Fractal Propagators and the Asymptotic Sectors of Quantum Field Theory

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Abstract

Propagators for charged fermions no longer follow the prescription of perturbative quantum field theory (QFT) in the far infrared (IR) and ultraviolet (UV) sectors of particle physics. They acquire a fractal structure from radiative corrections contributed by gauge bosons. Here we show how fractal propagators in QFT may be analyzed using fractional field theory on space-times having minimal deviations from four-dimensionality ($D = 4 - \varepsilon$, ε << 1). An intriguing consequence of this approach is the emergence of classical gravity as long-range and ultra-weak excitation of the Higgs condensate.

Key words: Standard Model, Fractal Propagator, Fractional Field Theory, Higgs Condensate, Continuous Dimension, Electroweak Model, Classical Gravity.

1. Introduction and Motivation

The free-fermion propagator in QFT determines the probability amplitude for a fermion to travel between different space-time locations. It is given by [1-2]

$$S_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \exp[-ip \cdot (x-y)] S_F(p) \tag{1}$$

in which

$$S_F(p) = \frac{\gamma^\mu p_\mu + m}{p^2 - m^2 + i0^+} = \frac{1}{\gamma^\mu p_\mu - m + i0^+} \tag{2}$$

This formula successfully applies to both the IR regime of quantum electrodynamics (QED) and the UV limit of quantum chromodynamics (QCD), where the approximation of nearly free-fermions holds well. In contrast, at distance scales where the radiative contribution of soft photons to electron self-interaction becomes relevant and is accounted for, the propagator changes to [3-4]

$$S(p) = \left(\frac{m}{i\Lambda}\right)^{\gamma} \Gamma(1+\gamma) \frac{\gamma^{\mu} p_{\mu} + m}{(p^2 - m^2 + i0^+)^{(1+\gamma)}} \tag{3}$$

Here, the fractional "anomalous" exponent $\gamma = \frac{\alpha}{\pi}$ is related to the low-energy value of the fine structure constant α, Λ is an arbitrary high-energy scale and $\Gamma(...)$ stands for the Gamma function. Surveying the history of publications on this topic reveals the limitations of conventional QFT in dealing with non-perturbative aspects of particle physics [3-4].

Let

$$S^{-1}(p) = \frac{(p^2 - m^2 + i0^+)^{(1+\gamma)}}{\gamma^{\mu} p_{\mu} + m} f\left(\frac{\Lambda}{m}\right) \approx (\gamma^{\mu} p_{\mu} - m + i0^+) f\left(\frac{\Lambda}{m}\right) \tag{4a}$$

$$f\left(\frac{\Lambda}{m}\right) = \left(\frac{i\Lambda}{m}\right)^{\gamma} \tag{4b}$$

represent the inverse propagator entering (3). Relation (4) explicitly factors out the contribution of the standard inverse propagator $(\gamma^{\mu} p_{\mu} - m + i0^+)$ and the interpolating function $f(\Lambda/m) = (i\Lambda/m)^{\gamma}$ expressed in terms of two widely separated mass scales $m << \Lambda$ and fractional exponent γ.

This analysis is, however, not limited to the QED of charged fermions. Similar reasoning indicates that both scalar and gauge bosons of the Standard Model (SM) cannot be realistically approximated as excitations of free fields. In particular [1-2],

a) Higgs and Yang-Mills theories are *nonlinear dynamic models* which exhibit self-interaction, with the possible exception of the deep UV sector where they become ultra-weakly coupled or "trivial".

b) In general, the contribution of fermionic loops (and hypothetical new degrees of freedom arising beyond SM) cannot be fully balanced without invoking precise cancellation of competing diagrams ("fine tuning").

c) Although the SM is perturbatively renormalizable and free from anomalies, anomalous propagators and their corresponding behavior can still occur whenever conditions fall outside perturbation theory.

It is reasonable, on these grounds, to posit that inverse propagators acting at the boundaries of QFT are well approximated by their conventional form times a generic interpolating function, as in [5-6]

$$S_s^{-1}(p) \approx (p^2 - m^2 + i0^+)\, f(\frac{p^2}{p_0^2}) \text{ (scalars)} \tag{5a}$$

$$S_b^{-1}(p) \approx g_{\mu\nu}^{-1}(p^2 - m^2 + i0^+)\, f(\frac{p^2}{p_0^2}) \text{ (vector bosons, Feynman gauge)} \tag{5b}$$

$$S_f^{-1}(p) \approx (\gamma^\mu p_\mu - m + i0^+)\, f(\frac{p}{p_0}) \text{ (fermions)} \tag{5c}$$

