we have seen, with the *equation*, see (1.1)

$$(5.7) \qquad T(x,D)U(x) = f(x), \quad x \in \Omega$$

and thus with the *mapping*, see (1.2)

$$(5.8) \qquad \mathcal{C}^m(\Omega) \stackrel{T(x,D)}{\longrightarrow} \mathcal{C}^0(\Omega)$$

In view of the above Proposition, we extend it to the mapping

$$(5.9) \qquad \mathcal{C}^m_{nd}(\Omega) \stackrel{T(x,D)}{\longrightarrow} \mathcal{C}^0_{nd}(\Omega)$$

which extension is obvious, since $\mathcal{C}^0(\Omega) \subset \mathcal{C}^0_{nd}(\Omega)$ and $\mathcal{C}^m \subset (\Omega)\mathcal{C}^m_{nd}(\Omega)$, thus we have the *commutative* diagram

$$(5.10) \qquad \begin{array}{ccc} \mathcal{C}^m(\Omega) & \stackrel{T(x,D)}{\longrightarrow} & \mathcal{C}^0(\Omega) \\ \downarrow \subset & & \downarrow \subset \\ \mathcal{C}^m_{nd}(\Omega) & \stackrel{T(x,D)}{\longrightarrow} & \mathcal{C}^0_{nd}(\Omega) \end{array}$$

Now it is convenient, from the point of view of order completion, to go to the mapping

$$(5.11) \qquad \mathcal{M}^m_T(\Omega) \stackrel{T}{\longrightarrow} \mathcal{M}^0(\Omega)$$

constructed as follows. For $0 \leq l \leq \infty$, we define the *equivalence* relation $\approx$ on $\mathcal{C}^l_{nd}(\Omega)$ by

$$(5.12) \qquad u \approx v \quad \Longleftrightarrow \quad \exists\, \Gamma \subset \Omega \text{ closed, nowhere dense } : \ u = v \text{ on } \Gamma$$

and then

$$(5.13) \qquad \mathcal{M}^0(\Omega) = \mathcal{C}^0(\Omega)/\approx$$

Further, we define the *equivalence* relation $\approx_T$ on $\mathcal{C}^m(\Omega)$ by

$$(5.14) \qquad u \approx v \quad \Longleftrightarrow \quad T(x,D)u \approx T(x,D)v$$

and then

$$(5.15) \qquad \mathcal{M}^m_T(\Omega) = \mathcal{C}^m(\Omega)/\approx_T$$

And now, the commutative diagram (5.10) obviously extends to the *commutative* diagram, see (5.11)

$$
(5.16) \qquad
\begin{array}{ccc}
\mathcal{C}^m(\Omega) & \stackrel{T(x,D)}{\longrightarrow} & \mathcal{C}^0(\Omega) \\
\downarrow \subset & & \downarrow \subset \\
\mathcal{C}^m_{nd}(\Omega) & \stackrel{T(x,D)}{\longrightarrow} & \mathcal{C}^0_{nd}(\Omega) \\
\downarrow & & \downarrow \\
\mathcal{M}^m_T(\Omega) & \stackrel{T}{\longrightarrow} & \mathcal{M}^0(\Omega)
\end{array}
$$

to which the 1937 Dedekind order completion method of MacNeille is applied, [1], leading to the commutative diagram

$$
(5.17) \qquad
\begin{array}{ccc}
\mathcal{C}^m(\Omega) & \stackrel{T(x,D)}{\longrightarrow} & \mathcal{C}^0(\Omega) \\
\downarrow \subset & & \downarrow \subset \\
\mathcal{C}^m_{nd}(\Omega) & \stackrel{T(x,D)}{\longrightarrow} & \mathcal{C}^0_{nd}(\Omega) \\
\downarrow & & \downarrow \\
\mathcal{M}^m_T(\Omega) & \stackrel{T}{\longrightarrow} & \mathcal{M}^0(\Omega) \\
\downarrow & & \downarrow \\
\hat{\mathcal{M}}^m_T(\Omega) & \stackrel{\hat{T}}{\longrightarrow} & \hat{\mathcal{M}}^0(\Omega)
\end{array}
$$

Example 3

Here are several further examples of PDEs or systems of PDEs which do not have solutions or are proved not to have solutions, be they local or global, classical, weak, distributional, Sobolev, or any usual generalized ones.

In view of Example 1, the PDE

$$
(5.18) \qquad (D_t + \lambda D_y)U(t,y) = f(y - \lambda t), \quad x = (t,y) \in \Omega \subseteq \mathbb{R}^2
$$

where $\lambda \in \mathbb{R}$ is given, cannot have solutions $U \in \mathcal{C}^1_{nd}(\Omega)$, for arbitrary $f \in \mathcal{C}^0(\Omega)$.

