Electro-Osmosis With Corrected Solution of Poisson-Boltzmann Equation That Satisfies Charge Conservation Principle

Rajib Chakraborty*

89/3 Tanupukur Road,

Dhakuria, Calcutta 700 031,

West Bengal, India

(Dated: September 13, 2014)

Abstract

We derive the electro-osmotic velocity profile in a micro-channel using a recently corrected charge density distribution within an electrolytic solution. Previous distribution did not take care of charge conservation principle while solving Poisson-Boltzmann equation and needed modification, hence the velocity profile also needs modification that we do here. Helmholtz-Smoluchowskii velocity scale is redefined, which accommodates Debye length parameter in it, unlike old definition.

PACS numbers: 47.61.-k, 82.45.-h, 47.65.-d, 85.85.+j

* mecrajib@gmail.com

Electro-osmosis concerns driving an electrolytic solution through a narrow channel using an externally applied axial electric field, exploiting a possible non-trivial charge density distribution (ρ_e) in the solution. It has a very old history [1, 2] and finds applications in many fields [3–5]. Now, ρ_e appears in the body force term in the fluid momentum equation, which takes a very simple form for a flow with small Reynolds number (typical for a narrow channel) and when the flow is steady and hydrodynamically fully developed. For a rectangular geometry, the momentum equation is given by [6, 7]:

$$0 = \mu \frac{d^2 v}{dx^2} + E_y \rho_e \tag{1}$$

ρ_e and fluid velocity v vary essentially along the smallest side ('$2a$') of geometry (x direction); E_y is external electric field, applied in y direction; μ is uniform viscosity. Now, solving Poisson-Boltzmann (PB) equation we obtain ρ_e as spatial function that is needed to solve the above equation. However, the old solution [8, 9] did not take care of charge conservation principle properly. This author attempted to remove that discrepancy in Ref [10, 11], and obtained a velocity profile using it in Ref [7]. However, that formulation for ρ_e still had problem, for it was not satisfying Poisson's equation in general. Finally, the correct expression for ρ_e (scaled) has been developed in Ref [12], and is given by,

$$\rho_e^* = \frac{1}{2\sinh(\kappa)} \left[q_0 \kappa \cosh(\kappa\eta) - \delta \sinh(\kappa\eta) \right] \tag{2}$$

Meaning of symbols can be found in Ref [7, 12]. We mention them briefly:

$$\kappa \equiv a/\lambda_D; \quad \eta \equiv x/a; \quad \rho_e^* \equiv \rho_e/\rho_0; \quad \rho_0 \equiv \left(\epsilon\kappa^2\zeta/a^2\right) \tag{3}$$

Where, λ_D is Debye length [8]; ϵ is permittivity of liquid; ζ is a suitable scale for electrostatic potential ψ, ($\zeta > 0$).

δ is the potential difference between walls at $\eta = +1$ and $\eta = -1$. If Q_0 is the net charge present in liquid (in a cross-section, per unit axial length), $\int_{-1}^{+1} \rho_e^* d\eta = Q_0/\rho_0 \equiv q_0$, (using Eq. 3). Now, using Eq. 3 and Eq. 2, we can write Eq. 1 as,

$$\begin{aligned} \frac{d^2 v}{d\eta^2} &= -\left(\frac{a^2 E_y \rho_0}{\mu}\right) \rho_e^* \\ &= -\left(\frac{\epsilon\zeta E_y}{\mu}\right) \kappa^2 \rho_e^* \\ &= -M\kappa^2 \left[q_0 \kappa \cosh(\kappa\eta) - \delta \sinh(\kappa\eta) \right] \end{aligned} \tag{4}$$

$$\text{Where, } M \equiv \left(\frac{\epsilon\zeta E_y}{2\mu \sinh(\kappa)}\right) \tag{5}$$

Intergrating Eq. 4 twice w.r.t η we get,

$$v = -M\left[q_0\kappa\cosh(\kappa\eta) - \delta\sinh(\kappa\eta)\right] + C_1\eta + C_2 \tag{6}$$

We use no-slip conditions at both walls, i.e. $v = 0$ at $\eta = \pm 1$. Hence,

$$0 = -M\left[q_0\kappa\cosh(\kappa) - \delta\sinh(\kappa)\right] + C_1 + C_2 \tag{7}$$

$$0 = -M\left[q_0\kappa\cosh(\kappa) + \delta\sinh(\kappa)\right] - C_1 + C_2 \tag{8}$$

