Tail behavior of the generalized exponential and Maxwell distributions*

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Abstract: Motivated by Finner et al. (2008), the asymptotic behavior of the probability density function (pdf) and the cumulative distribution function (cdf) of the generalized exponential and Maxwell distributions are studied. Specially, we consider the asymptotic behavior of the ratio of the pdfs (cdfs) of the generalized exponential and Student's t-distributions (likewise for the Maxwell and Student's t-distributions) as the degrees of freedom parameter approach infinity in an appropriate way. As by products, Mills' ratios for the generalized exponential and Maxwell distributions are gained. Moreover, we illustrate some examples to indicate the application of our results in extreme value theory.

Keywords: Generalized exponential distribution; Maxwell distribution; Mills' ratio; Student's t-distribution.

Mathematics Subject Classification(2010): 60F15, 60G70

1 Introduction

The generalized exponential and Maxwell distributions are quickly becoming preferred probability distributions in economics and duration analysis due to both models of subfamilies of the general gamma are simpler and more flexible than the general gamma distribution, see Morteza and Alireza (2010) for details.

Generalized exponential distribution (GE for short) was introduced by Gupta and Kundu (1999). The generalized exponential distribution can be found important applications in survival analysis, lifetime analysis of product, reliability engineering and geophysical signal time-frequency analysis. It is also observed that the generalized exponential distribution is

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quite flexible and can be used quite effectively in analyzing positive lifetime data in place of well-known gamma, Weibull or log-normal distributions, for details see Gupta and Kundu (2001) and Gupta and Kundu (2007). At the same time, the GE distribution has a nice physical interpretation also. Suppose, there are n-components in a parallel system and the lifetime distribution of each component is independent and identically distributed. If the lifetime distribution of each component is GE, then the lifetime distribution of the system is also GE.

The probability density function (pdf) of the generalized exponential random variable is given by

$$g_{\lambda,\alpha}(x) = \alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x},\ x>0, \tag{1.1}$$

where $\alpha,\ \lambda > 0$. Let $G_{\lambda,\alpha}(\cdot)$ denote the corresponding cumulative distribution function (cdf). When the parameter $\alpha = 1$, it coincides with the exponential distribution (lifetime distribution).

The pdf of the Maxwell random variable is given by

$$m_\sigma(x) = \sqrt{\frac{2}{\pi}}\frac{x^2}{\sigma^3}\exp(-\frac{x^2}{2\sigma^2}),\ x>0, \tag{1.2}$$

where $\sigma > 0$. Let $M_\sigma(\cdot)$ denote the corresponding cumulative distribution function (cdf). The Maxwell distribution have a variety of areas that range from chemistry to physics, particularly in statistical mechanics. It has also attracted interesting applications in molecular speeds, ideal gases close to thermodynamic equilibrium, negligible quantum effects, and non-relativistic speeds, describing the distribution of the momenta and energy of the molecules and studying gases.

Finner et al. (2008) investigated the tail behavior of the Student's t-distribution with respect to the normal distribution when the degrees of freedom goes to infinity in an appropriate way. They established some interesting results consisting of the asymptotic behavior of the ratio of the densities, a large deviation theorem and asymptotic Mills' ratio of the Student's t-distribution. The study regarding tail property of distribution can be found in the recent references as follows. Peng et al. (2009) investigated tail property of the general error distribution and obtained associated asymptotic Mills-type ratio. Peng et al. (2009) discussed asymptotic ratios of the cdfs (pdfs) of the standard Laplace and the Student's t-distributions and standard logistic distributions. Lin and Peng (2010) considered the tail behavior of the short-tailed symmetric distribution and obtained the corresponding limiting distribution of maxima with i.i.d. random variable. Liao et al. (2013) studied tail behaviors, subexponentiality and extreme value distribution of logarithmic skew-normal distribution. Lin and Jiang (2012) considered a generalization of the short-tailed symmetric distribution and derived the corresponding asymptotic tail behavior, Mills' ratio and asymptotic distribution of the partial maximum.

