The Definition of Density in General Relativity

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Abstract

According to general relativity the geometry of space depends on the distribution of matter or energy fields. The relation between the locally defined geometry parameters and the volume elements depends on curvature. Thus integration of local properties like energy density, defined in the Euclidean tangent space, does not lead to correct integral data like total energy. To obtain integral conservation, a correction term must be added to account for the curvature of space. This correction term is the equivalent of potential energy in Newtonian gravitation. With this correction the formation of singularities by gravitational collapse does no longer occur and the so called dark energy finds its natural explanation as potential energy of matter itself.

[1]

1 Introduction

The theory of General Relativity (GRT) is presently regarded as a valid description of gravitation. For weak fields it has been impressively confirmed by observations like the aberration of light rays passing close to the sun, by gravitational red shift or by the perihelion advance of Mercury. But for strong fields it leads to very weird results, to an irresistible collapse of matter and the occurrence of singularities. Many attempts have been made to overcome this problem, but as been shown long years ago [1][2], singularities are unavoidable, as long as we assume that the energy density, or better to say, the trace of the energy tensor is always positive. But it is just this assumption, which is questionable.

In conventional treatments of gravitational collapse it is assumed that the components of the energy tensor can be described by densities of energy and momentum, quantities that are defined as the amount of the respective quantity per volume, as we are accustomed from Minkowskian space-time. But in GRT the volume is no longer a fixed quantity. The relations between the limiting geometrical parameters and the enclosed volume depend on the curvature of space.

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Thus there is no unique relation between the locally measured density of some property, as we would determine it in Euclidean space, and the corresponding integral data. For a property to be integrally conserved it is not sufficient that the flux through the boundaries vanishes, but in addition the local density must be defined in such a way that the curvature of space is taken into account. This will invalidate the positivity condition of energy.

In the next section we will first give simple two-dimensional example to demonstrate the problem. Then a way is proposed, how by adding a correction term to the energy tensor the problem can be solved and integral energy conservation is reestablished. In the following section the modification will be applied to the gravitational collapse to show that the problem of singularities does no longer occur. Finally we give some hints, how the modification may change our general view of the development of the universe.

2 A two-dimensional example

To demonstrate the problem that occurs in curved geometry with the definition of matter density and its integration to obtain the total mass, we consider a spherical surface with constant matter distribution. In Euclidean geometry we would describe such a surface by spherical coordinates centered at some point at distance a from the surface.

Any point on the surface is defined then by its angular coordinates ϑ and φ. The line element on the surface is defined by

$$ds^2 = a^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{1}$$

But in the framework of differential geometry, which is the mathematical basis of GRT, we cannot use coordinates defined at some point away from the surface. Instead we have to use geometrical parameters, locally defined within the surface. Introducing the radial parameter $r = a\sin\vartheta$ the line element reads

$$ds^2 = h(r)dr^2 + r^2 d\varphi^2 \tag{2}$$

with $h(r) = 1/(1 - r^2/a^2)$. The quantity r in this case denotes the value, at which the length of a circle around some reference point is $2\pi r$. It is reduced to the radial distance in the flat field limit. But the area enclosed in a circle at $r = r_0$ depends on the radius of curvature. It is given by

$$A = 2\pi \int_0^{r_0} \sqrt{h(r)}\, r\, dr. \tag{3}$$

Thus, if the surface is homogeneously covered with matter particles and their number is conserved, their surface density must be adjusted, when the curvature of the surface changes. In this case the density ϱ_0, which we would measure in the flat space limit, must be replaced by $\varrho_0/\sqrt{h(r)}$.

3 Conserved density in curved space

In GRT the situation is more intricate. The energy fields are represented by a tensor, so that all components will be affected by curvature corrections and on the other hand the energy fields themselves are regarded as the source of curvature. If we want to express the energy tensor by the densities of energy fields, in the Einstein field equation

$$R_{ij} - 1/2\, R\, g_{ij} = \kappa\, T_{ij}, \tag{4}$$

which relates the geometry, expressed by the metric tensor g_{ij} and its derivatives contained in the Ricci tensor R_{ij} to the energy tensor T_{ij}, the right hand side implicitly depends on the geometry.

