# An origin of transitivity and other useful relation's properties

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ABSTRACT. Usually itemizing relations' properties those of them are always pointed out – some appearance of reflexivity, symmetry and transitivity. Also it is not so clear whether they are introduced artificially – i.e. axiomatically, rather for the sake of convenience or it may be done due to inartificial reasons. At the same time an origin of them is not so clear – whether they appear chaotically and independently on each other or there should be rigorous association between them. It is shown here that request of relation's reversibility leads to these properties' presence or absence. Often symmetry appearances are defined by using of ambiguous way. In fact, anti-symmetry is not direct negation for symmetry – there is also something that may be called as asymmetry or it may be something else. To avoid it here there was found the unified method of their definition. The same thing may be told about reflexivity and it was shown that just the only intransitivity may be represented as direct negation of transitivity.

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## 1. Introduction

Usually the main properties of relations – such as some appearances of reflexivity, symmetry and transitivity, are noticed. But often their origin is not too plain to see. Perhaps it is due to their acquisition – they are rather obtained by using of some empiric observations. Meanwhile, it seems that they may be obtained as a result of discourses in general theory. So, here was initiated an attempt to proceed it.

To get transitivity condition definition of subset notion may be used biconditional expression

$$A \subset B \leftrightarrow \{x: x \in A \Rightarrow x \in B\}. \tag{1.1}$$

It is important here that there is single-valued correspondence between directions of implication and of inclusion one.

## 2. Semantic invariance

It is clear that there is nothing standing in the way for set and their elements to be notated by using assignment them some attributes – such as names, numbers etc., e.g., writings $X_1$ and $X_2$ may be represented as

$$\left.\begin{aligned} X_1 &= \{\langle x,1\rangle\} = \{\langle x,1\rangle : x \in X \wedge 1 \in \mathbb{N}\} \\ X_2 &= \{\langle x,2\rangle\} = \{\langle x,2\rangle : x \in X \wedge 2 \in \mathbb{N}\} \end{aligned}\right\}.$$

Direct sum of these sets forms a *set family* and summands themselves are turned out to be enumerated by some *index set* $I = \{i\}$. Here the index set is $I = \{1,2\} \subset \mathbb{N}$, that is index set is a subset of natural numbers set, but it also could be some another one set. Similar expression defines Cartesian product of two sets

$$X \times Y = \{\langle x,y\rangle : x \in X \wedge y \in Y\}.$$

Comparison between the expressions lets formally define set family as

$$\boldsymbol{X}_I = \coprod_{i\in I} X_i = X \times I \neq \emptyset \leftarrow I \neq \emptyset. \tag{2.1}$$

Usage of **fat** *italic* means here the notation of set family. So, direct sum appears here as the *co-product*. And it needs to get an agreement of which set is written first (indexed or indexing[1]). As long as set family is determined by this manner, i.e. not to be empty, index set cannot be null – otherwise family is empty

$$\coprod_{i\notin\emptyset} X_i = X \times \emptyset = \emptyset. \tag{2.2}$$

Generally, since multiplicands sequence order remains product should not change – it doesn't depend on kind of index set. It is consequence of ZF axiom of substitution. So, this statement is an expression of some property that probably may be called *semantic invariance*. However, it needs to find criteria of such invariance.

To solve this, it needs to write an expression that would unify formulae (2.1) with (2.2) one. One can notice that index set is not any special and it can be indexed by initial set – result shouldn't has changed

$$\boldsymbol{X} = X \times I \times X. \tag{2.3}$$

To be convinced that case described by formula (2.3) is unique, it needs to write more general expression – while index set indexing by some another set $Y \neq X$. As a result instead of binary Cartesian product it leads to *ternary* expression $\boldsymbol{X} = X \times I \times Y$. In order to get it, this expression should involve union of formulae (2.1) and (2.2); the set $I$ should be represented as set intersection $J \cap K$. Set union is not valid to do this because it is empty just when both operands are equal to null. So we may write

$$X \times J \cap K \times Y = \boldsymbol{X}_J \cap \boldsymbol{Y}_K^* \leftarrow I = J \cap K. \tag{2.4}$$

[1] Because disjoint set union is non-commutative as Cartesian product itself, excluding the case of self-indication.