Mills (1926) gave the following inequality and Mills' ratio of note for the normal distribution $\Phi(x)$ with the pdf $\phi(x)$:

$$x^{-1}(1+x^{-2})^{-1}\phi(x) < 1-\Phi(x) < x^{-1}\phi(x),\ \text{for } x>0, \tag{1.3}$$

and

$$\frac{1-\Phi(x)}{\phi(x)} \sim \frac{1}{x}, \text{ as } x \to \infty. \tag{1.4}$$

To derive the main results, we need the following one result.

TheoremA (Corollary 1.1, Finner et al. (2008)). *If* $\lim_{v\to\infty} x_v^4/v = \beta \in [0,\infty)$ *and* $\lim_{v\to\infty} x_v = \infty$, *we have*

$$\frac{F_v(-x_v)}{f_v(x_v)} \sim \frac{1}{x_v} \ (v \to \infty), \tag{1.5}$$

where $f_v(x)$ *and* $F_v(x)$ *denote the pdf and the cdf of the Student's t-distribution with degree of freedom* v, *respectively.*

The main objective of this article is to extend the above result for the generalized exponential and Maxwell distributions. The rest of the paper is organized as follows. In Section 2, the tail of the generalized exponential distribution compared with the Student's t-distribution are described and a Mills' ratio is obtained. The similarity for the Maxwell distribution is discussed in Section 3. Note that the respective tail behaviors controlled by the scale parameters λ in (1.1) and σ in (1.2). For the Student's t-distribution, the scale is the degrees of freedom parameter v controlling its tail. Therefore, it is of interest to know how the tails of the three distributions contrast between them.

2 Tail behavior of the generalized exponential distribution

With the preceding notation, for fixed v, λ, and α, it is easy to check that

$$\frac{f_v(x)}{g_{\lambda,\alpha}(x)} \to \infty, \ \frac{1-F_v(x)}{1-G_{\lambda,\alpha}(x)} \to \infty \tag{2.1}$$

as $x \to \infty$.

Here, we describe the tail of the generalized exponential distribution compared to the Student's t-distribution when the degree of freedom of the latter approaches infinity in an appropriate way.

Theorem 2.1. *For* $x > 0$, *suppose* $v = v(x)$, $\lambda = \lambda(x)$ *such that*

$$\lambda = \frac{x}{2}, \ \lim_{v\to\infty}\left(\frac{x^4}{v} - 4\log x\right) = \beta \in [0,\infty) \tag{2.2}$$

holds. For fixed $\alpha > 0$, *we have*

$$\lim_{v\to\infty} \frac{f_v(x)}{g_{\lambda,\alpha}(x)} = \frac{2}{\sqrt{2\pi\alpha}} \exp\left(\frac{\beta}{4}\right). \tag{2.3}$$

Proof. Observing that

$$\frac{f_v(x)}{g_{\lambda,\alpha}(x)} = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})}\frac{1}{\sqrt{v\pi}}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}\frac{1}{\alpha\lambda}(1-\exp(-\lambda x))^{-(\alpha-1)}\exp(\lambda x) \tag{2.4}$$

and since

$$\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}\exp(-x+\theta(x)), \text{ as } x>0, \tag{2.5}$$

where $\theta(x) = \frac{\xi}{12x}$, $0<\xi<1$, we have

$$\frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})}\frac{1}{\sqrt{v\pi}} \to \frac{1}{\sqrt{2\pi}}, \text{ as } v\to\infty. \tag{2.6}$$

Put

$$k_{\alpha,\lambda,v}(x) = \frac{1}{\lambda}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}(1-\exp(-\lambda x))^{-(\alpha-1)}\exp(\lambda x).$$

