The formula of next Mersenne prime

Oh Jung Uk

Abstract

For Mersenne prime of $2^{6n+1}-1$ type, if a Mersenne prime is $2^{6p+1}-1$, just next Mersenne prime is $2^{6x+1}-1$ then the following equation is satisfied.

$$x = p + \frac{3}{2} + \frac{1}{2}\sum_{k=p+1}^{x} \frac{\pi\beta\left(2^{6k+1}-1\right)+1}{\pi\beta\left(2^{6k+1}-1\right)-1} + \frac{1}{\pi}\sum_{k=p+1}^{x}\sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k+1}-1\right)}{\pi\beta\left(2^{6k+1}-1\right)-1}\right)}{m}$$

$$\text{where, } \beta\left(2^{6k+1}-1\right) = \tau\left(2^{6k+1}-1\right)-2, \ldots$$

Mersenne prime of $2^{6n-1}-1$ type is omitted in abstract.

1. Introduction

Mersenne prime is useful for finding big prime. Because N of 2^N-1 Mersenne prime is $6n \pm 1$, we study to express the formula of just next Mersenne prime of an arbitrary Mersenne prime by using $\rho(N)$ defined in paper "The formula of $\pi(N)$" [1] of myself. And, we study to express the formula of the sequence of Mersenne prime by using the formula of the above.

2. The formula of next Mersenne prime

Definition 1. We apply same definition of paper "The formula of $\pi(N)$" [1] of myself.

Definition 2. Let us define $\pi_m(2^N-1)$ as the number of Mersenne prime of 2^N-1 or less.

Theorem 1. First next Mersenne prime

If 2^N-1 is Mersenne prime then $N=6n\pm 1$.

If $2^{6p+1}-1$ is an arbitrary Mersenne prime of $2^{6n+1}-1$ type and if $2^{6x+1}-1$ is the first Mersenne prime of $2^{6n+1}-1$ type after $2^{6p+1}-1$ then the following formula is satisfied.

$$x=p+1+\sum_{k=p+1}^{x-1}\rho\left(2^{6k+1}-1\right)=p+1+\sum_{k=p+1}^{x}\rho\left(2^{6k+1}-1\right)$$

$$=p+1+\frac{1}{2}\sum_{k=p+1}^{x-1}\frac{\pi\beta\left(2^{6k+1}-1\right)+1}{\pi\beta\left(2^{6k+1}-1\right)-1}+\frac{1}{\pi}\sum_{k=p+1}^{x-1}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k+1}-1\right)}{\pi\beta\left(2^{6k+1}-1\right)-1}\right)}{m}$$

$$=p+\frac{3}{2}+\frac{1}{2}\sum_{k=p+1}^{x}\frac{\pi\beta\left(2^{6k+1}-1\right)+1}{\pi\beta\left(2^{6k+1}-1\right)-1}+\frac{1}{\pi}\sum_{k=p+1}^{x}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k+1}-1\right)}{\pi\beta\left(2^{6k+1}-1\right)-1}\right)}{m}$$

where

$$\beta\left(2^{6k+1}-1\right)=\tau\left(2^{6k+1}-1\right)-2=\sum_{p=1}^{2^{6k+1}-1}\left(\left[\frac{2^{6k+1}-1}{p}\right]-\left[\frac{2^{6k+1}-2}{p}\right]\right)-2$$

If $2^{6p-1}-1$ is an arbitrary mersenne prime of $2^{6n-1}-1$ type and if $2^{6x-1}-1$ is the first twin prime of $2^{6n-1}-1$ type after $2^{6p-1}-1$ then the following formula is satisfied.

$$x=p+1+\sum_{k=p+1}^{x-1}\rho\left(2^{6k-1}-1\right)=p+1+\sum_{k=p+1}^{x}\rho\left(2^{6k-1}-1\right)$$

$$=p+1+\frac{1}{2}\sum_{k=p+1}^{x-1}\frac{\pi\beta\left(2^{6k-1}-1\right)+1}{\pi\beta\left(2^{6k-1}-1\right)-1}+\frac{1}{\pi}\sum_{k=p+1}^{x-1}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1}-1\right)}{\pi\beta\left(2^{6k-1}-1\right)-1}\right)}{m}$$

