Proof of Infinite Number of Triplet Primes

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Abstract

This paper presents a complete and exhaustive proof that an Infinite Number of Triplet Primes exist. The approach to this proof uses same logic that Euclid used to prove there are an infinite number of prime numbers. Then we prove that if p > 1 and d > 0 are integers, that p and p + d are both primes if and only if for integer n (see reference 1 and 2):

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{\mathrm{d}}\mathrm{d}!}{\mathrm{p} + \mathrm{d}}\right) + \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{p}+\mathrm{d}}$$

We use this proof and Euclid logic to prove only an infinite number of Triplet Primes exist. However we shall begin by assuming that a finite number of Triplet Primes exist, we shall prove a contradiction to the assumption of a finite number, which will prove that an infinite number of Triplet Primes exist.

Introduction

In number theory, a Prime Triplet is:

A set of three prime numbers of the form (p, p + 2, p + 6) or (p, p + 4, p + 6), see reference 3.

Another way of describing prime triples is that they are three consecutive primes, such that the first and the last differ by six. For example:

(5, 7, 11), (7, 11, 13), (11, 13, 17), (13, 17, 19), (17, 19, 23), (37, 41, 43), (41, 43, 47), (67, 71, 73), (97, 101, 103), and (101, 103, 107).

It is conjectured that there are infinitely many such primes, but has not been proven yet, this paper will provide the proof. In fact the Hardy-Littlewood prime k-tuple conjecture (see reference 4) suggests that the number less than x of each of the forms

$$(p, p+2, p+6) \text{ and } (p, p+4, p+6)$$

is approximately:

$$\frac{9}{2}\prod_{p\geq 5}\frac{p^2(p-3)}{(p-1)^3}\int_2^x \frac{dx}{(\log x)^3} \doteq 2.858248596\int_2^x \frac{dx}{(\log x)^3}$$

The actual numbers less than 100,000,000 are 55,600 and 55,556 respectively. The Hardy-Littlewood estimate above is 55,490 which is very close to the actual numbers.

Proof:

We shall use Euclid's logic that he used to prove our assumption that there are an infinite number of prime numbers to prove there are an infinite number Triplet Primes.

First we shall assume there are only a finite number of n Triplet primes, specifically:

1) Say, p_1, $p_1 + 2$, $p_1 + 6$, p_2, $p_2 + 4$, $p_2 + 6$. . . . , $p_{n-1,}$ $p_{n-1} + 2$, $p_{n-1} + 6$, $p_{n,}$ $p_n + 4$, $p_n + 6$

2) Let $N = p_1(p_1 + 2)(p_1 + 6)p_2(p_2 + 4)(p_2 + 6)$ . . . , $p_{n-1}(p_{n-1} + 2)(p_{n-1} + 6)p_n(p_n + 4)(p_n + 6) +1$

By the fundamental theorem of arithmetic, N is divisible by some prime q. Since N is the product of all existing Prime Triplet primes plus 1, then this prime q cannot be among the p_i, $p_i + 2$, $p_i + 6$, p_i, and p_i, $p_i + 4$, $p_i + 6$ that make up the n Prime Triplet primes since by assumption these are all the Prime Triplet primes that exist and N is not divisible by any of the p_i, $p_i + 2$, $p_i + 6$, p_i, and p_i, $p_i + 4$, $p_i + 6$ Prime Triplet primes. N is clearly seen not to be divisible by any of the p_i, $p_i + 2$, $p_i + 6$, p_i, and $p_i + 4$, $p_i + 6$ Prime Triplet primes. First we know that 2 is a prime number that is not in the set of finite Prime Triplet primes since if $p_i = 2$, then $p_i + 2$, $p_i + 4$, $p_i + 6$ are all even numbers with $p_i = 2$.