Then, for sufficiently large v, we have

$$\begin{aligned}
\log(k_{\alpha,\lambda,v}(x)) &= -\log\lambda+\lambda x-\frac{v+1}{2}\log\left(1+\frac{x^2}{v}\right)-(\alpha-1)\log(1-\exp(-\lambda x)) \\
&= -\log\lambda+\lambda x-\frac{v+1}{2}\log\left(1+\frac{x^2}{v}\right)+o(1) \\
&= -\log\lambda+\lambda x-\frac{v+1}{2}\left(\frac{x^2}{v}-\frac{x^4}{2v^2}+O\left(\frac{x^6}{v^3}\right)\right) \\
&= \log 2+\frac{x^2}{2}-\frac{x^2}{2}+\left(\frac{v+1}{4v}\frac{x^4}{v}-\log x\right)-\frac{x^2}{2v}+O\left(\frac{x^6}{v^2}\right) \\
&= \log 2+\frac{\beta}{4}+o(1),
\end{aligned}$$

and by (2.6) we complete the proof of Theorem 2.1. □

The following result is the corresponding large deviation theorem.

Theorem 2.2. *Under the condition of Theorem* 2.1, *we have*

$$\lim_{v\to\infty}\frac{1-F_v(x)}{1-G_{\lambda,\alpha}(x)} = \frac{1}{\sqrt{2\pi}\alpha}\exp\left(\frac{\beta}{4}\right). \tag{2.7}$$

Proof. We present a result for the Student's t distribution in Soms (1983, 1984):

$$\left(\frac{1}{x}+\frac{x}{v}\right)\left(1-\frac{v}{v+2}\frac{1}{x^2}\right) < \frac{1-F_v(x)}{f_v(x)} < \left(\frac{1}{x}+\frac{x}{v}\right) \tag{2.8}$$

for all $x>0$ and $v>0$. Utilizing (2.2), we have

$$\lim_{v\to\infty}\frac{g_{\lambda,\alpha}(x)}{1-G_{\lambda,\alpha}(x)} = \frac{x}{2}r_\alpha(x), \tag{2.9}$$

where

$$r_\alpha(x) = \left(1 - \exp\left(-\frac{x^2}{2}\right)\right)^{\alpha-1} \left(1 + O\left(\exp\left(-\frac{x^2}{2}\right)\right)\right)^{-1}$$

with

$$\lim_{x\to\infty} r_\alpha(x) = 1.$$

Note that

$$\frac{1 - F_v(x)}{1 - G_{\lambda,\alpha}(x)} = \frac{1 - F_v(x)}{f_v(x)} \frac{f_v(x)}{g_{\lambda,\alpha}(x)} \frac{g_{\lambda,\alpha}(x)}{1 - G_{\lambda,\alpha}(x)}, \tag{2.10}$$

which combining with (2.8) and (2.9), we have

$$\left(1 + \frac{x^2}{v}\right)\left(1 - \frac{v}{v+2}\frac{1}{x^2}\right)\frac{r_\alpha(x)}{2}\frac{f_v(x)}{g_{\lambda,\alpha}(x)} < \frac{1 - F_v(x)}{1 - G_{\lambda,\alpha}(x)} < \left(1 + \frac{x^2}{v}\right)\frac{r_\alpha(x)}{2}\frac{f_v(x)}{g_{\lambda,\alpha}(x)}, \tag{2.11}$$

then by using Theorem 2.1 and (2.2), the conclusion can be deduced. □

Corollary 2.1. *Under the condition of Theorem* 2.1*, we have*

$$\frac{1 - G_{\lambda,\alpha}(x_v)}{g_{\lambda,\alpha}(x_v)} \sim \frac{2}{x_v}\ (v \to \infty). \tag{2.12}$$

Proof. Noting that (2.2) implies $\lim_{v\to\infty} x^2/v = 0$, the result follows directly Theorem A, Theorem 2.1 and Theorem 2.2. □

Remark 2.1. The Mills ratios such as (2.12), (1.4), (1.5) are extremely important in considering some behavior of economic and financial data. Further, the hazard rate (failure rate) is equal to the reciprocal of Mills ratio.