$$=p+\frac{3}{2}+\frac{1}{2}\sum_{k=p+1}^{x}\frac{\pi\beta\left(2^{6k-1}-1\right)+1}{\pi\beta\left(2^{6k-1}-1\right)-1}+\frac{1}{\pi}\sum_{k=p+1}^{x}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1}-1\right)}{\pi\beta\left(2^{6k-1}-1\right)-1}\right)}{m}$$

where

$$\beta\left(2^{6k-1}-1\right)=\tau\left(2^{6k-1}-1\right)-2=\sum_{p=1}^{2^{6k-1}-1}\left(\left[\frac{2^{6k-1}-1}{p}\right]-\left[\frac{2^{6k-1}-2}{p}\right]\right)-2$$

Proof 1.

If a Mersenne number $2^N - 1$ should be a Mersenne prime then N should be also prime. So, N should be $N \equiv \pm 1 (mod\ 6)$ except 2,3. That is, N should be $N = 6n \pm 1$ type.

$2^{6n\pm 1} - 1 \equiv 2 - 1 \equiv 1 (mod\ 6)$ because $2^{6n\pm 1} \equiv 2 (mod\ 6)$.

Let us define $2^{6p+1} - 1$ as an arbitrary Mersenne prime of $2^{6n+1} - 1$ type and let us define $2^{6x+1} - 1$ as the first Mersenne prime of $2^{6n+1} - 1$ type after $2^{6p+1} - 1$. According to "definition 4 in paper The formula of $\pi(N)$" [1] of myself, $\rho(2^{6k+1} - 1) = 1$ because $2^{6k+1} - 1$ for k $(p + 1 < k < x)$ is composite. And $\rho(2^{6x+1} - 1) = 0$ because $2^{6x+1} - 1$ is prime. Therefore,

$$x = \sum_{k=1}^{x} 1 = \sum_{k=1}^{p} 1 + \sum_{k=p+1}^{x-1} 1 + \sum_{k=x}^{x} 1 = p + \sum_{k=p+1}^{x-1} \rho(2^{6k+1} - 1) + 1 = p + 1 + \sum_{k=p+1}^{x-1} \rho(2^{6k+1} - 1)$$

$$= \sum_{k=1}^{p} 1 + \sum_{k=p+1}^{x-1} 1 + \sum_{k=x}^{x} 1 + \sum_{k=x}^{x} 0 = p + 1 + \sum_{k=p+1}^{x-1} \rho(2^{6k+1} - 1) + \sum_{k=x}^{x} \rho(2^{6k+1} - 1)$$

$$= p + 1 + \sum_{k=p+1}^{x} \rho(2^{6k+1} - 1)$$

And,for $p < k < x$, $\rho(2^{6k+1} - 1) = \left[\frac{\beta(2^{6k+1}-1)}{\beta(2^{6k+1}-1)-w}\right]$, $1 < \frac{\beta(2^{6k+1}-1)}{\beta(2^{6k+1}-1)-w} < 2$,that is,

$\frac{\beta(2^{6k+1}-1)}{\beta(2^{6k+1}-1)-w} \in \overline{\mathbb{R}}$,so, according to "theorem 3 in paper The formula of $\pi(N)$" [1] of myself

$$x = p + 1 + \sum_{k=p+1}^{x-1} \rho(2^{6k+1} - 1)$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \left\{ \frac{\pi\beta(2^{6k+1} - 1)}{\pi\beta(2^{6k+1} - 1) - 1} - \frac{1}{2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(2^{6k+1} - 1)}{\pi\beta(2^{6k+1} - 1) - 1}\right)}{m} \right\}$$

$$= p + 1 + \frac{1}{2} \sum_{k=p+1}^{x-1} \frac{\pi\beta(2^{6k+1} - 1) + 1}{\pi\beta(2^{6k+1} - 1) - 1} + \frac{1}{\pi} \sum_{k=p+1}^{x-1} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta(2^{6k+1} - 1)}{\pi\beta(2^{6k+1} - 1) - 1}\right)}{m}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho(2^{6k+1} - 1) + \sum_{k=x}^{x} 0 = p + 1 + \sum_{k=p+1}^{x-1} \rho(2^{6k+1} - 1) + \sum_{k=x}^{x} \rho(2^{6k+1} - 1)$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho\left(2^{6k+1} - 1\right)$$

