When $\pi(n)$ does not divide n

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Abstract

Let $\pi(n)$ denote the prime-counting function and let

$$f(n) = |\lfloor \log n - \lfloor \log n \rfloor - 0.1 \rfloor| \left\lfloor \frac{\lfloor n / \lfloor \log n - 1 \rfloor \rfloor \lfloor \log n - 1 \rfloor}{n} \right\rfloor .$$

In this paper we prove that if n is an integer ≥ 60184 and $f(n) = 0$, then $\pi(n)$ does not divide n. We also show that if $n \geq 60184$ and $\pi(n)$ divides n, then $f(n) = 1$. In addition, we prove that if $n \geq 60184$ and $n/\pi(n)$ is an integer, then n is a multiple of $\lfloor \log n - 1 \rfloor$ located in the interval $[e^{\lfloor \log n - 1 \rfloor + 1}, e^{\lfloor \log n - 1 \rfloor + 1.1}]$. This allows us to show that if c is any fixed integer ≥ 12, then in the interval $[e^c, e^{c+0.1}]$ there is always an integer n such that $\pi(n)$ divides n.

Let S denote the sequence of integers generated by the function $d(n) = n/\pi(n)$ (where $n \in \mathbb{Z}$ and $n > 1$) and let S_k denote the kth term of sequence S. Here we ask the question whether there are infinitely many positive integers k such that $S_k = S_{k+1}$.

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0 Notation

Throughout this paper the number n is always a positive integer. Moreover, we use the following notation:

- $|\cdot|$ (absolute value)
- $\lceil\cdot\rceil$ (ceiling function)
- $|$ (divides)
- $\nmid$ (does not divide)
- $\lfloor\cdot\rfloor$ (floor function)
- $\mathrm{frac}(\cdot)$ (fractional part)
- $\log$ (natural logarithm)

1 Introduction

Determining how prime numbers are distributed among natural numbers is one of the most difficult mathematical problems. This explains why the prime-counting function $\pi(n)$ (which counts the number of primes less than or equal to a given number n) has been one of the main objects of study in Mathematics for centuries.

In [2] Gaitanas obtains an explicit formula for $\pi(n)$ that holds infinitely often. His proof is based on the fact that the function $d(n) = n/\pi(n)$ takes on every integer value greater than 1 (as proved by Golomb [3]) and on the fact that $x/(\log x - 0.5) < \pi(x) < x/(\log x - 1.5)$ for $x \geq 67$ (as shown by Rosser and Schoenfeld [4]). In this paper we find alternative expressions that are valid for infinitely many positive integers n, and we also prove, among other results, that if $n \geq 60184$ and

$$\left|\lfloor \log n - \lfloor \log n \rfloor - 0.1 \rfloor\right| \left\lfloor \frac{\lfloor n/\lfloor \log n - 1 \rfloor \rfloor \lfloor \log n - 1 \rfloor}{n} \right\rfloor$$

equals 0, then $\pi(n)$ does not divide n.

We will place emphasis on the following three theorems, which were proved by Golomb, Dusart, and Gaitanas respectively:

Theorem 1.1 [3]**.** The function $d(n) = n/\pi(n)$ takes on every integer value greater than 1. ■

Theorem 1.2 [1]**.** If n is an integer ≥ 60184, then

$$\frac{n}{\log n - 1} < \pi(n) < \frac{n}{\log n - 1.1}. \quad ■$$

Remark 1.3. Dusart's paper states that for $x \geq 60184$ we have $x/(\log x - 1) \leq \pi(x) \leq x/(\log x - 1.1)$, but since $\log n$ is always irrational when n is an integer > 1, we can state his theorem the way we did. ◀

Theorem 1.4 [2]**.** The formula

$$\pi(n) = \frac{n}{\lfloor \log n - 0.5 \rfloor}$$

is valid for infinitely many positive integers n. ■

2 Main results

We are now ready to prove our main results:

Theorem 2.1. The formula

$$\pi(n) = \frac{n}{\lfloor \log n - 1 \rfloor}$$

holds for infinitely many positive integers n. ■

Proof. According to Theorem 1.2, for $n \geq 60184$ we have

$$\frac{n}{\log n - 1} < \pi(n) < \frac{n}{\log n - 1.1} \Rightarrow \frac{\log n - 1.1}{n} < \frac{1}{\pi(n)} < \frac{\log n - 1}{n}.$$

If we multiply by n, we get

$$\log n - 1.1 < \frac{n}{\pi(n)} < \log n - 1. \quad (1)$$

Since $\log n - 1.1$ and $\log n - 1$ are both irrational (for $n > 1$), inequality (1) implies that when $n/\pi(n)$ is an integer we must have

$$\frac{n}{\pi(n)} = \lfloor \log n - 1 \rfloor = \lfloor \log n - 1.1 \rfloor + 1 = \lceil \log n - 1.1 \rceil = \lceil \log n - 1 \rceil - 1. \quad (2)$$

Taking Theorem 1.2 and equality (2) into account, we can say that for every $n \geq 60184$ when $n/\pi(n)$ is an integer we must have

$$\frac{n}{\pi(n)} = \lfloor \log n - 1 \rfloor \Rightarrow \pi(n) = \frac{n}{\lfloor \log n - 1 \rfloor}.$$

Since Theorem 1.1 implies that $n/\pi(n)$ is an integer infinitely often, it follows that there are infinitely many positive integers n such that $\pi(n) = n/\lfloor \log n - 1 \rfloor$. ■

In fact, the following theorem follows from Theorems 1.1, from Gaitana's proof of Theorem 1.4, and from the proof of Theorem 2.1:

Theorem 2.2. For every $n \geq 60184$ when $n/\pi(n)$ is an integer we must have

$$\begin{aligned} \frac{n}{\pi(n)} &= \lceil \log n - 1.5 \rceil = \lfloor \log n - 0.5 \rfloor = \lfloor \log n - 1 \rfloor = \\ &= \lfloor \log n - 1.1 \rfloor + 1 = \lceil \log n - 1.1 \rceil = \lceil \log n - 1 \rceil - 1. \end{aligned} \tag{3}$$

In other words, for $n \geq 60184$ when $n/\pi(n)$ is an integer we must have

$$\begin{aligned} \pi(n) = \frac{n}{\lceil \log n - 1.5 \rceil} = \frac{n}{\lfloor \log n - 0.5 \rfloor} = \frac{n}{\lfloor \log n - 1 \rfloor} = \frac{n}{\lfloor \log n - 1.1 \rfloor + 1} = \\ = \frac{n}{\lceil \log n - 1.1 \rceil} = \frac{n}{\lceil \log n - 1 \rceil - 1}. \end{aligned}$$ ■

Theorem 2.3. Let n be an integer ≥ 60184. If $\operatorname{frac}(\log n) = \log n - \lfloor \log n \rfloor > 0.1$, then $\pi(n) \nmid n$ (that is to say, $n/\pi(n)$ is not an integer). ■

Proof. According to Theorem 2.2, if $n \geq 60184$ and $n/\pi(n)$ is an integer, then

$$\frac{n}{\pi(n)} = \lfloor \log n - 1 \rfloor = \lceil \log n - 1.1 \rceil.$$

In other words, for $n \geq 60184$ when $n/\pi(n)$ is an integer we have

$$\begin{aligned} \lfloor \log n - 1 \rfloor &= \lceil \log n - 1.1 \rceil \\ \lfloor \log n - 1 \rfloor &= \lceil \log n - 1 - 0.1 \rceil \\ \operatorname{frac}(\log n - 1) &\leq 0.1 \\ \log n - 1 - \lfloor \log n - 1 \rfloor &\leq 0.1 \end{aligned}$$

$$\begin{aligned}\log n - \lfloor \log n - 1 \rfloor &\le 1.1\\ \operatorname{frac}(\log n) &\le 0.1\\ \log n - \lfloor \log n \rfloor &\le 0.1.\end{aligned}$$

Suppose that P is the statement '$n/\pi(n)$ is an integer' and Q is the statement '$\log n - \lfloor \log n \rfloor \le 0.1$'. According to propositional logic, the fact that $P \to Q$ implies that $\neg Q \to \neg P$. ■

Similar theorems can be proved by using Theorem 2.2 and equality (3).