In the following part, we will consider the asymptotic behavior of the ratio of densities of Maxwell distribution and Students' t distribution as the degrees of freedom tends to infinity in some way. Note that for the fixed σ, v, easily check that

$$\frac{f_v(x)}{m_\sigma(x)} \to \infty, \ \frac{1 - F_v(x)}{1 - M_\sigma(x)} = \frac{F_v(-x)}{M_\sigma(-x)} \to \infty \tag{2.13}$$

as $x \to \infty$.

3 Tail behavior of the Maxwell distribution

Theorem 3.1. *Let* $\sigma = \sigma(x)$, $v = v(x)$ *such that*

$$\sigma = \left(\frac{v}{v+1}\right)^{1/2}, \ \lim_{v\to\infty}\left(\frac{x_v^4}{v} - 8\log x_v\right) = \beta \in [0, \infty), \tag{3.1}$$

then for $x > 0$*, we have*

$$\lim_{v\to\infty} \frac{f_v(x)}{m_\sigma(x)} = \frac{1}{2}\exp\left(\frac{\beta}{4}\right).$$

Proof. Note that

$$\frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})}\frac{1}{\sqrt{2v}} \to \frac{1}{2}, \text{ as } v \to \infty, \tag{3.2}$$

and

$$\frac{f_v(x)}{m_\sigma(x)} = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})}\frac{1}{\sqrt{2v}}\frac{\sigma^3}{x^2}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}} \exp\left(\frac{x^2}{2\sigma^2}\right). \tag{3.3}$$

Let

$$h_{\sigma,v}(x) = \frac{\sigma^3}{x^2}\left(1+\frac{x^2}{v}\right)^{-\frac{v+2}{2}} \exp\left(\frac{x^2}{2\sigma^2}\right),$$

then for sufficiently large v, we obtain

$$\begin{aligned}
\log(h_{\sigma,v}(x)) &= 3\log\sigma - 2\log x + \frac{x^2}{2\sigma^2} - \frac{v+1}{2}\log\left(1+\frac{x^2}{v}\right) \\
&= 3\log\sigma - 2\log x + \frac{x^2}{2\sigma^2} - \frac{v+1}{2}\left(\frac{x^2}{v} - \frac{x^4}{2v^2} + O\left(\frac{x^6}{v^3}\right)\right) \\
&= \frac{3}{2}\log\frac{v}{v+1} + \left(\frac{v+1}{4v}\frac{x^4}{v} - 2\log x\right) + \frac{v+1}{2v}x^2 - \frac{v+1}{2v}x^2 + O\left(\frac{x^6}{v^2}\right) \\
&= \frac{\beta}{4} + o(1),
\end{aligned}$$

which combining with (3.2) finish the proof of theorem 3.1. □

For the asymptotic behavior of Students' t-distribution and Maxwell distribution, we have the following large deviation theorem.

Theorem 3.2. *Under the condition of theorem* 3.1*, we have*

$$\lim_{v\to\infty}\frac{F_v(-x)}{M_\sigma(-x)} = \frac{1}{2\sigma^2}\exp\left(\frac{\beta}{4}\right). \tag{3.4}$$

Proof. By integration by parts, we can obtain the following inequalities which is used for the proof of the theorem, *i.e.*

$$\sigma^2\frac{1}{x}\left(1+\frac{\sigma}{x}\right)^{-1} < \frac{M_\sigma(-x)}{m_{\sigma^2}(x^2)} < \sigma^2\frac{1}{x}\left(1+\frac{\sigma}{x}\right). \tag{3.5}$$

Since the similar deduction of above equalities can be found in Lin and Peng (2010), we omit that process. By combining both chains of inequalities (2.8) and (3.5), it follows

$$\begin{aligned}
&\frac{1}{\sigma^2}\left(1+\frac{x^2}{v}\right)\left(1+\frac{\sigma}{x}\right)^{-1}\left(1-\frac{v}{v+2}\frac{1}{x^2}\right)\frac{f_v(x)}{m_\sigma(x)} < \frac{F_v(-x)}{M_\sigma(-x)} \\
&< \frac{1}{\sigma^2}\left(1+\frac{x^2}{v}\right)\left(1+\frac{\sigma^2}{x^2}\right)\frac{f_v(x)}{m_\sigma(x)}.
\end{aligned} \tag{3.6}$$