Here, p_0 represents an arbitrary reference IR or UV momentum scale. In particular, the IR regime of massive scalar field theory is characterized by [5-6]

$$p_0 = p_{IR} < p < \Lambda \tag{6}$$

subject to the constraint

$$\frac{p_{IR}}{p} = \frac{p}{\Lambda} \Rightarrow p_{IR} = \frac{p^2}{\Lambda} \tag{7}$$

Near and below the lower limit of range (6), the scaling ratio (7) behaves as

$$\lim_{p \to p_{IR}} \left(\frac{p}{p_{IR}}\right)^2 = 1 \quad (p \neq 0) \tag{8}$$

$$\lim_{p \to 0} \left(\frac{p}{p_{IR}}\right)^2 = 0 \quad (p < p_{IR}) \tag{9}$$

The object of this article is to further understand the structure and dynamic implications of the inverse propagator (5) using *fractional field theory* (FFT). We choose to work in space-times with arbitrarily small deviations from four-dimensionality ($D = 4 - \varepsilon$, $\varepsilon << 1$) since they minimally violate from the conventional framework of QFT and the SM [7-8]. These spaces are referred below to as "*minimal fractal manifolds*" (MFM). To avoid overloading the text with excessive information, we direct the reader to [8-10, 13-14] for an introductory review on the motivation and results of FFT. An intriguing outcome of our analysis is the emergence of classical gravity as long-range and ultra-weak excitation of the Higgs condensate.

The paper is organized as follows: section 2 introduces the concept of fractal propagator starting from the fractional Klein-Gordon equation; the connection between fractal propagators and FFT is presented in section 3. Building on these premises, section 4 derives the link between fractal propagators and classical gravity, where the latter emerges as long-range and ultra-weak excitation of the Higgs condensate.

2. The fractal propagator concept

Consider the stationary fractional Klein-Gordon equation in one space dimension [11]

$$(D_x^{\beta} + m^2)\varphi = \rho(x) \tag{10}$$

where D_x^{β} is the differential operator of non-integer index β, $\rho(x)$ is a time-independent point source of strength g

$$\rho(x) = g\,\delta(x) \tag{11}$$

The choice $\beta = 2$ recovers the standard Klein-Gordon equation. The Green function can be evaluated taking the Laplace transform of (10), which leads to

$$G(m^2, p, \beta) = (p^{\beta} + m^2)^{-1} \tag{12}$$

If $\beta = 2 + \varepsilon$ with ε << 1, we obtain

$$G(m^2, p, 2+\varepsilon) = (p^{2+\varepsilon} + m^2)^{-1} \tag{13}$$

The solution of (10) may be explicitly expanded in Mittag-Leffler (ML) functions [11]

$$\varphi(x)=\sum_{k=0}^{[2+\varepsilon]}\{a_k x^{2+\varepsilon-k}E_{2+\varepsilon,3+\varepsilon-k}(-m^2x^{2+\varepsilon})+\int_0^x E_{2+\varepsilon,3+\varepsilon-k}(-m^2(x-x')^{2+\varepsilon}(x-x')^{1+\varepsilon}\rho(x')\,dx'\} \quad (14)$$

(14) represents a generalization of the Yukawa short-range solution in exactly four-dimensional spacetime ($\varepsilon=0$)

$$\varphi_Y(x)=\frac{g}{4\pi}\frac{\exp(-mx)}{x} \quad (15)$$

where the presence of ML functions signals the onset of long-range spatial correlations in the behavior of the scalar field $\varphi(x)$ [11-12].

3. Fractal propagators in FFT

Let us now take a detour and return to the conventional formulation of particle propagators in QFT [1-2]. The propagator for free massive spinless fields expressed in dimensionless form reads

$$S_s^*(\frac{p}{p_0})=\frac{S_b(xp_0)}{p_0^2}=\int\frac{d^4p}{(2\pi)^4}\frac{p_0^2}{p_0^4}\exp(-i\frac{p}{p_0}xp_0)\frac{1}{p^2-m^2+i0^+} \quad (16)$$

or

$$S_s^*(\frac{p}{p_0})=\int\frac{d^4(\frac{p}{p_0})}{(2\pi)^4}\exp(-ipx)\frac{1}{(\frac{p}{p_0})^2-(\frac{m}{p_0})^2+i0^+} \quad (17)$$

We introduce the inverse propagator in momentum space as

$$S_s^{-1}(\frac{p}{p_0}) = (\frac{p}{p_0})^2 - (\frac{m}{p_0})^2 + i0^+ \tag{18}$$

Using the line of arguments presented section 2, the inverse propagator acting on the minimal fractal manifold (MFM) is given by

$$S_s^{-1}(\frac{p}{p_0}, \varepsilon) = (\frac{p}{p_0})^{2(1+\varepsilon)} - (\frac{m}{p_0})^2 + i0^+ \tag{19}$$

