Dirac-like equation and the excited states of the nucleon

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Abstract- We use a Dirac-like equation with a linear potential in order to evaluate the energy levels of the excited states of the Y-shaped color-electric flux tubes, describing the quarks interactions inside the nucleon. We have neglected the bare-masses of the quarks, and the obtained results reproduce the energy of the centroid of the excited states of the nucleon as well as the possible creation of nucleon-antinucleon pairs.

1 – Introduction

The non-abelian character of the Quantum Chromodynamics (QCD) demands that at low energies, the lines of force of the color-electric fields to attract each other (in a Faraday-like description of the phenomenon), so that we have a color-electric flux tube connecting a quark-antiquark pair [1,2,3,4,5]. As was pointed out by Sakumichi and Suganuma [4], quark confinement is interpreted with a linear interquark potential acting at long distances. Also as was quoted in [4], the linear confining potential is considered to be caused by "one-dimensional squeezing" of the interquark color-electric flux, which is also shown by lattice QCD studies (please see also [3]).

Indeed, Nambu [5] was the first theorist to propose that quarks, inside the hadrons, are tied together by strings. He draws this conclusion based in the evidence that the masses of strongly interacting particles increased without limits as their internal angular momentum increased (please see also [2]). As quoted in [2], lattice QCD have demonstrated [6] that Nambu's conjecture was essentially correct:- in chromodynamics, a string-like chromoelectric flux tube forms between distant quarks, leading to their confinement with an energy proportional to the distance between them.

As quarks are spin-½ particles, it seems that the more appropriate treatment of their interactions is through the Dirac equation. There are examples in the literature where the Dirac equation has been solved in 1+1 dimensions, with the interactions represented by a linear potential [7,8,9,10].

In this work we intend to look at the energy levels of the excited states of the nucleon, by considering the three quarks tied by linear potentials, in an Y-type flux tube formation [11,12]. We are going to do this by solving a Dirac-like equation, where the interaction between quarks is described by a static linear potential.

2 – Dirac equation and the linear potential

Let us consider an arm of the Y-shaped flux tube with a potential, V= ar, being a the string tension. Neglecting the bare masses of the quarks, we can write the "Dirac hamiltonian"

$$\mathbf{H} = \boldsymbol{\alpha}\, \mathbf{p} + \boldsymbol{\beta}\, a\, \mathbf{r}. \tag{1}$$

We also have the Dirac-like equation

$$\mathbf{H}\Psi = E\Psi. \tag{2}$$

As is usually in treating Dirac equation (please see [13]), we are going to take the square of the operator **H** acting on the wave function Ψ. We have

$$\mathbf{H}^2\, \Psi \equiv (\mathbf{H}*\mathbf{H})\Psi = [(\boldsymbol{\alpha}\mathbf{p} + \boldsymbol{\beta} a\mathbf{r})*(\boldsymbol{\alpha}\mathbf{p} + \boldsymbol{\beta} a\mathbf{r})]\Psi = E^2\Psi. \tag{3}$$

Besides to take in account that the operators **p** and **r** satisfies the usual commutation relation of quantum mechanics, we need to define the products in the (**α**,**β**) space. Next we establish the rules to these products.

$$\boldsymbol{\alpha}*\boldsymbol{\alpha} = \boldsymbol{\beta}*\boldsymbol{\beta} = 1. \tag{4A}$$

$$\boldsymbol{\alpha}*\boldsymbol{\beta} = |\boldsymbol{\alpha}|\,|\boldsymbol{\beta}| \sin(\pi/2)\,(-i) = -i. \tag{4B}$$

In the case of the "cross-product" we can consider two possibilities:

$$\boldsymbol{\alpha}*\boldsymbol{\beta} = \boldsymbol{\beta}*\boldsymbol{\alpha}, \tag{4C}$$

in the commuting case, and

$$\boldsymbol{\alpha}*\boldsymbol{\beta} = -\boldsymbol{\beta}*\boldsymbol{\alpha}, \tag{4D}$$

in the anti-commuting one.

3 – Dirac equation and the linear potential: the commuting case

In the commuting case, the Dirac equation reads

$$(\mathbf{p}^2 + a^2\,\mathbf{r}^2 - i\mathbf{p}a\mathbf{r} - ia\mathbf{r}\mathbf{p})\Psi = E^2\Psi. \tag{5}$$

Let us seek for a trying wave function which satisfies equation (5).
By taking

$$\mathbf{p} = id/dr \qquad \text{and} \qquad \Psi = \exp(-Kr^2), \tag{6}$$

and performing the indicated calculations, we have from (5)

$$[(a^2 - 4K^2 - 4Ka)\,r^2]\,\Psi = [\,E^2 - (2K + a)]\,\Psi. \tag{7}$$

Equation (7) implies that

$$a^2 - 4K^2 - 4Ka = 0, \tag{8A}$$

$$E^2 = 2K + a. \tag{8B}$$

From the roots of equation (8A), we take the positive one, namely

$$2K_+ = a(\sqrt{2} - 1). \tag{9}$$

Inserting (9) into (8B), we obtain

$$E^2 = 2K_+ + a = \sqrt{2}\, a. \qquad (10)$$

It would be interesting to estimate the value of the string constant a, appearing in (10). This can be done as follows. We write the Hamiltonian

$$\mathcal{H} = p + ar. \qquad (11)$$

Besides this we consider the uncertainty relation

$$r\, p = ½. \;\; (\hbar=1) \qquad (12)$$

Putting (12) into (11) we get

$$\mathcal{H} = p + a/(2p). \qquad (13)$$

Differentiating $\mathcal{H}$ with respect to p, by taking $d\mathcal{H}/dp|_{p0} = 0$, we have

$$a = 2\, p_0^2 = 2\, m^2 = 2\, M^2/9. \qquad (14)$$

In (14) we took m, the constituent mass of the quark, as being equal to one third of the nucleon mass (M).