Here new symbol $\boldsymbol{Y}_K^*$ was introduced and it is determined by formulae

$$\left.\begin{array}{c}\boldsymbol{Y}_K^* = K \times Y \\ \boldsymbol{Y}_k = Y \times K \\ \boldsymbol{Y}_K^* \ncong \boldsymbol{Y}_K\end{array}\right\}. \qquad (2.5)$$

They denote family that appears to be *dual* to initial one in sense of multiplicands sequence changing. The family $\boldsymbol{Y}_k$ is *covariant* to set $Y$ and family $\boldsymbol{Y}_K^*$ is *contra-variant* to it. Index below the symbol points out an indexing set. No doubt that both families are equipotential each other.

However, availability of indices in families' notation still makes them non-invariant – genuine invariant objects shouldn't contain them. In fact, presence of indices in left part of equality (2.4) is stipulated by difference between indexing sets that forms their intersection

$$X \times J \cap K \times Y \ncong X \times K \cap J \times Y = \boldsymbol{X}_K \cap \boldsymbol{Y}_J^*. \qquad (2.6)$$

It is worth to pay an attention that there is "non-commutativity" of indexing sets intersection – but generally this is contradiction. One of the ways to avoid the contradiction is to presume indexing sets to be disjoint. But such indexing is empty. The other way – presuming indexing sets to be coincident – it makes their intersection to be *idempotent*

$$X \times I \cap I \times Y = \boldsymbol{X}_I \cap \boldsymbol{Y}_I^*. \qquad (2.7)$$

It is not altered by indices replacement – it makes indices to be unnecessary. Obviously, there is no other way to avoid contradiction of non-commutative intersection. Anyway, such invariance is inherent to the case describing by formula (2.3). In new terms it looks like

$$\left.\begin{array}{c}\mathrm{id}_X = X \times I \cap I \times X = \boldsymbol{X}_I \cap \boldsymbol{X}_I^* \\ \mathrm{id}_Y = Y \times I \cap I \times Y = \boldsymbol{Y}_I \cap \boldsymbol{Y}_I^*\end{array}\right\}. \qquad (2.8)$$

These formulae determine set *diagonals*. Obviously, diagonal is the set of all diagonal ordered pairs that may be formed among elements of random (disordered) set. In addition, every set may be indexed by itself. So, one can write

$$\left.\begin{array}{c}\mathrm{id}_X = X \times X \cap X \times X = X^3 = \boldsymbol{X} \cap \boldsymbol{X}^* \\ \mathrm{id}_Y = Y \times Y \cap Y \times Y = Y^3 = \boldsymbol{Y} \cap \boldsymbol{Y}^*\end{array}\right\}. \qquad (2.9)$$

The last ones have quite not indices – it makes them semantically invariant initially.

## 3. Subsets of Cartesian product

As it widely known, any subset of Cartesian product is called *correspondence* between set-multiplicands. If multiplicands coincides each other, correspondence is called *relation* that is established on a set. So, binary relation is always determined on Cartesian square.

To satisfy semantic invariance, it is strongly recommended observing them as subsets of families and as some result of correspondences multiplication – correspondences *composition*. According to discourses that were described above, someone may write

$$\left.\begin{array}{c}\beta \subseteq A \times I = \boldsymbol{A} \\ \gamma \subseteq I \times B = \boldsymbol{B}^*\end{array}\right\}.$$

Therefore, compositions are

$$\left.\begin{array}{c}\xi = \beta \circ \gamma = \{\langle x, y\rangle : \langle x, i\rangle \in \beta \wedge \langle i, y\rangle \in \gamma\} \\ \xi^+ = \gamma \circ \beta = \{\langle x, y\rangle : \langle x, i\rangle \in \gamma \wedge \langle i, y\rangle \in \beta\}\end{array}\right\}. \quad (3.1)$$

