Proof of Beal's Conjecture

Stephen Marshall

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Abstract: This paper presents a complete and exhaustive proof of the Beal Conjecture. The approach to this proof uses the Fundamental Theorem of Arithmetic as the basis for the proof of the Beal Conjecture. The Fundamental Theorem of Arithmetic states that every number greater than 1 is either prime itself or is unique product of prime numbers. The prime factorization of every number greater than 1 is used throughout every section of the proof of the Beal Conjecture. Without the Fundamental Theorem of Arithmetic, this approach to proving the Beal Conjecture would not be possible.

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1. Introduction

In 1997 an amateur mathematician and Texas banker named Andrew Beal discovered the Beal Conjecture from running extensive computer programs that always gave results consistent with the conjecture. From these large number of computer solutions, with all solutions satisfying the conditions of the conjecture gave Beal the confidence to formally propose the Beal Conjecture and offer prize money for a proof of the conjecture. The Beal Conjecture states that the only solutions to the equation $A^x + B^y = C^z$, when A, B, C, are positive integers, and x, y, and z are positive integers greater than 2, are those in which A, B, and C have a common prime factor. The truth of the Beal Conjecture implies Fermat's Last Theorem is given with a solution to the Beal's Conjecture. Fermat's Last Theorem states that there are no solutions to the equation $a^n + b^n = c^n$ where a, b, and c are positive integers and n is a positive integer greater than 2.

More than three hundred years ago, Pierre de Fermat claimed he had a proof but did not leave a record of it. The theorem was proved in the 1990s by Andrew Wiles, together with Richard Taylor. Both the Beal Conjecture and Fermat's Last Theorem are typical of many statements in number theory: easy to say, but extremely difficult to prove.

2. The Proof

The Beal Conjecture states the following:

If $A^x + B^y = C^z$, where A, B, C, x, y and z are positive integers and x, y and z are all greater than 2, then A, B and C must have a common prime factor.

We will prove for every positive integer for A, B, and C and for every x, y and z that are greater than 2 that the Beal Conjecture is true. This proof is based on the Fundamental Theorem of Arithmetic; therefore we will begin the proof of Beal's Conjecture with a formal statement of the Fundamental Theorem of Arithmetic below:

Fundamental Theorem of Arithmetic:

In Number Theory, the **fundamental theorem of arithmetic**, also called the **unique factorization theorem** or the **unique-prime-factorization theorem**, states that every number greater than 1 is either prime itself or is the product of prime numbers, and that, although the order of the primes in the second case is arbitrary, the primes themselves are not. For example,

$$1200 = 2^4 \times 3^1 \times 5^2 = 3 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5 = 5 \times 2 \times 3 \times 2 \times 5 \times 2 \times 2 = \cdots \text{ etc.}$$

The theorem is stating two things: first, that 1200 *can* be represented as a product of primes, and second, no matter how this is done, there will always be four 2s, one 3, two 5s, and no other primes in the product.

The requirement that the factors be prime is necessary: factorizations containing composite numbers may not be unique (e.g. $12 = 2 \times 6 = 3 \times 4$).

Proof of Beal's Conjecture:

First we shall assume that the Beal Conjecture is false, specifically:

If $A^x + B^y = C^z$, where A, B, C, x, y and z are positive integers and x, y and z are all greater than 2, then A, B and C do not always have a common prime factor.

We start with $A^x + B^y = C^z$

Factoring the left side, $A^x(1 + B^y/A^x) = C^z$

Since A, B, and C are all positive integers then A^x, B^y, and C^z are all positive integers. Since C^z is a positive integer, then $(1 + B^y/A^x)$ must be a positive integer (**first possibility)** or a fraction that is a rational number having a factor of A^x in its denominator so by reducing $A^x(1 + B^y/A^x)$ it is equal to the integer C^z (**second possibility)**.

Proof of first Possibility:

For the first case if $(1 + B^y/A^x)$ = positive integer

Then, B^y/A^x must be an integer, which implies that B^y and A^x have a common prime factor in accordance with (IAW) the Fundamental Theorem of Arithmetic. Furthermore, B^y must be divisible by A^x since B^y/A^x is an integer, then A^x must be reduced to 1 for B^y/A^x to be reduced to a positive integer.