Since $p_i + 2$, $p_i + 4$, $p_i + 6$ are even, they are greater than 2, so they cannot be prime, therefore, 2 cannot be included in the finite set of Prime Triplet primes. We also know that 2 is the only even prime number, therefore, for the finite set of Prime Triplet primes all of the p_i, $p_i + 2$, $p_i + 6$, p_i, and p_i, $p_i + 4$, $p_i + 6$ are odd numbers. Since the product of odd numbers is always odd, then the product of all the p_i, $p_i + 2$, $p_i + 6$, p_i, and p_i, $p_i + 4$, $p_i + 6$ in our finite set of Prime Triplet primes is an odd number. Since N is product of all the p_i, $p_i + 2$, $p_i + 6$, p_i, and p_i, $p_i + 4$, $p_i + 6 + 1$, then N is an even number, and since all the p_i, $p_i + 2$, $p_i + 6$, p_i, and p_i, $p_i + 4$, $p_i + 6$ are odd numbers and N is even, then N is not divisible by any of the p_i, $p_i + 2$, $p_i + 6$, p_i, and p_i, $p_i + 4$, $p_i + 6$ Prime Triplet primes. Therefore, q must be another prime number that does not exist in the finite set of Prime Triplet prime numbers. Therefore, since this proof could be repeated an infinite number of times we have proven that an infinite number of prime numbers q exist outside of our finite set of Prime Triplet primes.

Now we must prove that two sets of three of these infinite prime numbers, q, are Prime Triplet primes. We will pick a prime number p_i and p_j from the infinite set of primes outside our finite set of Prime Triplet primes and we will need to prove that there exist primes $p_i + 2$, $p_i + 6$, and $p_j + 4$, $p_j + 6$ are also prime, where $i \neq j$, except when $i = j$ only if both forms of Prime Triplets are overlapping Triplets forming a Prime Quadruplet. Both Triplets, p_i, $p_i + 2$, $p_i + 6$, p_i, and p_j, $p_j + 4$, $p_j + 6$ do not exist in the finite set of Prime Triplet primes, since p_i and p_j do not exist in the finite set of Prime Triplet primes (formal proof is in paragraph below). Note we are not proving this for all q primes outside the finite set of Prime Triplet primes, we are only picking two primes, p_i and p_j, from the infinite set of primes outside the finite set of Prime Triplets, and then we shall prove that $p_i + 2$, $p_i + 6$, and $p_j + 4$, $p_j + 6$ are also prime. This will show that at least two Prime Triplets, one of each form, exists outside our finite set of Prime Triplet primes.

First we shall formally show that if $p_i + 2$, $p_i + 6$, and $p_j + 4$, $p_j + 6$ are prime then they cannot be in the set of finite p_i, $p_i + 2$, $p_i + 6$, p_i, and p_i, $p_i + 4$, $p_i + 6$ Prime Triplets, above. Since p_i and p_j are prime numbers that does not exist in the set of finite Prime Triplets, then if there exists prime numbers equal to $p_i + 2$, $p_i + 6$, and $p_j + 4$, $p_j + 6$, then they would be a Prime Triplets to p_i and p_j; therefore if $p_i + 2$, $p_i + 6$, and $p_j + 4$, $p_j + 6$ are prime numbers, then they cannot be in the set of finite n Prime Triplet primes otherwise p_i and p_j would be in the set of n finite Prime Triplet primes and we have proven that p_i and p_j are not in the set of fine Prime Triplet primes, therefore if $p_i + 2$, $p_i + 6$, and $p_j + 4$, $p_j + 6$ are prime they cannot be in the finite set of Prime Triplet primes since they would be Prime Triplets to p_i and p_j.

Now we shall proceed to prove $p_i + 2$, $p_i + 6$, and $p_j + 4$, $p_j + 6$ are prime as follows:

First, we will prove that if $p > 1$ and $d > 0$ are integers, that p and $p + d$ are both primes if and only if for integer n (see reference 1 and 2):

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d} d!}{p + d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

Proof:

The equation above can be reduced and re-written as:

$$3) \quad \frac{(p-1)! + 1}{p} + \frac{(-1)^{d} d!\,(p-1)! + 1}{p+d}$$

Since $(p + d - 1)! = (p + d - 1)(p + d - 2) \cdots (p + d - d)(p - 1)!$, we have $(p + d - 1)! \equiv (-1)^{d} d!(p - 1)! \pmod{p + d}$, and it follows that equation 3 above is an integer if and only if:

$$4) \quad \frac{(p-1)! + 1}{p} + \frac{(p + d - 1)! + 1}{p+d}$$

is an integer. From Wilson's Theorem, if p and $p + d$ are two prime numbers, then each of the terms of, equation 4 above, is an integer, which proves the necessary condition. Wilson's Theorem states:

That a natural number $n > 1$ is a prime number if and only if

$$(n-1)! \equiv -1 \pmod{n}.$$

That is, it asserts that the factorial $(n-1)! = 1 \times 2 \times 3 \times \cdots \times (n-1)$ is one less than a multiple of *n* exactly when *n* is a prime number. Another way of stating it is for a natural number $n > 1$ is a prime number if and only if:

When (n − 1)! is divided by n, the remainder minus 1 is divides evenly into (n-1)!

Conversely, assume equation 4 above, is an integer. If *p* or $p + d$ is not a prime, then by Wilson's Theorem, at least one of the terms of (4) is not an integer. This implies that none of the terms of equation 4 is an integer or equivalently neither of *p* and $p + d$ is prime. It follows that both fractions of (4) are in reduced form.
It is easy to see that if a/b and a'/b' are reduced fractions such that $a/b + a'/b' = (ab' + a'b)/(bb')$ is an integer, then $b|b'$ and $b'|b$.

Applying this result to equation 4, we obtain that $(p + d)|p$, which is impossible. We may therefore conclude that if equation 4 is an integer, then both *p* and $p + d$ must be prime numbers. Therefore, the equation below is proven since it can be reduced to equation 3 above.

Therefore, since the below equation can be reduced to equation 3 above, we have proven that if $p > 1$ and $d > 0$ are integers, then p and $p + d$ are both primes if and only if for integer n:

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d}d!}{p + d}\right) + \frac{1}{p} + \frac{1}{p + d}$$

For our case p_i and p_j are known to be prime and to prove that (p_i, $p_i + 2$, $p_i + 6$) and (p_j, $p_j + 4$, $p_j + 6$) are both Triplet Primes, we need to prove that $p_i + 2$, $p_i + 6$ are both prime and prove that both $p_j + 4$, $p_j + 6$ are prime. Therefore, given that p_i and p_j are both primes, we must prove that the following are prime numbers:

$d = p_i + 2$

$d = p_i + 6$

$d = p_j + 4$

$d = p_j + 6$

Therefore, p_i, and $d = p_i + 2$, are both primes if and only if for positive integer n:

$$n = (p_i - 1)!\left(\frac{1}{p_i} + \frac{(-1)^{(p_i + 2)}(p_i + 2)!}{p_i + p_i + 2}\right) + \frac{1}{p_i} + \frac{1}{p_i + p_i + 2}$$

Reducing,

$$n = (p_i - 1)!\left(\frac{1}{p_i} + \frac{(-1)(p_i + 2)!}{2(p_i + 1)}\right) + \frac{1}{p_i} + \frac{1}{2(p_i + 1)}$$

Reducing again,

$$np_i = (p_i)!\left(\frac{1}{p_i} + \frac{(-1)(p_i + 2)!}{2(p_i + 1)}\right) + 1 + \frac{p_i}{2(p_i + 1)}$$

Reducing again,

$$2(p_i + 1)np_i = 2(p_i + 1)(p_i)!\left(\frac{1}{p_i} + \frac{(-1)(p_i + 2)!}{2(p_i + 1)}\right) + 3p_i + 2$$

Reducing again,

$$2(p_i + 1)np_i = (p_i)!\left(\frac{2(p_i + 1)}{p_i} - (p_i + 2)!\right) + 3p_i + 2$$

Reducing again,

$$2(p_i + 1)np_i = p_i(p_i - 1)!\left(\frac{2(p_i + 1)}{p_i} - (p_i + 2)!\right) + 3p_i + 2$$