One differs from another by sequential order of multiplicands. But it's too early to talk about semantic invariance of such constructions – to realize it there must be a possibility to simplify these expressions by elements of indexing sets. But expected phenomena may appear while both indexing sets fully coincide

$$\left.\begin{array}{c}\xi = \beta \circ \gamma = \{\langle x, y\rangle : \langle x, i\rangle \in \beta \wedge \langle i, y\rangle \in \gamma\} \subseteq A \times B = \boldsymbol{A} \\ \xi^+ = \gamma \circ \beta = \{\langle x, y\rangle : \langle x, i\rangle \in \gamma \wedge \langle i, y\rangle \in \beta\} \subseteq B \times A = \boldsymbol{A}^* \\ J = K = I\end{array}\right\}. \quad (3.2)$$

Forming in every multiplicand Cartesian square, there may form their subsets – set diagonals. And according to equality (2.8) one may write

$$\left.\begin{array}{c}\mathrm{id}_A = \{\langle x, x\rangle : x \in A\} = \{\langle x, x\rangle : \langle x, i\rangle \in \beta \wedge \langle i, x\rangle \in \beta^{-1}\} \\ \mathrm{id}_B = \{\langle y, y\rangle : y \in B\} = \{\langle y, y\rangle : \langle y, i\rangle \in \gamma \wedge \langle i, y\rangle \in \gamma^{-1}\} \\ x \neq y \Rightarrow \mathrm{id}_A \neq \mathrm{id}_B\end{array}\right\}. \quad (3.3)$$

Symbols $\beta^{-1}$ and $\gamma^{-1}$ are introduced here to denote correspondences that are *reversal* to initial ones. They are determined in accordance with expressions

$$\left.\begin{array}{c}\beta = \{\langle x, i\rangle : \langle x, i\rangle \in \beta\} \subseteq \boldsymbol{A}_I \\ \beta^{-1} = \{\langle i, x\rangle : \langle x, i\rangle \in \beta\} \subseteq \boldsymbol{A}_I^* \\ \gamma = \{\langle i, y\rangle : \langle i, y\rangle \in \gamma\} \subseteq \boldsymbol{B}_I^* \\ \gamma^{-1} = \{\langle y, i\rangle : \langle i, y\rangle \in \gamma\} \subseteq \boldsymbol{B}_I\end{array}\right\}. \quad (3.4)$$

Due to be indistinctive of permutable diagonal's elements one may say that they are wholly coincide with their reverse correspondences

$$\left.\begin{array}{c}\mathrm{id}_A = \mathrm{id}_A^{-1} = \{\langle x, x\rangle : \langle x, x\rangle \in \mathrm{id}_A\} \\ \mathrm{id}_B = \mathrm{id}_B^{-1} = \{\langle y, y\rangle : \langle y, y\rangle \in \mathrm{id}_B\}\end{array}\right\}. \quad (3.5)$$

Also, it is easy to show that they are some idempotent among correspondences multiplication

$$\left.\begin{array}{c}\mathrm{id}_A = \mathrm{id}_A \circ \mathrm{id}_A \\ \mathrm{id}_B = \mathrm{id}_B \circ \mathrm{id}_B\end{array}\right\}. \quad (3.6)$$

And even one can suppose that they are *identity* elements of such multiplication. But there is one thing standing on the way of this expectation. One may notice that the truth is in expressions

$$\left.\begin{array}{r}\beta \circ \mathrm{id}_A = \{\langle x,y\rangle : \langle x,y\rangle \wedge \langle y,y\rangle\} = \{\langle x,y\rangle : \langle x,y\rangle \in \beta\} = \beta \\ \mathrm{id}_B \circ \beta = \{\langle x,y\rangle : \langle x,x\rangle \wedge \langle x,y\rangle\} = \{\langle x,y\rangle : \langle x,y\rangle \in \beta\} = \beta\end{array}\right\}.$$

Consequently we have equalities

$$\beta \circ \mathrm{id}_A = \mathrm{id}_B \circ \beta = \beta. \tag{3.7}$$

They are too similar to *identities*, but they just play roles of *right* and *left* ones – their compositions are not commutative. In addition, identity element is unique – otherwise it will lead to contradiction

$$\left.\begin{array}{c}\mathrm{id}_A \neq \mathrm{id}_B \\ \mathrm{id}_A = \mathrm{id}_A \circ \mathrm{id}_B = \mathrm{id}_B\end{array}\right\}.$$

And one can assert that just following implication occurs

$$A \neq B \mapsto \mathrm{id}_A \neq \mathrm{id}_A \circ \mathrm{id}_B \neq \mathrm{id}_B. \tag{3.8}$$

Compositions $\mathrm{id}_A \circ \beta$ and $\beta \circ \mathrm{id}_B$ are not defined. So diagonals are *quasi-idempotent* among relation's multiplication.