From Eq. 7 and Eq. 8 we solve for C_1 and C_2 and get,

$$C_1 = -M\delta\sinh(\kappa) \tag{9}$$

$$C_2 = Mq_0\kappa\cosh(\kappa) \tag{10}$$

Using Eq. 9 and Eq. 10 in Eq. 6 and rearranging terms,

$$\begin{aligned} v &= M\left[q_0\kappa\left(\cosh(\kappa) - \cosh(\kappa\eta)\right) - \delta\left(\eta\sinh(\kappa) - \sinh(\kappa\eta)\right)\right] \\ &= M\left[q_0\kappa\cosh(\kappa)\left(1 - \frac{\cosh(\kappa\eta)}{\cosh(\kappa)}\right) - \delta\sinh(\kappa)\left(\eta - \frac{\sinh(\kappa\eta)}{\sinh(\kappa)}\right)\right] \\ &= M\kappa\cosh(\kappa)\left[q_0\left(1 - \frac{\cosh(\kappa\eta)}{\cosh(\kappa)}\right) - \delta\frac{\tanh(\kappa)}{\kappa}\left(\eta - \frac{\sinh(\kappa\eta)}{\sinh(\kappa)}\right)\right] \end{aligned} \tag{11}$$

Now, using Eq. 5 we get,

$$\begin{aligned} M\kappa\cosh(\kappa) &= \frac{\epsilon\zeta E_y}{2\mu\sinh(\kappa)}\kappa\cosh(\kappa) \\ &= \lambda_{||}\frac{\epsilon\zeta E_0\kappa}{2\mu\tanh(\kappa)} \\ \text{Where, } \lambda_{||} &\equiv \frac{E_y}{E_0} \text{ ,with } E_0 > 0 \end{aligned} \tag{12}$$

Let us define corrected Helmholtz-Smoluchowskii velocity scale $v_{H.S.Corr}$ by,

$$v_{H.S.Corr} \equiv \frac{\epsilon\zeta E_0\kappa}{2\mu\tanh(\kappa)} \tag{13}$$

It differs from old Helmholtz-Smoluchowskii velocity scale, see Ref [8]. Let $\bar{v} \equiv v/v_{H.S.Corr}$. Finally we arrive at,

$$\bar{v} = \lambda_{||}\left[q_0\left(1 - \frac{\cosh(\kappa\eta)}{\cosh(\kappa)}\right) - \delta\frac{\tanh(\kappa)}{\kappa}\left(\eta - \frac{\sinh(\kappa\eta)}{\sinh(\kappa)}\right)\right] \tag{14}$$

When we reverse E_y, velocity field must reverse too. In the above equation, the sign of $\lambda_{||}$ changes when we reverse E_y, and hence $\bar{v}$ changes sign as expected; in old works it was not possible to capture this reversal of direction in the scaled velocity, because $v_{H.S}$ was not defined properly. This author made an attempt to correct it in Ref [7], however ρ_e was still not correct there. We can control the fluid flow easily using the potential difference.

---

[1] J. Lyklema, Colloids and Surfaces A: Physicochem. Eng. Aspects **222**, 5 (2003).

[2] S. Wall, Current Opinion in Colloid & Interface Sci. **15**, 119 (2010).

[3] C. Zhao and C. Yang, Microfluid Nanofluid (2012), 10.1007/s10404-012-0971-1.

[4] H. Stone, A. Stroock, and A. Ajdari, Annu. Rev. Fluid Mech. **36**, 381 (2004).

[5] R. B. Schoch, J. Han, and P. Renaud, Rev. Mod. Phys. **80**, 839 (2008).

[6] S. S. Dukhin, Advances in Colloid and Interface Science **35**, 173 (1991).

[7] R. Chakraborty, "Plug-flow does not exist in electro-osmosis," http://vixra.org/abs/1406.0031 (2014).

[8] S. Chakraborty and D. Paul, Journal of Physics D: Applied Physics **39**, 5364 (2006).

[9] H. Bruus, *Theoretical Microfluidics* (Oxford University Press Inc., New York, United States, 2008).

[10] R. Chakraborty, "Mystery of missing co-ions solved," http://vixra.org/abs/1408.0186 (2014).

[11] R. Chakraborty, "Electric triple-layer theory," http://vixra.org/abs/1405.0354 (2014).

[12] R. Chakraborty, "Physical solution of poisson-boltzmann equation," http://vixra.org/abs/1408.0093 (2014).