Factoring B^y/A^x, then

$$B^y/A^x = \frac{(B_1)(B_2)(B_3) \ldots\ldots\ldots (B_{y-2})(B_{y-1})(B_y)}{(A_1)(A_2)(A_3) \ldots\ldots (A_{x-2})(A_{x-1})(A_x)}$$

Since $B = B_1 = B_2 = B_{y-1} = B_y$ and $A = A_1 = A_2 = A_{x-1} = A_x$, then none of the series of B's or A's have common prime factors since in our assumption we assumed that Beal's Conjecture was false and A, B and C do not always have a common prime factor. Therefore, B^y and A^x do not always have common prime factors.

However, IAW the **Fundamental Theorem of Arithmetic** every integer > 1 must be a unique series of prime factors (including A and B which are positive integers), therefore B^y/A^x can only be an integer if A and B have common prime factors. Therefore since B^y/A^x cannot be an integer according to our assumption that Beal's Conjecture is false and A, B and C do not always have a common prime factor. However B^y/A^x must be an integer, therefore B^y and A^x must have common prime factors, which also means that A and B must have common prime factors since all $B = B_1 = B_2 = B_{y-1} = B_y$ and $A = A_1 = A_2 = A_{x-1} = A_x$. This also implies that our assumption that A, B, and C do not always have a common prime factor is false and Beal's Conjecture must be true for this first possibility. This proof also depends on our proof on page 8 that C has prime factor with A and B, which we save until last to prove.

Proof of second Possibility:

The only other possibility is for $(1 + B^y/A^x)$ to be a fraction that is a rational number and has a factor of A^x in its denominator so by reducing $A^x(1 + B^y/A^x)$ it is equal to the integer C^z.

Let $A^x{}_F$ = a factor of A^x

Therefore, $(1 + B^y/A^x) = N/A^x{}_F$, where N = a positive integer

Reducing, $A^x{}_F + (B^yA^x{}_F)/A^x = N$

$A^x{}_F$ is a positive integer since it is a factor of A^x which is a positive integer.

Therefore, $(B^yA^x{}_F)/A^x$ must be an integer, then rearranging, $(B^yA^x{}_F)/A^x = (B^y)/(A^x/A^x{}_F)$

Then using $(B^y)/(A^x/A^x{}_F)$ = integer

Let A^{X-R} = the multiplication of series of prime factors remaining for A^x after reducing $A^x/A^x{}_F$ to an integer. This follows from Fundamental Theorem of Arithmetic that the remainder of A^x after factoring out $A^x{}_F$ must be a unique series of prime factors, and since $A^x{}_F$ = a factor of A^x, then $A^x{}_F$ which divides evenly into A^x can be reduced to 1 in the denominator leaving an integer in the numerator, and this integer in the numerator according to the Fundamental Theorem of Arithmetic must also be a unique series of prime factors, which we are calling A^{X-R}

Therefore, $(B^y)/(A^x/A^x{}_F) = (B^y)/(A^{X-R})$ = integer.

However, according to our original assumption, A^{X-R} cannot have a prime factor with B^y since the prime factors for A^{X-R} is a subset of the prime factors for A^x and according to our assumption that Beal's Conjecture is false and B and A do not always have common prime factors, then B^y and A^x do not always have common prime factors. This implies that A^{X-R} do not always have common prime factors with B^y.

More specifically,

$$(B^y)/(A^{X-R}) = \frac{(B_1)(B_2)(B_3) \ldots\ldots\ldots (B_{y-2})(B_{y-1})(B_y)}{(A_1)(A_2)(A_3) \ldots\ldots\ldots (A_{x-R-1})(A_{x-R})}$$

None of the A's 1 through X-R have a common prime factor with any of the B's 1 through Y since $B = B_1 = B_2 = B_y$ and $A = A_1 = A_2 = A_{x-R-1}$ and then A_{x-R} does not have to be equal to A. According to our original assumption A, B, and C do not always have common prime factors. Therefore, A^{X-R} so not always have common prime factors with B^y. However, for our second and final possibility, we have shown earlier that $(B^y)/(A^x/A^x{}_F)$ must be a positive integer. Since $(B^y)/(A^x/A^x{}_F) = (B^y)/(A^{X-R})$ = integer. Therefore, since:

$$(B^y)/(A^{X-R}) = \frac{(B_1)(B_2)(B_3) \ldots\ldots\ldots (B_{y-2})(B_{y-1})(B_y)}{(A_1)(A_2)(A_3) \ldots\ldots\ldots (A_{x-R-1})(A_{x-R})} = \text{integer}$$ (see page 6 for case when all $A_i=1$)

Therefore, since $B = B_1 = B_2 = B_y$ and $A = A_1 = A_2 = A_{x-R-1}$, then A_{x-R} does not have to be equal to A, but since $(B^y)/(A^{X-R})$ = integer, then A_{x-R} must be a factor of B. Additionally, A and B must have a common prime factor for $(B^y)/(A^{X-R})$ to be an integer, note all A_{i-R-1} must have prime factors with all B_i to be reduced to 1 in the denominator and be equal to an integer. Furthermore, IAW the **Fundamental Theorem of Arithmetic** every integer > 1 must be a unique series of prime factors, therefore $(B^y)/(A^{X-R})$ is a unique series of prime factors that can only be an integer if A and B have a common prime factor. This proof also depends on our proof on page 8 that C has prime factor with A and B, which we save until last to prove.

