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CURVATURE OF n-DIMENSIONAL ELLIPSOIDS EMBEDDED IN R^{n+1}

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SUMMARY: The curvature tensor and scalar is computed for n up to 6 with the computer algebra system STENSOR. From that new empirical material, formulae for any n are deduced. For the special case of a sphere, these coincide with wellknown results.

DEFINITIONS:

The ellipsoid: $\Sigma_{a=1}^{n+1}(x^a/A_a)^2 = 1$ is parametrized as: $x^a = A_a f^a(q^1,...,q^n)$, where $-1 \le f^a \le 1$ and $f^a f^a = 1$, where from now on summation is implied, and a,b..=1,...,n+1 and i,j..=1,...,n.

THE FOLLOWING OBJECTS LEAD TO THE CURVATURE: DEFINITION:

Metric on the ellipsoid, induced by euclidean metric in R^{n+1}: $g_{ij} = x^a_{,i}\, x^a_{,j}$
(",i" indicate differentiation w. r. to coordinate q^i)

The (un-normalized) vector field normal to the ellipsoid: $m_a = e_{aijk..l}\, x^i_{,1} ... x^l_{,n}$

The second fundamental form of the surface: $a_{ij} = x^a_{,i,j}\, m_a/\sqrt{g}$

The Riemann tensor, through Gauss equations: $R_{ijkl} = a_{j[k}\, a_{l]i}$
-which gives the Ricci curvature scalar in the usual way.

COMPUTATION: Eg the definition of g is put in by: (pdef g) <x ^a ,i<<x ^a ,j< ie after the PDEF command type the formula just as above, but with "<" around tensors. Repeated "A" mean summation, and "^" contravariance. Output is as above. STENSOR can also compute indicially, but here we choose to make explicit both n and all f-components, whereby the components of "a" and "R" etc are computed according to the definitions. The results for the individual n=2,..,6 (CPU-time ≈30h on VAX11/750) suggest for arbitrary n the following:

RESULTS: With the parametrization $f^a = (\Pi_{i=1}^{a-1} \cos q^i) \sin q^a$, we obtain for the determinant:

$g = \det g_{ij} = W\, g_s \cos^2 q^{n-1} \cos^4 q^{n-2} ... \cos^{2(n-1)} q^1$ where $W = \Pi A^2_a$ and $g_s = \Sigma_{a=1}^{n+1}(f^a/A_a)^2$.

The 2:nd fundamental form turned out to be diagonal. Indeed,it is shown that any parametrization that makes the metric diagonal in the spherical case, diagonalizes a_{ij} in all cases!

The Riemann tensor became: $R_{ijkl} = \varkappa_{(i)}\, \varkappa_{(j)}\, \delta_{j[k}\, \delta_{l]i}$ where $\varkappa_{(i)} = (\Pi_{k=1}^{i-1} \cos^2 q^k)/\sqrt{g_s}$

Explicit form of the Ricci scalar curvature: $R = -2\, \lambda\, W^{-1}\, g_s^{-2}$,

where $\lambda =$ (sum of all (n-2)-plets of $\{A^2_1, A^2_2, ..., A^2_n\}$) +
+(sum of all (n-3)-plets of $\{A^2_3, ..., A^2_{n+1}\}$) $(A^2_2 - A^2_1)$ $\cos^2 q^1$ +
+(sum of all (n-3)-plets of $\{A^2_1, A^2_4, ..., A^2_{n+1}\}$) $(A^2_3 - A^2_2)$ $\cos^2 q^1 \cos^2 q^2$ + ...+
+(sum of all (n-3)-plets of $\{A^2_1, A^2_2, ..., A^2_{n-1}\}$) $(A^2_{n+1} - A^2_n)$ $\cos^2 q^1 \cos^2 q^2 ... \cos^2 q^n$.

With "m-plet" we mean a product, selected from n factors without putting items back and disregarding order. The possible number of m-plets, ie terms, is then $\binom{n}{m}$. For a n-sphere of radius A, the axis differences make just the first part of λ remain, which contain $\binom{n}{n-2}$ identical terms $A^{2(n-2)}$. $W = A^{2n}$ and $g_s = 1/A^2$, so $R = -n(n-1)/A^2$ as expected. One application in mind is to use this ellipsoid as internal space in Kaluza-Klein theory.

Sine is excluded from λ to better show the structure. In the actual computation, the trigonometric simplifier[2] in STENSOR carefully selected such combinations of sine and cosine, that made λ and especially intermediate expressions s.a. g^{ij} considerably shorter. This optimal simplifier was necessary already from n=4 so as not to exhaust memory storage.

[1] Hornfeldt "STENSOR reference manual", Preprint, University of Stockholm (1986).
[2] Hornfeldt "A Sum-substitutor used as Trigonometric Simplifier", EUROCAM 82, ed Calmet, Springer-Verlag, Lecture Notes in Computer Science, vol.144, p188-195 (1982).