In addition, inside diagonals there may be formed *pseudo-invariant* fragments of these *quasi-invariants*

$$\left.\begin{array}{r}\mathrm{Id}_A \supseteq A \times J \cap K \times A = \boldsymbol{A}_J \cap \boldsymbol{A}_K^* \\ \mathrm{id}_{AB} \supseteq B \times J \cap K \times B = \boldsymbol{B}_J \cap \boldsymbol{B}_K^*\end{array}\right\}. \tag{3.9}$$

It is in accordance with the discourse of previous section.

Also it is easy to show that such single-value equalities occur

$$\left.\begin{array}{r}(\beta^{-1})^{-1} = (\beta^{+})^{+} = \beta \\ (\beta \circ \gamma)^{-1} = (\beta^{-1} \circ \gamma^{-1})^{+} = \gamma^{-1} \circ \beta^{-1}\end{array}\right\}. \tag{3.10}$$

Particularly, for *inclusion* relation there may be written formulae

$$\left.\begin{array}{r}\subset^{-1} = (\supset^{-1})^{+} = \supset \\ (\subset \gamma)^{-1} = (\subset^{-1} \gamma^{-1})^{-1} = \gamma^{-1} \supset\end{array}\right\}. \tag{3.11}$$

But to avoid contradictions it needs to make it carefully.

Generally, it's not possible to say that even composition $\beta \circ \beta^{-1}$ is commutative, because in accordance with definition (3.2) we may write

$$\left.\begin{array}{c}\beta \circ \beta^{-1} = \{\langle x,x\rangle : \langle x,j\rangle \in \beta \wedge \langle k,x\rangle \in \beta^{-1}\} \subseteq \boldsymbol{A}_I = \boldsymbol{A}_J \cap A_K^* \\ \beta^{-1} \circ \beta = \{\langle y,y\rangle : \langle y,k\rangle \in \beta^{-1} \wedge \langle j,y\rangle \in \beta\} \subseteq \boldsymbol{B}_I^* = \boldsymbol{B}_J \cap \boldsymbol{B}_K^* \\ \mathrm{id}_A \neq \beta \circ \beta^{-1} \neq \beta^{-1} \circ \beta \neq \mathrm{id}_B\end{array}\right\}. \tag{3.12}$$

Even though correspondences formed at the Cartesian square are quite coincide, composition of such relation is not necessarily commutative

$$\left.\begin{array}{c}\rho\circ\rho^{-1}=\{\langle x,x\rangle:\langle x,j\rangle\in\rho\wedge\langle k,x\rangle\in\rho^{-1}\}\subseteq\boldsymbol{A}_I=\boldsymbol{A}_J\cap A_K^*\\ \rho^{-1}\circ\rho=\{\langle y,y\rangle:\langle y,k\rangle\in\rho^{-1}\wedge\langle j,y\rangle\in\rho\}\subseteq\boldsymbol{A}_I^*=\boldsymbol{A}_J\cap\boldsymbol{A}_K^*\\ \mathrm{id}\neq\rho\circ\rho^{-1}\neq\rho^{-1}\circ\rho\neq\mathrm{id}\end{array}\right\}. \quad (3.13)$$

Equalities (3.7) for relations look like

$$\rho\circ\mathrm{id}=\mathrm{id}\circ\rho=\rho. \quad (3.14)$$

It becomes "almost[2]" *genuine* double-sided *identity*, its composition becomes commutative. There may also be determined *squaring* of relations. Obviously, it is commutative composition

$$\rho^2=\rho\circ\rho=\{\langle x,y\rangle:\langle x,i\rangle\in\rho\wedge\langle i,y\rangle\in\rho\}\subseteq\boldsymbol{A}. \quad (3.15)$$