The question is, how the correction term, which accounts for the curvature of space, can be defined in a way compatible with the covariant tensorial description of GRT. All information which is locally available is contained in the metric tensor and in the local density fields, as they would be measured in the limiting case of vanishing curvature. The flat space tensor $\hat{T}_{ij}$ has to be replaced by a quantity of the form

$$T_{ij} = \hat{T}_{ij} + \lambda\, g_{ij}, \tag{5}$$

where λ is a scalar property, depending on local parameters and on the curvature.

The parameters which can be used to define λ are the total energy density as defined in Minkowskian tangent space, $\hat{T} = \hat{T}_{ij}\hat{g}^{ij}$, and a quantity which expresses the relation between the volume elements of curved space and the corresponding Euclidean tangent space. The relation of the three-space tensor densities is given by $\sqrt{g^{(3)}/\hat{g}^{(3)}}$, where $g^{(3)}$ and $\hat{g}^{(3)}$ denote the spatial subdeterminants of the metric tensor in curved resp. Euclidean tangent space. Using these quantities, the function λ is defined as

$$\lambda = \hat{T}(\sqrt{\hat{g}^{(3)}/g^{(3)}} - 1) \tag{6}$$

The physical meaning of the correction term becomes immediately clear, if we consider the curvature of space caused by a large central mass. The spatial line element in this case is given by

$$ds^2 = h(r)dr^2 + r^2(d\vartheta^2 + cos^2\vartheta d\varphi^2) \tag{7}$$

with $h(r) = 1/(1-2GM/c^2/r)$ (G is the gravitational constant and M the mass of the central object). If there is a matter field at distance r with flat space density ϱ, in the weak field limit the value of λ is

$$\lambda \approx -GM\varrho/c^2/r, \tag{8}$$

just the potential energy as we know it from Newtonian physics. That this quantity appears as a negative pressure in the energy tensor is nothing special of GRT. This tension occurs in Newtonian gravity, too. Work has to be done to expand a gas against the mutual attraction of the constituting particles. Normally we do not recognize this negative pressure, as the positive kinetic pressure is higher by many orders of magnitude.

Correcting the energy tensor by the influence of curvature on the conserved density is nothing more than including potential energy. The new term is not some additional field, but a genuine part of every conserved density field in curved geometry.

4 Gravitational collapse

While in weak gravitational fields the influence of potential energy is negligibly small, correcting the energy density leads to dramatic changes in the description of gravitational collapse. Equilibria of collapsed stars or galaxies are strongly altered, when potential energy is taken into account.

Such equilibria are described by a radially symmetric static solution of the field equations, the Tolman-Oppenheimer-Volkov (TOV) equation [3], which gives the relation between the density and pressure distribution, when the equation of state is known:

$$dP/dr = -(\varrho + P)(m(r) + 4\pi r^3 P)/(r^2 - 2r \cdot m(r)). \tag{9}$$

(with$G = c = 1$) The function $m(r)$ is the integral

$$m(r) = 4\pi \int_0^r \varrho\, r^2 dr, \tag{10}$$

which is related to the quantity $h(r)$ appearing in the definition of the line element by $h(r) = 1/(1 - 2m(r)/r)$.

Solutions of this equation give the strange result that the pressure gradient must be infinite, when $2m(r)/r$ reaches the value 1. The function $h(r)$ is singular at this point, well known as the Schwarzschild radius [4]. No equilibrium can be reached, when this state of gravitational collapse is reached and the system should shrink to a singular state.

But when potential energy is taken into account the situation is altered. The density ϱ has to be replaced by $\varrho/\sqrt{h}$, so that the function $m(r)$ now has to be determined from the differential equation $dm/dr = 4\pi r^2 \varrho \sqrt{1 - 2m/r}$ with $m(0) = 0$. This leads to the result that the Schwarzschild radius cannot be reached for any reasonable equation of state.