$$+ \sum_{k=x}^{x} \left\{ \frac{\pi\beta\left(2^{6k+1} - 1\right)}{\pi\beta(2^{6k+1} - 1) - 1} - \frac{1}{2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k+1} - 1\right)}{\pi\beta(2^{6k+1} - 1) - 1}\right)}{m} + \frac{1}{2} \right\}$$

$$= p + 1 + \sum_{k=p+1}^{x} \left\{ \frac{\pi\beta\left(2^{6k+1} - 1\right)}{\pi\beta(2^{6k+1} - 1) - 1} - \frac{1}{2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k+1} - 1\right)}{\pi\beta(2^{6k+1} - 1) - 1}\right)}{m} \right\} + \frac{1}{2}$$

$$= p + \frac{3}{2} + \frac{1}{2} \sum_{k=p+1}^{x} \frac{\pi\beta\left(2^{6k+1} - 1\right) + 1}{\pi\beta(2^{6k+1} - 1) - 1} + \frac{1}{\pi} \sum_{k=p+1}^{x} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k+1} - 1\right)}{\pi\beta(2^{6k+1} - 1) - 1}\right)}{m}$$

In the above formula, if we express $\beta_{gold}(6k - 1)$ by using only $\beta(N) = \tau(N) - 2$ which is the most simple function in “theorem 2 in paper The formula of $\pi(N)$” [1] of myself then

$$\beta\left(2^{6k+1} - 1\right) = \tau\left(2^{6k+1} - 1\right) - 2 = \sum_{p=1}^{2^{6k+1}-1} \left(\left[\frac{2^{6k+1} - 1}{p} \right] - \left[\frac{2^{6k+1} - 2}{p} \right] \right) - 2$$

Let us define $2^{6p-1} - 1$ as an arbitrary Mersenne prime of $2^{6n-1} - 1$ type and let us define $2^{6x-1} - 1$ as the first Mersenne prime of $2^{6n-1} - 1$ type after $2^{6p-1} - 1$. Because the same reason of the above(We omit the detail proof)

$$x = p + 1 + \sum_{k=p+1}^{x-1} \rho\left(2^{6k-1} - 1\right)$$

$$= p + 1 + \frac{1}{2} \sum_{k=p+1}^{x-1} \frac{\pi\beta\left(2^{6k-1} - 1\right) + 1}{\pi\beta(2^{6k-1} - 1) - 1} + \frac{1}{\pi} \sum_{k=p+1}^{x-1} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1} - 1\right)}{\pi\beta(2^{6k-1} - 1) - 1}\right)}{m}$$

$$= p + 1 + \sum_{k=p+1}^{x} \rho\left(2^{6k-1} - 1\right)$$

$$= p + \frac{3}{2} + \frac{1}{2} \sum_{k=p+1}^{x} \frac{\pi\beta\left(2^{6k-1} - 1\right) + 1}{\pi\beta(2^{6k-1} - 1) - 1} + \frac{1}{\pi} \sum_{k=p+1}^{x} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1} - 1\right)}{\pi\beta(2^{6k-1} - 1) - 1}\right)}{m}$$

And, according to “theorem 2 in paper The formula of $\pi(N)$” [1] of myself

$$\beta\left(2^{6k-1} - 1\right) = \tau\left(2^{6k-1} - 1\right) - 2 = \sum_{p=1}^{2^{6k-1}-1} \left(\left[\frac{2^{6k-1} - 1}{p} \right] - \left[\frac{2^{6k-1} - 2}{p} \right] \right) - 2 \blacksquare$$

Theorem 2.　Sequence of Mersenne prime

If we define the sequence of Mersenne prime of $2^{6n+1}-1$ type as $\{2^{6p_1+1}-1, 2^{6p_2+1}-1, \ldots\}$ then the following formula is always true for all positive integer n

$$p_{n+1} = p_n + 1 + \sum_{k=p_n+1}^{p_{n+1}} \rho\left(2^{6k+1}-1\right)$$

If we define the sequence of Mersenne prime of $2^{6n-1}-1$ type as $\{2^{6p_1-1}-1, 2^{6p_2-1}-1, \ldots\}$ then the following formula is always true for all positive integer n

$$p_{n+1} = p_n + 1 + \sum_{k=p_n+1}^{p_{n+1}} \rho\left(2^{6k-1}-1\right)$$

Proof 2.