It is subset of semantic invariant family, so there is a possibility to expect the same from such composition. Generalizing inductively, there may be determined *arbitrary powering* by using formulae

$$\left.\begin{array}{c}\underbrace{\rho\circ\rho\circ\ldots\circ\rho}_{n+1}=\underbrace{\rho\circ\rho\circ\cdots\circ\rho}_{n}\circ\rho\\ \rho^{n+1}=\rho^n\circ\rho\end{array}\right\}. \quad (3.16)$$

There is not such possibility for correspondences in general because similar expression looks like

$$\beta\circ\beta=\{\langle x,y\rangle:\langle x,i\rangle\in\beta\wedge\langle i,y\rangle\in\beta\}\subseteq\boldsymbol{A}_J\cap\boldsymbol{B}_K^*. \quad (1.3.17)$$

Obviously, they do not satisfy to pointed requirements.

## 4. Relation's properties that may exist together

Semantic invariance for expression (3.15) leads it to simplifying by index set elements. Perhaps such relation $\tau$, satisfying this requirement, exists, so, expression (3.15) transfers to implication or inclusion

$$\left.\begin{array}{c}\langle x,i\rangle\in\tau\wedge\langle i,y\rangle\in\tau\Rightarrow\langle x,y\rangle\in\tau\\ x\tau i\wedge i\tau y\Rightarrow x\tau y\\ \{\langle x,y\rangle:\langle x,i\rangle\in\tau\wedge\langle i,y\rangle\in\tau\}\subseteq\{\langle x,y\rangle:\langle x,y\rangle\in\tau\}\\ \tau\circ\tau\subseteq\tau\end{array}\right\}. \quad (4.1)$$

As it widely known, such relation $\tau$ is called *transitive*. Direction of implication fully correlates with inclusion's one in accordance with formula (1.1).

Generalizing inductively, there may be seen that arbitrary power of transitive relation is transitive too

$\tau^n\subseteq\tau$. (4.2)

[2] The sense of it and usage of quotation marks will be clear a little later.

However, there are no objective criteria of relation's appearance to be transitive, all the more – it is not obvious that relation which is reversal to transitive one is transitive too. But just as it was made for initial relation, transitivity of reversal relation is defined by implication

$$\langle y,i\rangle \in \tau^{-1} \wedge \langle i,x\rangle \in \tau^{-1} \Rightarrow \langle y,x\rangle \in \tau^{-1}.$$

Comparison with implication (4.1) shows that such conversion depends on both conversion of initial relation and sequential order changing of conjunction operands – it is possible but it is not universal due to non-commutativity of ordered pair forming. Still, behavior of implication direction is not quite clear – may be (or may not) it is changing due to such procedure. Nevertheless, one may suppose that in any case reversal relation is transitive if initial one is *reversible*. And it may be written as

$$\left.\begin{array}{c}\langle x,i\rangle \in \sigma \Rightarrow \langle i,x\rangle \in \sigma^{-1}\\ x\sigma i \Rightarrow i\sigma^{-1}x\\ \sigma \subseteq \sigma^{-1}\end{array}\right\}. \tag{4.3}$$

Due to idem-potency of set intersection that inclusion may be written as

$$\sigma = \sigma \cap \sigma \subseteq \sigma \cap \sigma^{-1}.$$

Because intersection is always a subset of both its operands, it occurs in unique case – equality

$$\left.\begin{array}{c}\langle x,i\rangle \in \sigma \Leftrightarrow \langle i,x\rangle \in \sigma^{-1}\\ x\sigma i \Leftrightarrow i\sigma^{-1}x\\ \sigma = \sigma^{-1}\end{array}\right\}. \tag{4.4}$$

Naturally, these relations are called *symmetrical*. But there is a danger to suppose mistakenly that the exclusive reason of reversibility is in relation symmetry.