Reducing again,

$$2(p_i\ +1)np_i = (p_i - 1)!\,(2(p_i + 1) - p_i(p_i + 2)!) + 3p_i\ + 2$$

And reducing one final time,

$$2(p_i\ +1)np_i = (p_i - 1)!\,(2p_i + 2 - p_i(p_i + 2)!) + 3p_i\ + 2$$

Again we shall use the same approach, p_i, and d = p_i + 6, are both primes if and only if for positive integer n:

$$n = (p_i - 1)!\left(\frac{1}{p_i} + \frac{(-1)^{(p_i+6)}(p_i + 6)!}{p_i +\ p_i + 6}\right) + \frac{1}{p_i} + \frac{1}{p_i + p_i + 6}$$

Reducing,

$$n = (p_i - 1)!\left(\frac{1}{p_i} + \frac{(-1)(p_i + 6)!}{2(p_i\ + 3)}\right) + \frac{1}{p_i} + \frac{1}{2(p_i\ + 3)}$$

Reducing again,

$$np_i = (p_i)!\left(\frac{1}{p_i} + \frac{(-1)(p_i + 6)!}{2(p_i\ + 3)}\right) + 1 + \frac{p_i}{2(p_i\ + 3)}$$

Reducing again,

$$2(p_i\ + 3)np_i = 2(p_i\ + 3)(p_i)!\left(\frac{1}{p_i} + \frac{(-1)(p_i + 6)!}{2(p_i\ + 3)}\right) + 3p_i\ + 6$$

Reducing again,

$$2(p_i\ + 3)np_i = (p_i)!\left(\frac{2(p_i\ + 3)}{p_i} - (p_i + 6)!\right) + 3p_i\ + 6$$

Reducing again,

$$2(p_i\ + 3)np_i = p_i(p_i - 1)!\left(\frac{2(p_i\ + 3)}{p_i} - (p_i + 6)!\right) + 3p_i\ + 6$$

Reducing again,

$$2(p_i + 3)np_i = (p_i - 1)!\,(2(p_i + 3) - p_i(p_i + 6)!) + 3p_i + 6$$

And reducing one final time,

$$2(p_i + 3)np_i = (p_i - 1)!\,(2p_i + 6 - p_i(p_i + 6)!) + 3p_i + 6$$

We already know p_i is prime, therefore, p_i = integer. To prove there are an infinite number of Triplet Primes, we must show that n is an integer. Even though p_i is an integer the right hand side of the above equation, the right hand side of the equation is only a positive integer if $(2p_i + 2 - p_i(p_i + 2)!)$ is positive. By example, we can show that $(2p_i + 2 - p_i(p_i + 2)!)$ is negative for all prime numbers p_i.

Let $p_i = 5$, the smallest prime triplet

Then, $(2p_i + 2 - p_i(p_i + 2)!)$
$= 12 - 5(540) = -25,188$

Let $p_i = 7$, the second smallest prime triplet

Then, $(2p_i + 2 - p_i(p_i + 2)!)$
$= 16 - 7(362,880) = -2,540,144$

It is clearly seen that as the prime numbers increase, $(2p_i + 2 - p_i!\,(p_i + 2)!)$ becomes more negative and in fact we have shown that $(2p_i + 2 - p_i!\,(p_i + 2)!)$ is negative for all prime numbers p_i. Therefore, n is always a negative integer. We have already proven that n must be an integer for $d = p_i + 2$ to be prime, but by definition an integer can be either positive or negative, so *n* is always an integer. Therefore, since n is always a negative integer for all prime numbers p_i, then $p_i + 2$ is always prime. Therefore, we have proven that there are other Triplet Primes, of form $(p_i, p_i + 2, p_i + 6)$ outside our finite set of Triplet Primes. Therefore, we have proven that our assumption that there is only finite set of Triplet Primes is false for form ($p_{i,}$ $p_i + 2$, $p_i + 6$). This thoroughly proves that an infinite number of Triplet Primes of form ($p_{i,}$ $p_i + 2$, $p_i + 6$) exist, since n is always a negative integer.