Similarly, the second order hyperbolic PDE

$$(5.19) \quad (D_t+\lambda D_y)(D_t+\mu D_y)U(t,y) = f(y-\lambda t), \ \ x=(t,y) \in \Omega \subseteq \mathbb{R}^2$$

where $\lambda, \mu \in \mathbb{R}$ is given, cannot have solutions $U \in \mathcal{C}^2_{nd}(\Omega)$, for arbitrary $f \in \mathcal{C}^0(\Omega)$.

Further, let Ω be the open unit ball in $\mathbb{R}^3$. Then it is known that there exist functions $f \in \mathcal{C}^0(\bar{\Omega})$ and dense subsets $S \subset \Omega$, such that every weak solution U on Ω of the PDE

$$(5.20) \quad \Delta U(x) = f(x), \quad x \in \Omega$$

is *not* $\mathcal{C}^2$-smoth at any point of S, thus in particular, we cannot have $U \in \mathcal{C}^2_{nd}(\Omega)$.

Let us now recall a classical 1928 result of Perron for systems of PDEs in two unknown functions U, V, of the form

$$(5.21) \quad \begin{aligned} &D_t U(t,y) - D_y U(t,y) - D_y V(t,y) = 0 \\ &a D_y U(t,y) - D_t V(t,y) + D_y V(t,y) + f(t+y) = 0 \end{aligned}$$

where $a \in \mathbb{R}$ is given, while $x=(t,y) \in [0,\infty) \times \mathbb{R}$, and the initial data hold

$$(5.22) \quad \begin{aligned} &U(0,y) = 0 \\ &V(0,y) = 0 \end{aligned}$$

for $y \in \mathbb{R}$.

Then it is known that the necessary and sufficient condition for the existence of a $\mathcal{C}^1$-smooth solution U, V is to have satisfied on of the three conditions :

$$(5.23) \quad f \in \mathcal{C}^0, \text{ if } a > 0$$

$$(5.24) \quad f \in \mathcal{C}^2, \text{ if } a = 0$$

$$(5.25) \quad f \text{ analytic, if } a < 0$$

Therefore, (5.21), (5.22) cannot have solutions U, V in $\mathcal{C}^1_{nd}$.

Two more examples.

It was shown that the PDE

$$(5.26) \qquad D_y(u(x,y) + iv(x,y)) + iyD_x(u(x,y) + iv(x,y)) =$$

$$= f(x,y) + ig(x,y), \quad (x,y) \in \mathbb{R}^2$$

does *not* have any distributional solutions in any neighbourhood of $(0,0) \in \mathbb{R}^2$, for certain $\mathcal{C}^\infty$-smooth f, g. Now, if we re-write (5.26) as the system

$$(5.27) \qquad \begin{aligned} &D_y u(x,y) - yD_X v(x,y) = f(x,y) \\ &D_y v(x,y) + yD_x u(x,y) = g(x,y) \end{aligned}$$

then it is covered by the above Theorem's extension to systems.

Last, the celebrated 1957 example of Hans Lewy is the PDE

$$(5.28) \qquad D_x U(x,y,z) + iD_y U(x,y,z) - 2i(x+iy)D_z U(x,y,z) =$$

$$= f(x,y,z), \quad (x,y,z) \in \mathbb{R}^3$$

which does *not* have distributional solutions in any neighbourhood of any point in $\mathbb{R}^3$, for a large class of $\mathcal{C}^\infty$-smooth functions f.

References

[1] Oberguggenberger M B, Rosinger E E : Solutions of Continuous Nonlinear PDEs through Order Completion. North-Holland Mathematics Studies, Vol. 181, Noth-Holland, Amstedam, 1994

[2] Rosinger E E : Hausdorff continuous solutions of arbitrary continuous nonlinear PDEs through the order completion method. arXiv:math/0405546

[3] Anguelov R, Rosinger E E : Hausdorff continuous solutions of nonlinear PDEs through the order completion method. arXiv:math/0406517

[4] Rosinger E E : How to solve smooth nonlinear PDEs in algebras of generalized functions with dense singularities. arXiv:math/0406594

[5] Rosinger E E : Can there be a general nonlinear PDE theory for the existence of solutions ? arXiv:math/0407026, updated version in http://hal.archives-ouvertes.fr/hal-00512015

[6] Anguelov R, Rosinger E E : Solving large classes of nonlinear systems of PDEs. arXiv:math/0505674

[7] Rosinger E E : Solving General Equations by Order Completion. arXiv:math/0608450

[8] Rosinger E E : Extending Mappings between Posets. arXiv:math/0609234

[9] Rosinger E E : Survey on Singularities and Differential Algebras of Generalized Functions : A Basic Dichotomic Sheaf Theoretic Singularity Test. Lambert Academic Publishing, 2013, ISBN : 978-3-659-50531-7, see also http://hal.archives-ouvertes.fr/hal-00510751, http://vixra.org/abs/1008.0025

[10] See in the References in [9] above, the related papers of Jan Harm van der Walt.