Let us define the sequence of Mersenne prime of $2^{6n+1}-1$ type as $\{2^{6p_1+1}-1, 2^{6p_2+1}-1, \ldots\}$,that is, $\{2^7-1, 2^{13}-1, \ldots\}$. If we define below (2.1) according to theorem 1 then

$$p_{n+1} = p_n + 1 + \sum_{k=p_n+1}^{p_{n+1}} \rho\left(2^{6k+1}-1\right) \text{------------- (2.1)}$$

When $n=1$, the first Mersenne prime is $2^7-1=127$ and $p_1=1$, the second Mersenne prime is $2^{13}-1=8191$ and $p_2=2$.

$$p_2 = 2 = p_1 + 1 + \sum_{k=p_1+1}^{p_2} \rho\left(2^{6k+1}-1\right) = 1 + 1 + \sum_{k=1+1}^{2} \rho\left(2^{6k+1}-1\right) = 1 + 1 + \rho(2^{6\times2+1}-1)$$

$$= 1 + 1 + 0 = 2$$

so, (2.1) is true when $n=1$

When $m=n$, if we suppose that (2.1) is true then

$$p_{m+1} = p_m + 1 + \sum_{k=p_m+1}^{p_{m+1}} \rho\left(2^{6k+1}-1\right) \text{------------- (2.2)}$$

Because $2^{6k+1}-1$ for $(p_{m+1} < k < p_{m+2})$ is composite number ,so $\rho\left(2^{6k+1}-1\right)=1$and because $2^{6p_{m+2}+1}-1$ is prime number,so $\rho(2^{6p_{m+2}+1}-1)=0$. Therefore,

$$\sum_{k=p_{m+1}+1}^{p_{m+2}} \rho\left(2^{6k+1}-1\right) = \sum_{k=p_{m+1}+1}^{p_{m+2}-1} \rho\left(2^{6k+1}-1\right) + \sum_{k=p_{m+2}}^{p_{m+2}} \rho\left(2^{6k+1}-1\right)$$

$$= \sum_{k=p_{m+1}+1}^{p_{m+2}-1} 1 + \rho(2^{6p_{m+2}+1}-1) = \sum_{k=p_{m+1}+1}^{p_{m+2}-1} 1 + 0 = p_{m+2} - p_{m+1} - 1, \text{so,}$$

$$\sum_{k=p_{m+1}+1}^{p_{m+2}} \rho(2^{6k+1}-1) = p_{m+2} - p_{m+1} - 1 \text{-------------} (2.3)$$

If we add

$$\sum_{k=p_{m+1}+1}^{p_{m+2}} \rho(2^{6k+1}-1)$$

to both sides of (2.2) then

$$p_{m+1} + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho(2^{6k+1}-1) = p_m + 1 + \sum_{k=p_m+1}^{p_{m+1}} \rho(2^{6k+1}-1) + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho(2^{6k+1}-1)$$

$$\text{-------------} (2.4)$$

If we substitute (2.3) to left side of (2.4) then

$$p_{m+1} + p_{m+2} - p_{m+1} - 1 = p_m + 1 + \sum_{k=p_m+1}^{p_{m+1}} \rho(2^{6k+1}-1) + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho(2^{6k+1}-1) \text{-------} (2.5)$$

Because $p_m = p_{m+1} - 1 - \sum_{k=p_m+1}^{p_{m+1}} \rho(2^{6k+1}-1)$

from (2.2), if we substitute this formula to p_m of (2.5) then

$$p_{m+2} - 1 = p_{m+1} - 1 - \sum_{k=p_m+1}^{p_{m+1}} \rho(2^{6k+1}-1) + 1 + \sum_{k=p_m+1}^{p_{m+1}} \rho(2^{6k+1}-1)$$

$$+ \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho(2^{6k+1}-1)$$

Therefore,

$$p_{m+2} = p_{m+1} + 1 + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho(2^{6k+1}-1) \text{-------} (2.6)$$

And, if we sustitue $m+1$ to m of (2.2) then

$$p_{m+1+1} = p_{m+1} + 1 + \sum_{k=p_{m+1}+1}^{p_{m+1+1}} \rho(2^{6k+1}-1) \rightarrow$$

$$p_{m+2} = p_{m+1} + 1 + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho(2^{6k+1}-1) \text{-------------} (2.7)$$