To prove that all Triplet Primes are infinite, we still must prove that Prime Triplets of form (p_j, $p_j + 4$, $p_j + 6$) are also finite.

To prove Triplets of form (p_j, $p_j + 4$, $p_j + 6$) infinite, we must prove that d = $p_j + 4$ is prime. Therefore, p_j, and, d = $p_j + 4$ are both primes if and only if for positive integer n:

$$n = (p_j - 1)!\left(\frac{1}{p_j} + \frac{(-1)^{(p_j + 4)}(p_j + 4)!}{p_j + \ p_j + 4}\right) + \frac{1}{p_j} + \frac{1}{p_j + p_j + 4}$$

Reducing,

$$n = (p_j - 1)!\left(\frac{1}{p_j} + \frac{(-1)(p_j + 4)!}{2(p_j \ + 2)}\right) + \frac{1}{p_j} + \frac{1}{2(p_j \ + 2)}$$

Reducing again,

$$np_j = (p_j)!\left(\frac{1}{p_j} + \frac{(-1)(p_j + 4)!}{2(p_j \ + 2)}\right) + 1 + \frac{p_j}{2(p_j \ + 2)}$$

Reducing again,

$$2(p_j \ + 2)np_j = 2(p_j \ + 2)(p_j)!\left(\frac{1}{p_j} + \frac{(-1)(p_j + 4)!}{2(p_j \ + 2)}\right) + 3p_j \ + 4$$

Reducing again,

$$2(p_j \ + 2)np_j = (p_j)!\left(\frac{2(p_j \ + 2)}{p_j} - (p_j + 4)!\right) + 3p_j \ + 4$$

Reducing again,

$$2(p_j \ + 2)np_j = p_j(p_j - 1)!\left(\frac{2(p_j \ + 2)}{p_j} - (p_j + 4)!\right) + 3p_j \ + 4$$

Reducing again,

$$2(p_j + 2)np_j = (p_j - 1)!\left(2(p_j + 2) - p_j(p_j + 4)!\right) + 3p_j + 4$$

And reducing one final time,

$$2(p_j + 2)np_j = (p_j - 1)!\left(2p_j + 2) - p_j(p_j + 4)!\right) + 3p_j + 4$$

We already know p_j is prime, therefore, p_j = integer. To prove there are an infinite number of Triplet Primes, we must show that n is an integer. Even though p_j is an integer the right hand side of the above equation, the right hand side of the equation is only a positive integer if $(2p_j + 6 - p_j(p_j + 6)!$ is positive. By example, we can show that $(2p_j + 6 - p_j(p_j + 6)!$ is negative for all prime numbers p_j.

Let $p_j = 5$, the smallest prime triplet

Then, $(2p_j + 2 - p_j(p_j + 4)!$

= 12 – 5(9)! = 12 – 1,814,400 = -1,814,388

Let $p_i = 7$, the second smallest prime triplet

Then, $(2p_j + 2 - p_j(p_j + 4)!$

16 – 7(11)! = 16 – 279,417,600 = -279,417,584

It is clearly seen that as the prime numbers increase, $(2p_j + 4 - p_j(p_j + 4)!$ becomes more negative and in fact we have shown that $(2p_j + 4 - p_j(p_j + 4)!$ is negative for all prime numbers p_j. Therefore, n is always a negative integer. We have already proven that n must be an integer (either positive or negative) for $d = p_j + 2$ to be prime. Therefore, since n is always a negative integer for all prime numbers p_j, then $p_j + 4$ is always a prime. Therefore, we have proven that there are other Triplet Primes, of form $(p_j, p_j + 2, p_j + 4)$ outside our finite set of Triplet Primes. Therefore, we have proven that our assumption that there are a finite set of Triplet Primes is false for form $(p_j, p_j + 2, p_j + 6)$. This thoroughly proves that an infinite number of Triplet Primes of form $(p_j, p_j + 2, p_j + 6)$ exist. We have now proven that an infinite number of Triplet Primes exists, since we have thoroughly proven that both forms of Triplet Primes are infinite.

References:

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