Now we shall address the case when either A, B, or C are equal to 1. For $A^x + B^y = C^z$ let A = 1, then $1 + B^y = C^z$, then $1 = C^z - B^y$, since Y and Z are both > 2, then since B and C > 1 it is only possible for $C^z - B^y$ to be greater than 1, or equal to 0 only if $C^z = B^y$. For example, the smallest number possible for either B or C is 2. Since Y and Z are both > 2, then the lowest integer for Y or Z is 3. Then if lowest integer for C or B is 2, then say B = 2, then $B^3 = 2^3 = 8$, which is > 1. If $C^z \neq B^y$ then the lowest integer C can be is 3, then $C^3 = 3^3 = 27$, and $C^z - B^y = 27 – 8 = 19 > 1$, so there is no solution for $A^x + B^y = C^z$ when A = 1.

Following the same logic we can show that if B = 1, then so there is no solution for $A^x + B^y = C^z$. For $A^x + B^y = C^z$ let B = 1, then $A^x + 1 = C^z$, then $1 = C^z - A^x$, since X and Z are both > 2, then since A and C > 1 it is only possible for $C^z - A^x$ to be greater than 1, or equal to 0 only if $C^z = A^x$. For example, the smallest number possible for either A or C is 2. Since X and Z are both > 2, then the lowest integer for X or Z is 3. Then if lowest integer for C or B is 2, then say A = 2, then $A^x = 2^3 = 8$, which is > 1. If $C^z \neq A^x$ then the lowest integer C can be is 3, then $C^3 = 3^3 = 27$, and $C^z - A^x = 27 – 8 = 19 > 1$, so there is no solution for $A^x + B^y = C^z$ when B = 1. Following similar logic, if C = 1, then $A^x + B^y = 1$, but the lowest integers that A and B can be is A = B = 1, but then $A^x + B^y = 1$ can be reduced to 1 + 1 = 1, since $2 \neq 1$, then we have shown that there is no solution for $A^x + B^y = C^z$ when C = 1. This also shows that if A = B = C = 1, then that there is no solution for $A^x + B^y = C^z$ when A = B = C = 1. Also if two of A, B, or C are equal to 1, then let A = B = 1. Then $A^x + B^y = C^z$ can be reduced to $1 + 1 = C^z$ and Z > 2, then the lowest integer Z can be is Z = 3. Then $1 + 1 = C^3$ but if C = 1, then 2 = 1, but $2 \neq 1$. If C = 2 then $1 + 1 = 2^z$. Again, the smallest integer for Z is Z – 3, then $1 + 1 = 2^3 = 8$. But $1 + 1 = 2 \neq 8$, and following the same logic for all C integers, C > 2 will not have solutions either. We have shown there is no solution for $A^x + B^y = C^z$ when A =B = 1.

Following similar logic as above for A = B =1, if any two of A, B, or C are equal to 1, then it can easily be shown that there is no solution for $A^x + B^y = C^z$ when any two combinations of A, B, or C are equal to 1.

The only remaining proof to completely prove that Beal's Conjecture is true is to show that C must have a common prime factor with A and B. We have already proven that A and B have a common prime factor and have shown that B^y and A^x have a common prime factor.

$A^x + B^y = C^z$

Let p be a common prime factor of B^y and A^x

Then $p(A^x/p + B^y/p) = C^z$

Reducing, $(A^x/p + B^y/p) = C^z/p$

Since p is a common prime factor of B^y and A^x, then A^x/p and B^y/p can both be reduced to integers. Therefore, since A^x/p and B^y/p are both integers then C^z/p is an integer, which is only possible if p is a common prime factor of C^z. Therefore, we have thoroughly proven that A, B, and C must have common prime factors.

3. Conclusion

Our assumption that A, B, and C do not always have a common prime factor is false and Beal's Conjecture must be true for this second possibility. Beal's Conjecture has already been proven for the first possibility, therefore Beal's Conjecture is proven true for all possibilities and for all A, B, and C positive integers and all x, y, z > 2.

References

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