Therefore, (2.1) is always true for all positive integer n, because (2.6) is same as (2.7),

Let us define the sequence of Mersenne prime of $2^{6n-1}-1$ type as $\{2^{6p_1-1}-1, 2^{6p_2-1}-1, \ldots\}$,that is, $\{2^5-1, 2^{11}-1, \ldots\}$. According to theorem 1

$$p_{n+1} = p_n + 1 + \sum_{k=p_n+1}^{p_{n+1}} \rho\left(2^{6k-1}-1\right) \text{-------------} \ (2.8)$$

If we progress the same proving process of $2^{6n+1}-1$ of the above (We omit the detail proof) then (2.8) is always true for all positive integer n, too. ■

Theorem 3. $\boldsymbol{\pi_m(2^N - 1)}$

For $0 < w < \frac{1}{2}, w \in \mathbb{R}, w = \frac{1}{e}, \frac{1}{\pi}, \ldots$

$$\pi_m(2^{6n+3} - 1) = 2n + 2 - \left\{\sum_{k=1}^{n} \rho\left(2^{6k-1} - 1\right) + \sum_{k=1}^{n} \rho\left(2^{6k+1} - 1\right)\right\}$$

$$= \pi_m(2^{6n+1} - 1) = \pi_m(2^{6n+2} - 1) = \pi_m(2^{6n+4} - 1)$$

$$= 2n + 2 - \frac{2}{3}\sum_{k=1}^{n}\left\{\frac{\beta\left(2^{6k-1} - 1\right)}{\beta(2^{6k-1} - 1) - w} + \frac{\beta\left(2^{6k+1} - 1\right)}{\beta(2^{6k+1} - 1) - w}\right\}$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left\{\frac{sin\left(\frac{2m\pi\beta\left(2^{6k-1} - 1\right)}{\beta(2^{6k-1} - 1) - w}\right) + sin\left(\frac{2m\pi\beta\left(2^{6k+1} - 1\right)}{\beta(2^{6k+1} - 1) - w}\right)}{m}\right\}$$

$$= 2n + 2 - \frac{2}{3}\sum_{k=1}^{n}\left\{\frac{\pi\beta\left(2^{6k-1} - 1\right)}{\pi\beta(2^{6k-1} - 1) - 1} + \frac{\pi\beta\left(2^{6k+1} - 1\right)}{\pi\beta(2^{6k+1} - 1) - 1}\right\}$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left\{\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1} - 1\right)}{\pi\beta(2^{6k-1} - 1) - 1}\right) + sin\left(\frac{2m\pi^2\beta\left(2^{6k+1} - 1\right)}{\pi\beta(2^{6k+1} - 1) - 1}\right)}{m}\right\}$$

$$= 2 + \frac{2n}{3} - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{1}{\pi\beta(2^{6k-1} - 1) - 1} + \frac{1}{\pi\beta(2^{6k+1} - 1) - 1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1} - 1\right)}{\pi\beta(2^{6k-1} - 1) - 1}\right) + sin\left(\frac{2m\pi^2\beta\left(2^{6k+1} - 1\right)}{\pi\beta(2^{6k+1} - 1) - 1}\right)}{m}\right)$$

$$= 2 + \frac{4n}{3} - \frac{1}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta\left(2^{6k-1} - 1\right) + 1}{\pi\beta(2^{6k-1} - 1) - 1} + \frac{\pi\beta\left(2^{6k+1} - 1\right) + 1}{\pi\beta(2^{6k+1} - 1) - 1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1} - 1\right)}{\pi\beta(2^{6k-1} - 1) - 1}\right) + sin\left(\frac{2m\pi^2\beta\left(2^{6k+1} - 1\right)}{\pi\beta(2^{6k+1} - 1) - 1}\right)}{m}\right)$$

Proof 3. If 2^N-1 is a prime number, then $1-\rho(2^N-1)=1$.
If 2^N-1 is 1 or a composite number then $1-\rho(2^N-1)=0$.