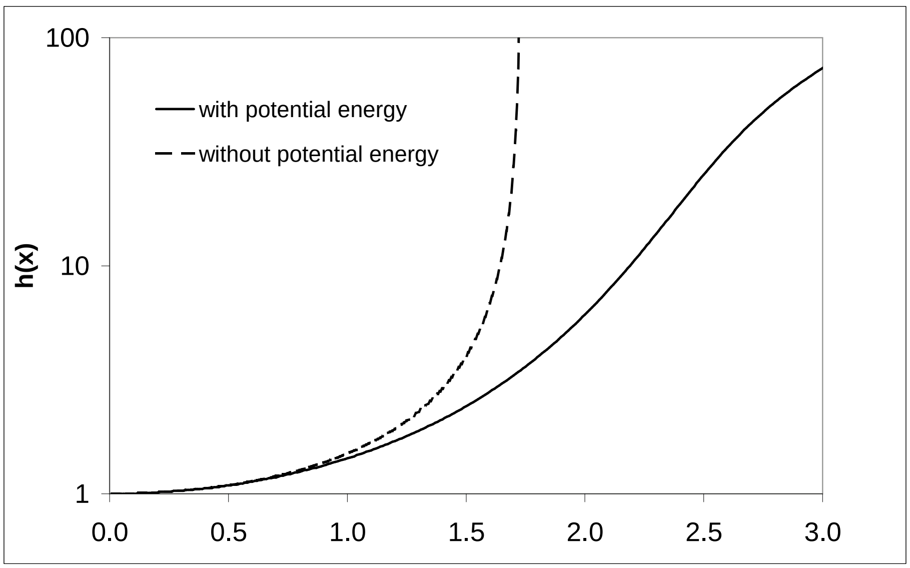
This can be best demonstrated considering as an example a system of constant density ϱ_0 and radial extension R. It is reasonable to assume that in every real isolated matter distribution the condition $d\varrho/dr < 0$ holds, so that with constant $\varrho = \varrho_0$ the maximum deviation of $h(r)$ from unity will be obtained. In this case introducing a normalized coordinate $x = r/R$ with $8\pi\varrho_0 R^2 = 1$ and the new variable $y = x/h$ instead of the differential equation for m(r) we have to solve the equation

$$dy/dx = 1 - x^{3/2} y^{1/2}. \tag{11}$$

Without potential energy we have

$$dy/dx = 1 - x^2. \tag{12}$$

The figure below shows the function h(x) for both cases. While without the curvature correction $h(x)$ exhibits a singularity at $x = \sqrt{3}$, it remains finite for all x, when potential energy is included. With other words: The quantity $2m(r)/r$ can never reach the value of one for any reasonable equation of state. All collapsing systems may find an equilibrium state without shrinking to singularities.



5 discussion

Inclusion of the curvature correction into the density definition of conserved quantities does not only avoid the occurrence of singularities in gravitational collapse. It generally reestablishes the existence of integral conservation laws. The interpretation of the correction term in the energy tensor as the density of potential energy shows the natural transition from GRT to the Newtonian limit with its fixed Euclidean geometry.

While formally the correction term $\lambda\, g_{ij}$ looks like the cosmological term introduced by Einstein, its properties are essentially different. The quantity λ is not a constant, but depends on the local energy or matter fields and on the local curvature. But what is discussed as dark energy with its negative pressure in modern cosmological models, may well be interpreted as the potential energy of matter itself. No additional dark energy is needed.

It should be mentioned in this context that Einstein's static world model would be stable, if the cosmological constant is interpreted in this way. With varying radius a of the universe the potential energy varies as ϱ/a, so that a virtual increase of a would produce a negative da/dt. The entire universe would be the only system at its Schwarzschild radius, the state at which the total matter energy is balanced by its own potential energy.

It has to be proved, if with the modification proposed here all the problems of GRT can be solved, but at least it shows a way, how by a simple redefinition of densities the problem of singularities can be eliminated.

References

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