Towards a closed-form formula for roots of polynomials

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Abstract

In the present work we make a small step towards finding explicit formula for polynomial roots. Explicit formulae for simple real roots, based on the Frobenius companion matrix, are derived. After some modifications the method can be used to compute multiple or complex roots.

1 Introduction

Solving algebraic equations has a long and fascinating history [1, 2]. In spite of great progress in analytical theory and numerical methods there is no general formula for roots of algebraic equations. In this context Ian Stewart made inspiring remark: *"there is no general formula for them, unless you invent new symbols specifically for the task"* [3].

In the present work we make a small step towards finding explicit formula for polynomial roots. In the next Section the Frobenius companion matrix A of an arbitrary polynomial $p(x)$ is described and some of its properties are reviewed. It is well known that due to the Cayley-Hamilton theorem the matrix A is a generalized root of the polynomial p, i.e. $p(A) = 0$. This suggests that this matrix should be in the focus of our search for a general formula for roots of polynomial equation $p(x) = 0$. In Section 3, taking advantage of properties of the companion matrix, explicit formulae for real simple roots of a polynomial are derived and an example is provided. The method can be extended for polynomials with multiple and complex roots. Our results are discussed in the last Section.

2 Polynomials, characteristic polynomial and the companion matrix

Let us consider a polynomial $p(x)$ of degree n with real or complex coefficients:

$$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots c_{n-1} x^{n-1} + x^n. \tag{1}$$

It is possible to construct a matrix A, such that the characteristic polynomial of which is $p(\lambda)$:

$$\det(A - \lambda I) = p(\lambda), \tag{2}$$

where I is the unit matrix. Indeed, the demanded matrix, known as the companion matrix introduced by Frobenius, is explicitly given by:

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & -c_0 \\ 1 & 0 & \ldots & 0 & -c_1 \\ 0 & 1 & \ldots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -c_{n-1} \end{pmatrix}. \tag{3}$$

Coefficients c_k of a characteristic polynomial $p(\lambda) = \det(B - \lambda I)$ of an arbitrary matrix B are [4]:

$$c_{n-1} = -\mathrm{Tr}(B), \tag{4a}$$

$$c_{n-2} = \tfrac{1}{2}\left((\mathrm{Tr}(B))^2 - \mathrm{Tr}\left(B^2\right)\right), \tag{4b}$$

$$c_{n-3} = \tfrac{1}{6}\left(-(\mathrm{Tr}(B))^3 + 3\mathrm{Tr}(B)\,\mathrm{Tr}\left(B^2\right) - 2\mathrm{Tr}\left(B^3\right)\right), \tag{4c}$$

the next coefficients, given by more and more complex formulae, are not provided here but can be found in [4].

Due to the Cayley-Hamilton theorem a matrix fulfills its characteristic equation. It follows that:

$$p(A) = 0, \tag{5}$$

and therefore the Frobenius companion matrix A is a generalized solution of polynomial equation $p(x) = 0$. It is important that it is possible to find explicit matrix solution for any polynomial equation. The good news is that the matrix A contains information about roots of the polynomial $p(x)$. Indeed, eigenvalues of A:

$$Au = \lambda u, \tag{6}$$

are roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$, and, due to construction of the companion matrix, roots of the polynomial defined in (1).

It seems, however, that the matrix solution of the equation $p(x) = 0$ is not very useful, since standard method to find eigenvalues and eigenvectors of a matrix is to solve the characteristic equation $\det(A - \lambda I) = 0$. Are we thus running in circles? Luckily, there are, however, two useful results described in the next Section.

3 Explicit formulae for real roots (the case of distinct roots)

Let us consider the eigenproblem (6) where A is the companion matrix of the polynomial (1). First of all, there are several very effective methods to compute eigenvalues and eigenvectors of a matrix, without explicit use of the characteristic equation [5].