(19) may be alternatively presented as

$$S_s^{-1}(\frac{p}{p_0}, \varepsilon) = [(\frac{p}{p_0})^2 - (m\frac{p_0^{\varepsilon-1}}{p^{\varepsilon}})^2 + i0^+](\frac{p}{p_0})^{2\varepsilon} \tag{20}$$

We proceed with the assumption that the far IR scale is set by the cosmological constant, that is,

$$p_{IR} = \Lambda_{cc}^{1/4} \tag{21a}$$

Following [8, 13-14], dimensional regularization applied in the context of FFT requires the far IR scale ($\Lambda_{cc}^{1/4}$), the electroweak scale (M_{EW}) and the far UV scale fixed by the Planck mass $(\Lambda_{UV} = M_{Pl})$ to satisfy the constraint

$$\frac{\Lambda_{cc}^{1/4}}{M_{EW}} = \frac{M_{EW}}{\Lambda_{UV}} = O(\varepsilon) \tag{21b}$$

We are now set to explore the IR region of field theory ranging from the electroweak scale $p_0 = M_{EW} << \Lambda_{UV}$ to the far scale of cosmic distances $M_{EW} > p >> \Lambda_{cc}^{1/4}$. It makes sense to revisit the arguments previously made, apply the formalism to the Higgs sector of the Standard Model ($m = m_H$) and cast (20) as

$$S_H^{-1}(\frac{p}{M_{EW}}, \varepsilon) = [(\frac{p}{M_{EW}})^2 - (m_H \frac{M_{EW}^{\varepsilon-1}}{p^{\varepsilon}})^2 + i0^+](\frac{p}{M_{EW}})^{2\varepsilon} \tag{22a}$$

Relation (22a) is well approximated by

$$S_H^{-1}(P, \varepsilon) \approx [P^2 - M_H^2(\varepsilon) + i0^+] P^{2\varepsilon} \tag{22b}$$

where the "effective" momentum and "effective" Higgs mass are respectively defined as

$$P = \frac{p}{M_{EW}} \tag{23}$$

$$\frac{m_H}{p^{\varepsilon} M_{EW}} = M_H(\varepsilon) \tag{24}$$

A glance at (21a-b), (22a-b), and (5) reveals that the interpolating function

$$f(\frac{p}{M_{EW}}) = (\frac{p}{M_{EW}})^{2\varepsilon} \tag{25}$$

exhibits the following limiting behavior as $\varepsilon << 1$, $\varepsilon \neq 0$

$$p = O(M_{EW}) = O(m_H) \Rightarrow \lim_{\frac{M_{EW}}{\Lambda_{UV}} \to \varepsilon} (\frac{p}{M_{EW}})^{2\varepsilon} = 1 \tag{26}$$

$$p \leq O(\Lambda_{cc}^{1/4}) << M_{EW} \Rightarrow \lim_{\frac{\Lambda_{cc}^{1/4}}{M_{EW}} \to \varepsilon} (\frac{p}{M_{EW}})^{2\varepsilon} = 0 \ , \quad \text{if} \ \ p/M_{EW} << \varepsilon \tag{27}$$

It is instructive to note here that, consistent with the principles of effective field theory, in the far IR limit (27), the effective Higgs mass ($M_H(\varepsilon)$) of (22) diverges and naturally decouples from physics occurring at very large distances.

Combined use of (25) and (27) yields

$$\lim_{\frac{\Lambda_{cc}^{1/4}}{M_{EW}}\to\varepsilon} f'(0) = \lim_{\frac{\Lambda_{cc}^{1/4}}{M_{EW}}\to\varepsilon} 2\varepsilon(\frac{p}{M_{EW}})^{2\varepsilon-1} \approx \lim_{\frac{\Lambda_{cc}^{1/4}}{M_{EW}}\to\varepsilon} \frac{2\varepsilon}{(\frac{p}{M_{EW}})} \approx \frac{2\varepsilon}{O(\varepsilon)} \approx O(1) \tag{28}$$

provided that p/M_{EW} does not fall too far below ε. We shall use (22) and (26-28) in the next section.

4. Classical gravity as long-range excitation of the Higgs condensate

An interesting proposal of [5-6] is that classical gravity may be modeled as long-range and ultra-weak excitation of the Higgs condensate. The approach developed here points in the same direction: minimal fractal manifolds (MFM) favor the onset of long-range coupling and the emergence of interpolating functions of the type (4b) and (25) in the expression of propagators.

Following [5-6], the connection between Newton's constant (G_N) and Fermi's constant (G_F) is given by

$$G_N = \frac{p_{IR}^2}{4\pi f'(0) m_H^2} G_F \tag{29}$$

Substituting (21a-b) and (28) in (29) leads to

$$G_N \sim 10^{-33} G_F \tag{30}$$

in good agreement with currently known observational values of the two constants.

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