Remark 2.4. We can also say that if $n \ge 60184$ and

$$n > e^{0.1+\lfloor \log n \rfloor},$$

then $\pi(n) \nmid n$. ◀

Remark 2.5. Because $\log n$ is irrational for $n > 1$, another way of stating Theorem 2.3 is by saying that if $n \ge 60184$ and the first digit to the right of the decimal point of $\log n$ is 1, 2, 3, 4, 5, 6, 7, 8, or 9, then $\pi(n) \nmid n$. Example:

$$\log 10^{31} = 71.38\ldots$$

The first digit after the decimal point of $\log 10^{31}$ (in red) is 3. This implies that $\pi(10^{31})$ does not divide 10^{31}. We can also say that if $n \ge 60184$ and $\pi(n)$ divides n, then the first digit after the decimal point of $\log n$ can only be 0.

Now, if y is a positive noninteger, then the first digit after the decimal point of y is equal to $\lfloor 10 \operatorname{frac}(y) \rfloor = \lfloor 10y - 10\lfloor y \rfloor \rfloor$. So, we can say that if $n \ge 60184$ and $\lfloor 10 \log n - 10\lfloor \log n \rfloor \rfloor \ne 0$, then $\pi(n) \nmid n$. On the other hand, if $n \ge 60184$ and $\pi(n)$ divides n, then $\lfloor 10 \log n - 10\lfloor \log n \rfloor \rfloor = 0$. ◀

The following theorem follows from Theorem 2.3:

Theorem 2.6. Let e be the base of the natural logarithm. If a is any integer ≥ 11 and n is any integer contained in the interval $[e^{a+0.1}, e^{a+1}]$, then $\pi(n) \nmid n$. (The number e^r is irrational when r is a rational number $\ne 0$.) ■

Example 2.7. Take $a = 18$. If n is any integer in the interval $[e^{18.1}, e^{19}]$, then $\pi(n) \nmid n$. ◀

Corollary 2.8. If a is any positive integer > 1, then $\pi(\lfloor e^a \rfloor) \nmid \lfloor e^a \rfloor$. ■

Proof. For $a \ge 12$ the proof follows from Theorem 2.6. On the other hand, $\lfloor e^a \rfloor / \pi(\lfloor e^a \rfloor)$ is not an integer whenever $2 \le a \le 11$, as shown in the following table:

a	$\lfloor e^a \rfloor / \pi(\lfloor e^a \rfloor)$
1	2
2	1.75
3	2.5
4	3.37...
5	4.35...
6	5.10...
7	5.98...
8	6.94...
9	7.95...
10	8.93...
11	9.89...

In other words, if $a \in \mathbb{Z}^+$, then $\pi(\lfloor e^a \rfloor) \mid \lfloor e^a \rfloor$ only when $a = 1$. ■

Theorem 2.9. Let n be an integer ≥ 60184 and let

$$f(n) = |\lfloor \log n - \lfloor \log n \rfloor - 0.1 \rfloor| \left\lfloor \frac{\lfloor n / \lfloor \log n - 1 \rfloor \rfloor \lfloor \log n - 1 \rfloor}{n} \right\rfloor.$$

If $f(n) = 0$, then $\pi(n) \nmid n$. On the other hand, if $\pi(n) \mid n$, then $f(n) = 1$. ■

Proof.

- **Part 1**

Suppose that

$$f(n) = g(n)h(n),$$

where

$$g(n) = |\lfloor \log n - \lfloor \log n \rfloor - 0.1 \rfloor|$$

and

$$h(n) = \left\lfloor \frac{\lfloor n / \lfloor \log n - 1 \rfloor \rfloor \lfloor \log n - 1 \rfloor}{n} \right\rfloor.$$