By utilizing Theorem 3.1 and (3.1), we have

$$\limsup_{v\to\infty} \frac{F_v(-x)}{M_\sigma(-x)} \le \frac{1}{2\sigma^2} \exp\left(\frac{\beta}{4}\right). \tag{3.7}$$

Analogously, for lower bound, we have

$$\liminf_{v\to\infty} \frac{F_v(-x)}{M_\sigma(-x)} \ge \frac{1}{2\sigma^2} \exp\left(\frac{\beta}{4}\right). \tag{3.8}$$

Combining with (3.7) and (3.8), the proof of the result is completed. $\square$

Corollary 3.1. *Under the condition of Theorem* 3.1*, we have*

$$\frac{M_\sigma(-x)}{m_\sigma(x)} \sim \frac{\sigma^2}{x_v} \ (v \to \infty).$$

Proof. Notice that (3.1) implies $\lim_{v\to\infty} x^2/v = 0$, the result follows directly Theorem A, Theorem 3.1 and Theorem 3.2. $\square$

Remark 3.1. For $\sigma > 0$, an application of Corollary 3.1 is to show that $M_\sigma \in D(\Lambda)$, i.e., there exist normalizing constants $a_n > 0$ and $b_n \in R$ such that $M_\sigma(a_n x + b_n) \to D(\Lambda)$, as $n \to \infty$, where $D(\Lambda)$ denotes the domain of attraction of $D(\Lambda) = \exp\{-e^{-x}\}$.

Since

$$\frac{(d/dx)m_\sigma(x)}{m_\sigma(x)} = \frac{1}{x}\left(2 - \frac{x^2}{\sigma^2}\right),$$

we have by Corollary 3.1 that

$$\frac{M_\sigma(-x)}{m_\sigma(x)} \frac{(d/dx)m_\sigma(x)}{m_\sigma(x)} \to -1$$

as $x \to \infty$, thus, by Proposition 1.18 in Resnick (1987) that $M_\sigma \in D(\Lambda)$.

The following corollary will give the representation of $M_\sigma(-x)$ by application of Corollary 3.1.

Corollary 3.2. *Under the condition of Corollary* 3.1*, we have*

$$M_\sigma(-x) = c(x) \exp\left\{-\int_1^x (g(t)/f(t))\, \mathrm{d}t\right\},$$

for sufficiently large x, where

$$c(x) \to \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2\sigma^2}\right),\ x \to \infty,$$

$$g(t) = 1 - \frac{\sigma^2}{t^2} \text{ and } f(t) = \frac{\sigma^2}{t}.$$

Proof. According to Corollary 3.1 and some simple calculation, the conclusion can be deduced. □

Remark 3.2. The representation of $M_\sigma(-x)$ given by Corollary 3.2 has some applications. A direct application of Corollary 3.2 is to show $M_\sigma \in D(\Lambda)$ for $\sigma > 0$, see Corollary 1.7 of Resnick (1987). We may also consider the uniform convergence of the distribution of normalized maxima (cf. Resnick, 1987, Sec. 2.4.2) and large deviation properties (cf. Resnick, 1987, Proposition 2.10) as the auxiliary function $f(t)$ plays an important role in these studies. Besides more significant application of Corollary 3.2 is to obtain the optimal normalized constants β_n satisfying

$$\sqrt{\frac{\pi}{2}}\frac{\sigma}{\beta_n}\exp\left(\frac{\beta_n}{2\sigma^2}\right) = n$$

and $\alpha_n = f(\beta_n)$ such that

$$C_1/\log n \le \sup|M_\sigma(a_n x + b_n) - \Lambda(x)| \le C_2/\log n,$$

for $n > n_0$, here C_1 and C_2 are constants depending on σ. As to the rate of convergence for the Maxwell distribution, it is difficult and is not in the scope of the present paper.

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