$$\text{So}, \pi_m(2^N-1)=\sum_{k=1}^{N}\{1-\rho(2^k-1)\}$$

If $N=6n+3$ then

$$\pi_m(2^N-1)=\pi_m(2^{6n+3}-1)$$

$$=\sum_{k=1}^{6n+3}\{1-\rho(2^k-1)\}=\sum_{k=1}^{6n+3}1-\sum_{k=1}^{6n+3}\rho(2^k-1)$$

$$=6n+3-\sum_{k=1}^{3}\rho(2^k-1)-\sum_{k=4}^{6n+3}\rho(2^k-1)$$

$$=6n+3-\{\rho(2^1-1)+\rho(2^2-1)+\rho(2^3-1)\}$$

$$-\sum_{k=1}^{n}\{\rho(2^{6k-2}-1)+\rho(2^{6k-1}-1)+\rho(2^{6k+0}-1)+\rho(2^{6k+1}-1)+\rho(2^{6k+2}-1)$$

$$+\rho(2^{6k+3}-1)\}$$

$\rho(2^1-1)=1$ and $2^2-1=3, 2^3-1=7$ is prime so $\rho(2^2-1)=0, \rho(2^3-1)=0$ and $6k-2, 6k+0, 6k+2, 6k+3$ is composite because the multiple of 2 or 3,so,
if N is a composite number then 2^N-1 is also composite number, so,
$\rho(2^{6k-2}-1)=1, \rho(2^{6k+0}-1)=1, \rho(2^{6k+2}-1)=1, \rho(2^{6k+3}-1)=1$. Therefore,

$$\pi_m(2^N-1)=\pi_m(2^{6n+3}-1)$$

$$=6n+3-\{1+0+0\}$$

$$-\left\{\sum_{k=1}^{n}1+\sum_{k=1}^{n}\rho(2^{6k-1}-1)+\sum_{k=1}^{n}1+\sum_{k=1}^{n}\rho(2^{6k+1}-1)+\sum_{k=1}^{n}1+\sum_{k=1}^{n}1\right\}$$

$$=6n+3-\{1\}-\left\{4\,n+\sum_{k=1}^{n}\rho(2^{6k-1}-1)+\sum_{k=1}^{n}\rho(2^{6k+1}-1)\right\}$$

$$=2n+2-\left\{\sum_{k=1}^{n}\rho(2^{6k-1}-1)+\sum_{k=1}^{n}\rho(2^{6k+1}-1)\right\} \text{------------- (3.1)}$$

And, $1-\rho(2^{6n+2}-1)=0, 1-\rho(2^{6n+3}-1)=0, 1-\rho(2^{6n+4}-1)=0$,so,
$\pi_m(2^{6n+3}-1)=\pi_m(2^{6n+1}-1)=\pi_m(2^{6n+2}-1)=\pi_m(2^{6n+4}-1)$.

Now,let us define $\mathbb{P}_{-}$ as a set of prime of $2^{6k-1}-1$type, $\mathbb{P}_{+}$ as a set of prime of $2^{6k+1}-1$ type, $\mathbb{C}_{-}$ as a set of composite of $2^{6k-1}-1$ type, $\mathbb{C}_{+}$ as a set of prime of $2^{6k+1}-1$ type,
and let us define

$$A=\frac{\beta\left(2^{6k-1}-1\right)}{\beta(2^{6k-1}-1)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta\left(2^{6k-1}-1\right)}{\beta(2^{6k-1}-1)-w}\right)}{m},$$

$$B=\frac{\beta\left(2^{6k+1}-1\right)}{\beta(2^{6k+1}-1)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta\left(2^{6k+1}-1\right)}{\beta(2^{6k+1}-1)-w}\right)}{m}$$

According to "theorem 3 in paper The formula of $\pi(N)$" [1] of myself,

if $2^{6k-1}-1\in\mathbb{C}_{-}$ then $\rho\left(2^{6k-1}-1\right)=A$, if $2^{6k-1}-1\in\mathbb{P}_{-}$ then $\rho\left(2^{6k-1}-1\right)=A+\frac{1}{2}$,

if $2^{6k+1}-1\in\mathbb{C}_{+}$ then $\rho\left(2^{6k+1}-1\right)=B$, if $2^{6k+1}-1\in\mathbb{P}_{+}$ then $\rho\left(2^{6k+1}-1\right)=B+\frac{1}{2}$ and