To begin with, if $n \geq 60184$, then $\log n - \lfloor \log n \rfloor$ can never be equal to 0.1. Now, when $\log n - \lfloor \log n \rfloor < 0.1$ we have $-1 < \log n - \lfloor \log n \rfloor - 0.1 < 0$ and hence $|\lfloor \log n - \lfloor \log n \rfloor - 0.1 \rfloor| = 1$. On the other hand, when $\log n - \lfloor \log n \rfloor > 0.1$ we have $0 < \log n - \lfloor \log n \rfloor - 0.1 < 1$ and hence $|\lfloor \log n - \lfloor \log n \rfloor - 0.1 \rfloor| = 0$. This means that if n is any integer ≥ 60184, then $g(n)$ equals either 0 or 1. We can also say that if $n \geq 60184$ and $g(n) = 0$, then $\log n - \lfloor \log n \rfloor > 0.1$, which implies that $\pi(n) \nmid n$ (according to Theorem 2.3). (This means that if $n \geq 60184$ and $\pi(n) \mid n$, then $g(n) = 1$.)

- **Part 2**

If $n \geq 60184$, then

$$\left\lfloor \frac{n}{\lfloor \log n - 1 \rfloor} \right\rfloor \leq \frac{n}{\lfloor \log n - 1 \rfloor},$$

which means that

$$\left\lfloor \left\lfloor \frac{n}{\lfloor \log n - 1 \rfloor} \right\rfloor \Big/ \frac{n}{\lfloor \log n - 1 \rfloor} \right\rfloor = \left\lfloor \frac{\lfloor n / \lfloor \log n - 1 \rfloor \rfloor \lfloor \log n - 1 \rfloor}{n} \right\rfloor = h(n)$$

equals either 0 or 1. If $h(n) = 0$, then n is not divisible by $\lfloor \log n - 1 \rfloor$, which implies that $\pi(n) \nmid n$ (according to Theorem 2.2). In other words, if $n \geq 60184$ and $h(n) = 0$, then $\pi(n) \nmid n$. (This means that if $n \geq 60184$ and $\pi(n) \mid n$, then $h(n) = 1$.)

- **Part 3**

There are two possible outputs for $g(n)$ (0 or 1) as well as two possible outputs for $h(n)$ (0 or 1). This means that for $n \geq 60184$ we have either

$$g(n)h(n) = 0 \cdot 0 = 0,$$

or

$$g(n)h(n) = 0 \cdot 1 = 0,$$

or

$$g(n)h(n) = 1 \cdot 0 = 0,$$

or

$$g(n)h(n) = 1 \cdot 1 = 1.$$

If $f(n) = g(n)h(n) = 0$, then at least one of the factors $g(n)$ and $h(n)$ equals 0, which implies that $\pi(n) \nmid n$ (see Part 1 and Part 2). This means that if $n \geq 60184$ and $f(n) = 0$, then $\pi(n) \nmid n$. Consequently, if $n \geq 60184$ and $\pi(n) \mid n$, then $f(n) = 1$. ■

Theorem 2.10. If $n \geq 60184$ and $n/\pi(n)$ is an integer, then n is a multiple of $\lfloor \log n - 1 \rfloor$ located in the interval $[e^{\lfloor \log n - 1 \rfloor + 1}, e^{\lfloor \log n - 1 \rfloor + 1.1}]$. ■

Proof. According to Theorems 2.2 and 2.3, if $n \geq 60184$ and $n/\pi(n)$ is an integer, then

$$\frac{n}{\pi(n)} = \lfloor \log n - 1 \rfloor \Rightarrow n = \pi(n)\lfloor \log n - 1 \rfloor$$

and

$$\operatorname{frac}(\log n) = \log n - \lfloor \log n \rfloor \leq 0.1.$$

The fact that $\operatorname{frac}(\log n) \leq 0.1$ implies that n is located in the interval

$$[e^k, e^{k+0.1}]$$

for some positive integer k. In other words, we have

$$e^k < n < e^{k+0.1} \Rightarrow k < \log n < k + 0.1 \Rightarrow k - 1 < \log n - 1 < k - 0.9,$$

which means that

$$\begin{aligned} k - 1 &= \lfloor \log n - 1 \rfloor \\ k &= \lfloor \log n - 1 \rfloor + 1. \end{aligned}$$