Also, one may guess that reversal relation is surely transitive if initial one is *reflexive* – it contains diagonal as a subset

$$\left.\begin{array}{c}\langle i,i\rangle \in \mathrm{id} \Rightarrow \langle i,i\rangle \in \rho\\ i\mathrm{id}i \Rightarrow i\rho i\\ \mathrm{id} \subseteq \rho\end{array}\right\}. \tag{4.5}$$

The base to think so is accordance with formula (3.11). Conversion of inclusion (4.5) makes a result of such procedure *pseudo-reflexive* – it is included in diagonal itself as a subset

$$\left.\begin{array}{c}\langle i,i\rangle \in \rho^{-1} \Rightarrow \langle i,i\rangle \in \mathrm{id}\\ i\rho^{-1}i \Rightarrow i\mathrm{id}i\\ \rho^{-1} \subseteq \mathrm{id}\end{array}\right\}. \tag{4.6}$$

It seems that transitivity of such relation appears automatically

$$\rho^{-1} \circ \rho^{-1} \subseteq \mathrm{id} \circ \rho^{-1} = \rho^{-1}.$$

But such discourses lead to *sophism* that consists in follows. Reversing reflexive symmetrical transitive relations – these are called *equivalences*, according to equality (4.4) it won't lead to any difference between initial relation and reversal one. But according to last discourse, such conversion will lead necessarily reflexive relation to pseudo-reflexive one[3]. And there is just the only way to avoid it – to assume it equal to diagonal. And if such discourse is universal, the unique equivalence that may exist is equality. Meanwhile, everyone can be assured empirically in falsity of this statement. It may be described formally. Introducing symbol $\varepsilon$ for equivalence, one may write

$$\left.\begin{array}{c}\varepsilon \supseteq \mathrm{id}\\ \varepsilon^{-1} = \varepsilon\\ \varepsilon^{2} \subseteq \varepsilon\end{array}\right\}. \tag{4.7}$$

Reversing first expression in accordance with formulae (3.11), we get

$$\varepsilon^{-1} \subseteq \mathrm{id}.$$

Together with the second one it leads to equality between initial relation and reversal one and both of them coincide with diagonal

$$\varepsilon^{-1} = \varepsilon = \mathrm{id}.$$

And the third one quite coincide with diagonal idem-potency condition

$$\varepsilon^{2} = \varepsilon = \mathrm{id}.$$

It is another "confirmation" of a statement that any equivalence is equality. As we didn't use concrete equivalence one may summarize that it has universal character.

But as usual, principal among circumstances that lead to sophism is failing to take into account some link in chain of discourses. It is almost sure that transitive symmetrical relation is equivalence – it is reflexive obligatorily. But there was failing to take into account that not as much relation itself as its intersection with reversal one plays huge role – this is inherent to reflexive relation and finally leads inclusion (4.3) to equality (4.4). It may be written as sequence

$$\left.\begin{array}{c}\sigma \cap \sigma^{-1} \supseteq \mathrm{id} \supset \emptyset\\ \langle x,j\rangle \in \sigma \wedge \langle x,k\rangle \in \sigma^{-1} \Leftarrow \langle x,x\rangle \in \mathrm{id} \Leftarrow \langle j,k\rangle\end{array}\right\}. \tag{4.8}$$

Changing of implication direction may be one of oppositions to symmetrical relations as some non-empty relations $\alpha$ and $\alpha^{-1}$

$$\left.\begin{array}{c}\emptyset \neq \alpha \neq \alpha^{-1} \neq \emptyset\\ \emptyset \subseteq \alpha \cap \alpha^{-1} \subseteq \mathrm{id}\\ \langle x,j\rangle \in \alpha \wedge \langle x,k\rangle \in \alpha^{-1} \Rightarrow (j = k)\end{array}\right\}. \tag{4.9}$$

Perhaps, it make possible for relation to be transitive not being reflexive, but probably pseudo-reflexive or even – *anti-reflexive*. The last ones is defined by its empty intersection with diagonal

[3] So, it couldn't be equivalence – it is always reflexive.