Secondly, we can use the Frobenius companion matrix and equations (4) to compute roots of a polynomial. Assume that all roots of the characteristic polynomial are distinct. Then eigenvalues of the matrix A^M are $\lambda_1^M, \ldots, \lambda_n^M$. We can now compute the sum of powers of eigenvalues of the companion matrix using Viète's formula and equation (4a) as $\mathrm{Tr}\left(A^M\right) = \sum_{i=1}^n \lambda_i^M$. This sum is always real. Indeed, $(a+bi)^M + (a-bi)^M = r^M e^{iM\varphi} + r^M e^{-iM\varphi} = 2r^M \cos(M\varphi)$ so the contribution from a complex root is real, however oscillatory.

Let us now assume that the root with the largest modulus, $\hat{\lambda}_1$, is real (it is also simple since all roots were assumed distinct). For M large enough contribution from $\left(\hat{\lambda}_1\right)^M$ will dominate the sum. Therefore we can compute $\hat{\lambda}_1$ as:

$$\hat{\lambda}_1 = \lim_{N\to\infty} \sqrt[2N+1]{T_{2N+1}}, \quad T_M \stackrel{df}{=} \mathrm{Tr}\left(A^M\right), \tag{7}$$

where (4a) and Viète's formula was used.

Suppose now that the root with the second-largest modulus, $\hat{\lambda}_2$, is real as well. Due to (4b) and the corresponding Viète's formula we get:

$$\hat{\lambda}_2 = \frac{1}{\hat{\lambda}_1} \lim_{N\to\infty} \sqrt[2N+1]{U_{2N+1}}, \quad U_M \stackrel{df}{=} \tfrac{1}{2}\left(T_M^2 - T_{2M}\right), \tag{8}$$

Under similar assumptions we get from (4c) and Viète's formula expression for the root with the third-largest modulus:

$$\hat{\lambda}_3 = \frac{1}{\hat{\lambda}_1\hat{\lambda}_2} \lim_{N\to\infty} \sqrt[2N+1]{W_{2N+1}}, \; W_M \stackrel{df}{=} -\tfrac{1}{6}\left(-T_M^3 + 3T_M T_{2M} - 2T_{3M}\right). \tag{9}$$

Below we provide example of convergence of the above formulae.

Example 1 *Consider polynomial* $p(x) = \left(x^2+1\right)(x-2)(x-3)(x+3.1) = x^5 - 1.9x^4 - 8.5x^3 + 16.7x^2 - 9.5x + 18.6$. *In this case roots with largest moduli are not well separated and thus the convergence to* $\hat{\lambda}_1$ *will be slow. The companion matrix is:*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & -18.6 \\ 1 & 0 & 0 & 0 & 9.5 \\ 0 & 1 & 0 & 0 & -16.7 \\ 0 & 0 & 1 & 0 & 8.5 \\ 0 & 0 & 0 & 1 & 1.9 \end{pmatrix}, \tag{10}$$

and we get:

N	$\sqrt[2N+1]{T_{2N+1}}$	N	$\sqrt[2N+1]{U_{2N+1}}$	N	$\sqrt[2N+1]{W_{2N+1}}$
25	$-3.087\,380\,731$	10	$-9.300\,044\,188\,24$	2	$-18.600\,018\,868\,2$
50	$-3.098\,860\,499\,81$	15	$-9.300\,000\,665\,59$	5	$-18.600\,000\,001\,4$
100	$-3.099\,978\,809\,84$	20	$-9.300\,000\,010\,11$	10	-18.6
200	$-3.099\,999\,984\,94$	25	$-9.300\,000\,000\,15$		

4 Discussion

In the present work we derived explicit formulae (7), (8), (9) for polynomial roots. More exactly, these formulae, based on the Frobenius companion matrix, can be used to compute real simple roots with largest moduli. Viète's formulae and expressions for coefficients c_k of the characteristic polynomial (4) have to be used as well (note, however, that for decreasing k these expressions become more and more complicated).

The formula (7) works well provided that the roots are well separated, other formulae behaving better. It is also possible, after carrying out transformation $x \to 1/x$, to compute real simple roots with smallest moduli.

Let us note finally that the method is related to Dandelin-Gräffe algorithm, discovered by Dandelin, Gräffe and Lobachevsky [6, 7], in which the polynomial roots are squared iteratively. It is thus possible to use methods elaborated for the Dandelin-Gräffe algorithm [7] to compute multiple or complex roots within our approach.

References

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