Inserting (14) into (10), we obtain

$$E^2 = 2\sqrt{2}\, M^2/9. \qquad (15)$$

The positive root of E^2 (eq.(15)) gives

$$E = 8^{1/4}\, M/3. \qquad (16)$$

We observe that (16) corresponds to the energy of the excited state of one arm of the Y-shaped description of the nucleon (please see reference [12]). Therefore the energy of the excited state of the nucleon is

$$E_{e\text{-nucleon}} = 3E = 8^{1/4}\, M. \tag{17}$$

Putting numbers in (17), we get

$$E_{e\text{-nucleon}} \cong 1.68\ M \cong 1.6\ \text{GeV}. \tag{18}$$

This value must be compared with that of 1.5 GeV of reference [12].

Meanwhile Brown and Rho [14], evaluated the energy difference between the centroid of the excited states and the ground state of the nucleon as being approximately 600 MeV. In order to compare the present result with that obtained by Brown and Rho [14], we write

$$\Delta E = E_{e\text{-nucleon}} - M \cong .68\ M \cong .64\ \text{GeV}. \tag{19}$$

We also may to compare (19) with the result we obtained in a previous work [15], namely

$$\hbar\omega = 2\ M/\pi \cong .64\ M \cong .60\ \text{GeV}. \tag{20}$$

4 – Dirac equation and the linear potential: the anti-commuting case

In the anti-commuting case (see(4D), the squared Dirac equation reads

$$(\mathbf{p}^2 + a^2\, \mathbf{r}^2 - i\mathbf{p}a\mathbf{r} + ia\mathbf{r}\mathbf{p})\Psi = E^2\, \Psi. \tag{21}$$

Inserting the trying wave function given by (6)into (21) leads to

$$[(a^2 - 4K^2)\ r^2]\ \Psi = [(E^2 - (2K + a)]\ \Psi. \tag{22}$$

Equation (22) implies that

$$a^2 = 4K^2 \qquad \text{and} \qquad E^2 = 2K + a. \tag{23}$$

The non-trivial solution of (23) gives

$$E^2 = 2a. \qquad (24)$$

If we use the evaluation of a obtained in (14), we get

$$E^2 = 4\, M^2/9. \qquad (25)$$

The positive root of (26) gives

$$E_{tot} = 3E = 2M. \qquad (26)$$

We can interpret (26) as the creation of a nucleon-anti-nucleon pair.

5 – The squared Hamiltonian

Let us consider the square of the Hamiltonian given by (11) acting on the wave function Ψ. We have

$$\mathcal{H}^2\, \Psi = (p + ar)^2\, \Psi = (p^2 + a^2r^2 + 2rpa)\, \Psi = E^2\, \Psi. \qquad (27)$$

By using the uncertainty relation (12) to fix the "cross-product" appearing in (27), we get

$$(p^2 + a^2r^2)\, \Psi = (E^2 - a)\, \Psi = (2\mathcal{H}_{HO})\, \Psi. \qquad (28)$$

In (28), we have identified the left hand term as being two times the harmonic-oscillator hamiltonian of frequency a. Pursuing further, we get

$$E^2 = 2(n + 1)a + a = (n + 1)2a, \quad n=0,1,2,\ldots \qquad (29)$$

We observe that the ground state of (29) coincides with the result we obtained in (24).

Inserting in (29)the value of the string tension we get in (14) and taking the positive root of E^2, we have

$$E_{tot} = 3E = (n + 1)^{1/2}\, 2M. \tag{30}$$

It is interesting to consider the states of (30) given by

$$N^2 = n + 1, \quad \text{with} \quad N^2 = 1,4,9,\ldots \tag{31}$$

These states correspond to the production of nucleons-anti-nucleons pairs.

6 – Concluding remarks

In this work we have considered solutions of the Dirac equation, where the interactions between quarks were attributed to the chromo-electric flux tubes, leading to a static linear potential. In doing this we have neglected the "free-quark" masses, as compared with the constituent-quark masses caused by the strong interactions (please see [16]).

A nice visualization of the manner in which QCD vacuum fluctuations are expelled from the interior region of a baryon (nucleon, for instance) is presented as an animation in reference [17]. There, a dynamic picture of the Y-shaped flux tube is shown.

Meanwhile we have considered in the present work an averaged description of this phenomenon, where the arms of the Y-shaped flux tube are described through a static linear potential. The size (R) of the arm of the Y-form flux tube can be estimated as it follows. We write

$$a\,R = M. \tag{32}$$

Inserting (14) in (32) we get

$$R = 4.5 / M \cong 0.95 \text{ fm}. \tag{33}$$

This value must be compared with an estimate of the nucleon radius, by using the MIT bag model (please see [18]), namely

$$R_{MIT} = 4 / M \cong 0.84 \text{ fm}. \qquad (34)$$

As was reported in [17], a key point of interest is the distance at which the flux tube formation occurs. Yet according to [17], the animation indicates that the transition to flux-tube formation occurs when the averaged distance of the quarks from the centre of the triangle is greater than 0.5fm. Therefore the value of R estimated in the present work, namely R ≅ 0.95 fm, agrees with the above statement.

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