■

Remark 2.11. Suppose that b is any fixed integer ≥ 12. Theorem 2.10 implies that if n is an integer in the interval $[e^b, e^{b+0.1}]$ and at the same time n is not a multiple of $b-1$, then $\pi(n) \nmid n$. This means that if $n \geq 60184$ and $\pi(n)$ divides n, then n is located in the interval $[e^b, e^{b+0.1}]$ for some positive integer b and n is a multiple of $b - 1$. ◀

The following theorem follows from Theorems 1.1 and 2.10 and from the fact that $n/\pi(n) < 11$ for $n \leq 60183$ (this fact can be checked using software):

Theorem 2.12. Let c be any fixed integer ≥ 12. In the interval $[e^c, e^{c+0.1}]$ there is always an integer n such that $\pi(n)$ divides n. In other words, in the interval $[e^c, e^{c+0.1}]$ there is always an integer n such that $\pi(n) = n/(c-1)$. ■

3 Conclusion and Further Discussion

The following are the main theorems of this paper:

Theorem 2.9. Let n be an integer ≥ 60184 and let

$$f(n) = |\lfloor \log n - \lfloor \log n \rfloor - 0.1 \rfloor| \left\lfloor \frac{\lfloor n/\lfloor \log n - 1 \rfloor \rfloor \lfloor \log n - 1 \rfloor}{n} \right\rfloor.$$

If $f(n) = 0$, then $\pi(n) \nmid n$. On the other hand, if $\pi(n) \mid n$, then $f(n) = 1$. ■

Theorem 2.10. If $n \geq 60184$ and $n/\pi(n)$ is an integer, then n is a multiple of $\lfloor \log n - 1 \rfloor$ located in the interval $[e^{\lfloor \log n - 1 \rfloor + 1}, e^{\lfloor \log n - 1 \rfloor + 1.1}]$. ■

Theorem 2.12. Let c be any fixed integer ≥ 12. In the interval $[e^c, e^{c+0.1}]$ there is always an integer n such that $\pi(n)$ divides n. In other words, in the interval $[e^c, e^{c+0.1}]$ there is always an integer n such that $\pi(n) = n/(c-1)$.■

We recall that Golomb [3] proved that for every integer $n > 1$ there exists a positive integer m such that $m/\pi(m) = n$. Suppose now that R is the sequence of numbers generated by the function $d(n) = n/\pi(n)$ ($n \in \mathbb{Z}$ and $n > 1$). In other words,

$$R = (2, \quad 1.5, \quad 2, \quad 1.66..., \quad 2, \quad 1.75, \quad 2, \quad 2.25, \quad 2.5, \quad \ldots).$$

Suppose also that S is the sequence of *integers* generated by the function $d(n) = n/\pi(n)$. In other words,

$$S = (2, \quad 2, \quad 2, \quad 2, \quad 3, \quad 3, \quad 3, \quad 4, \quad 4, \quad \ldots).$$

Motivated by Golomb's result and Theorem 2.12 we ask the following question:

Question 3.1. Are there infinitely many positive integers a such that in the interval $[e^a, e^{a+0.1}]$ there are at least two distinct positive integers n_1 and n_2 such that $\pi(n_1) \mid n_1$ and $\pi(n_2) \mid n_2$? In other words, are there infinitely many positive integers n that can be expressed as $m/\pi(m)$ in more than one way? ◄

Now, let S_k denote the kth term of sequence S. Clearly, Question 3.1 is equivalent to the following question:

Question 3.2. Are there infinitely many positive integers k such that $S_k = S_{k+1}$? ◄

References

[1] Dusart, P. "Estimates of Some Functions Over Primes without R.H." arXiv:1002.0442 [math.NT], 2010.

[2] Gaitanas, K. N. "An explicit formula for the prime counting function." arXiv:1311.1398 [math.NT], 2013.

[3] Golomb, S. W. "On the Ratio of N to $\pi(N)$." *The American Mathematical Monthly*. Vol. 69, No. 1, pp. 36–37, 1962.

[4] Rosser, J. B.; Schoenfeld, L. "Approximate formulas for some functions of prime numbers." *Illinois Journal of Mathematics*. Vol. 6, No. 1, pp. 64–94, 1962.