$$\rho \cap \text{id} = \emptyset. \quad (4.10)$$

In this terms reflexivity looks like

$$\rho \cap \text{id} = \text{id}. \quad (4.11)$$

Pseudo-reflexivity is

$$\rho \cap \text{id} = \rho. \quad (4.12)$$

Usually relations defining by formulae (4.9) are called anti-symmetrical. If in equality (4.11) to assume to be equality $\rho = \alpha \cap \alpha^{-1}$ it leads it into appearance $(\alpha \cap \alpha^{-1}) \cap \text{id} = \text{id}$, considering of intersection idem-potency leads to equality

$$\alpha \cap \alpha^{-1} = \text{id}. \quad (4.13)$$

This is criterion of anti-symmetric relation reflexivity; and this one makes left inclusion (4.9) proper. It's clear while intersection (4.9) is empty, we deal with disjoint relations $\alpha$ and $\alpha^{-1}$. As far as neither anti-symmetrical relation nor reversal one is not empty, there may be written inclusion $\emptyset = \alpha \cap \alpha^{-1} \cap \text{id} \subseteq \text{id}$. Surely, these criteria point out reflexive occurrence are indirect but it is not principal – more important thing is reflexive appearance of their intersection – it plays the main role in their conversion.

Coincidence of all kind invariances in diagonal compositions was also neglected. It may be expressed by equalities

$$[\text{id} \circ (\text{id})^{-1}]^{-1} = [\text{id} \circ (\text{id})^{-1}]^{+} = \text{id} \circ (\text{id})^{-1} = \text{id}. \quad (4.14)$$

And if another equivalence $\varepsilon$, which differs from diagonal, is established on a set, the last equality here is not fulfilled

$$\varepsilon \circ \varepsilon^{-1} = (\varepsilon \circ \varepsilon^{-1})^{-1} = (\varepsilon \circ \varepsilon^{-1})^{+} = \varepsilon^{-1} \circ \varepsilon \supset \text{id}. \quad (4.15)$$

To find explicit form of such equivalences there may note that according to formulae (3.10) some compositions of anti-symmetric relations are symmetric. Really, there may be written

$$\left.\begin{array}{l} (\alpha \circ \alpha^{-1})^{-1} = (\alpha^{-1})^{-1} \circ \alpha^{-1} = \alpha \circ \alpha^{-1} \\ (\alpha^{-1} \circ \alpha)^{-1} = \alpha^{-1} \circ (\alpha^{-1})^{-1} = \alpha^{-1} \circ \alpha \end{array}\right\}. \quad (4.16)$$

As opposed to diagonal composition commutativity of theses ones is probable but is quite not inherent. Its reason is that such compositions are not associated with each other by *inversion*. But they are associated by *transposition*. In general, summarizing, the last equality (4.15) dissociates into two inclusions. They might be written together as

$$\left.\begin{array}{c} \alpha \circ \alpha^{-1} \supset \text{id} \\ \alpha^{-1} \circ \alpha \subset \text{id} \\ (\alpha \circ \alpha^{-1})^{+} = \alpha^{-1} \circ \alpha \end{array}\right\}. \quad (4.15)$$

So, the first composition is reflexive, but the second one is pseudo-reflexive – that's why it is not equivalence. In spite of symmetry it is not transitive

$$\alpha^{-1} \circ \alpha \circ \alpha^{-1} \circ \alpha \supseteq \alpha^{-1} \circ \mathrm{id} \circ \alpha = \alpha^{-1} \circ \alpha.$$

Such inclusion direction is not proper to conjunction this expression is contradictive – it does not exist, instead of it its negation occurs

$$\alpha^{-1} \circ \alpha \circ \alpha^{-1} \circ \alpha \nsubseteq \alpha^{-1} \circ \mathrm{id} \circ \alpha = \alpha^{-1} \circ \alpha.$$

It may be named as *intransitivity* – for some relation[4] $\rho$ it is defined by negation of implication (4.1)

$$\left.\begin{array}{c} \langle x,i\rangle \in \rho \wedge \langle i,y\rangle \in \rho \nRightarrow \langle x,y\rangle \in \rho \\ x\rho i \wedge i\rho y \nRightarrow x\rho y \\ \{\langle x,y\rangle : \langle x,i\rangle \in \rho \wedge \langle i,y\rangle \in \rho\} \nsubseteq \{\langle x,y\rangle : \langle x,y\rangle \in \rho\} \\ \rho \circ \rho \nsubseteq \rho \end{array}\right\}. \tag{4.16}$$

Also it's easy to show that reflexive composition is genuine equivalence – transitive relation

$$\alpha \circ (\alpha \circ \alpha^{-1}) \circ \alpha^{-1} \subseteq \alpha \circ \mathrm{id} \circ \alpha^{-1} = \alpha \circ \alpha^{-1}. \tag{4.17}$$

In contrast to genuine equivalence this pseudo-reflexive symmetrical but non-transitive composition, perhaps, there may appropriately call *pseudo-equivalence*[5]. At the same time, probability to be equivalence is not excluded for both compositions. But anyway opposite inclusion for these compositions is not possible.

By congruence of definitions anti-symmetry is quite not direct negation of symmetry, though, reflexivity is not a direct negation of another its appearance. But non-transitivity is the negation for transitivity. Perhaps it is due to intersection which is used in their definitions – its reversibility is not defined. So, there may suppose existence of some "interstitial" symmetry appearance – some relations $\mu$ which are opposite to described ones above – implications (4.8) and (4.9)

$$\left.\begin{array}{c} \langle x,j\rangle \in \mu \wedge \langle x,k\rangle \in \mu^{-1} \Leftrightarrow (j \neq k) \\ \mu \cap \mu^{-1} = \emptyset \end{array}\right\}. \tag{4.18}$$

As a rule, such relations are called *asymmetrical*. It is easy to show that they are always anti-reflexive

$$\mu \cap \mathrm{id} = \emptyset. \tag{4.19}$$

Really, presence of inequality in the right part of bi-conditional (4.19) allows writing contraposition to implication (4.3)

$$\langle x,i\rangle \in \mu \Rightarrow \langle i,x\rangle \notin \mu^{-1}. \tag{4.20}$$

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[4] There may be written for *anti-transitivity*

$$\left.\begin{array}{c} \langle x,i\rangle \in \rho \wedge \langle i,y\rangle \in \rho \Rightarrow \langle x,y\rangle \notin \rho \\ x\rho i \wedge i\rho y \Rightarrow x\neg\rho y \\ \{\langle x,y\rangle : \langle x,i\rangle \in \rho \wedge \langle i,y\rangle \in \rho\} \subseteq \{\langle x,y\rangle : \langle x,y\rangle \notin \rho\} \\ \rho \circ \rho \subseteq \neg\rho \end{array}\right\}. \tag{4.16*}$$

There may be said that it is something akin to be nonempty subset of null. So it rather doesn't exist.

[5] But usually it is more known as *tolerance*.

It makes asymmetrical relations to be irreversible and it corresponds to inequalities $j \neq k \neq i$. The left part of definition (4.18) may be written as

$$\langle x, i\rangle \in \mu \wedge \langle i, x\rangle \notin \mu^{-1} \Rightarrow \langle x, x\rangle \notin \mu \wedge \langle i, i\rangle \in \mathrm{id}. \tag{4.21}$$

It corresponds to equality (4.19) for non-empty intersection operands. Hence, "converting" of asymmetrical relations is possible just like a negation of initial one

$$\mu^{-1} = \neg\mu. \tag{4.22}$$

Anti-symmetrical relations are free from condition (4.20) – instead of equality (4.22) there may be written inequalities

$$\alpha \neq \alpha^{-1} \neq \neg\alpha. \tag{4.23}$$

Therefore reversibility of the last one is possible. And it will be continued in further issue.

An example of asymmetric relation is an element membership to some disordered set. The only possible implication that might be formally written in this case is

$$(a \in A) \wedge (A \in^{-1} b) \Leftrightarrow (a \in A) \wedge (A \notin b) \Rightarrow (a \notin b). \tag{4.24}$$

That's why we couldn't talk about transitivity of asymmetrical relation. In this case the left part of inclusion (4.1) should be written as

$$\langle x, j\rangle \in \mu \wedge \langle j, y\rangle \in \mu.$$

But it's impossible because elements sequence order changing in these pairs leads to replacement of element $j$ by another element $k \neq j$. It is necessarily due to irreversibility of asymmetric relations.