let us express $\sum_{\mathbb{Z}}^{n}u(k)$ with the sum of $u(k)$, only if $u(k)\in\mathbb{Z}$ in $1\le k\le n$ for a certain $u(k)$, $\mathbb{Z}$

because $\mathbb{C}_{-}\cap\mathbb{P}_{-}=\emptyset, \mathbb{C}_{+}\cap\mathbb{P}_{+}=\emptyset$, so,

$$\sum_{k=1}^{n}\rho\left(2^{6k-1}-1\right)=\sum_{\mathbb{C}_{-}}^{n}\rho\left(2^{6k-1}-1\right)+\sum_{\mathbb{P}_{-}}^{n}\rho\left(2^{6k-1}-1\right),$$

$$\sum_{k=1}^{n}\rho\left(2^{6k+1}-1\right)=\sum_{\mathbb{C}_{+}}^{n}\rho\left(2^{6k+1}-1\right)+\sum_{\mathbb{P}_{+}}^{n}\rho\left(2^{6k+1}-1\right)$$

So, if we apply the above contents to (3.1) then

$$\pi_m(2^N-1)=2n+2$$

$$-\left\{\sum_{\mathbb{C}_{-}}^{n}\rho\left(2^{6k-1}-1\right)+\sum_{\mathbb{P}_{-}}^{n}\rho\left(2^{6k-1}-1\right)+\sum_{\mathbb{C}_{+}}^{n}\rho\left(2^{6k+1}-1\right)+\sum_{\mathbb{P}_{+}}^{n}\rho\left(2^{6k+1}-1\right)\right\}$$

$$=2n+2-\left\{\sum_{\mathbb{C}_{-}}^{n}A+\sum_{\mathbb{P}_{-}}^{n}\left(A+\frac{1}{2}\right)+\sum_{\mathbb{C}_{+}}^{n}B+\sum_{\mathbb{P}_{+}}^{n}\left(B+\frac{1}{2}\right)\right\}$$

$$=2n+2-\left\{\sum_{\mathbb{C}_{-}}^{n}A+\sum_{\mathbb{P}_{-}}^{n}A+\sum_{\mathbb{P}_{-}}^{n}\frac{1}{2}+\sum_{\mathbb{C}_{+}}^{n}B+\sum_{\mathbb{P}_{+}}^{n}B+\sum_{\mathbb{P}_{+}}^{n}\frac{1}{2}\right\}$$

$$=2n+2-\left\{\sum_{\mathbb{C}_{-}}^{n}A+\sum_{\mathbb{P}_{-}}^{n}A+\sum_{\mathbb{C}_{+}}^{n}B+\sum_{\mathbb{P}_{+}}^{n}B+\sum_{\mathbb{P}_{-}}^{n}\frac{1}{2}+\sum_{\mathbb{P}_{+}}^{n}\frac{1}{2}\right\} \text{------------- (3.2)}$$

$$\sum_{\mathbb{C}_-}^{n} A + \sum_{\mathbb{P}_-}^{n} A = \sum_{k=1}^{n} A \, , \sum_{\mathbb{C}_+}^{n} B + \sum_{\mathbb{P}_+}^{n} B = \sum_{k=1}^{n} B$$

so, if we apply this to (3.2) then

$$\pi_m(2^N - 1) = 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B + \sum_{\mathbb{P}_-}^{n} \frac{1}{2} + \sum_{\mathbb{P}_+}^{n} \frac{1}{2} \right\} \text{------------- (3.3)}$$

If we define $\pi_{m-}(2^N - 1)$ as the number of $2^{6n-1} - 1$ type prime number of $2^N - 1$ or less, $\pi_{m+}(2^N - 1)$ as the number of $2^{6n+1} - 1$ type prime number of $2^N - 1$ or less then $\pi_m(2^N - 1) = 2 + \pi_{m-}(2^N - 1) + \pi_{m+}(2^N - 1)$ because all Mersenne prime is $2^{6n-1} - 1$ or $2^{6n+1} - 1$ type except $2^2 - 1, 2^3 - 1$ and

$$\sum_{\mathbb{P}_-}^{n} \frac{1}{2} = \frac{1}{2} \sum_{\mathbb{P}_-}^{n} 1 = \frac{\pi_{m-}(2^N - 1)}{2} , \sum_{\mathbb{P}_+}^{n} \frac{1}{2} = \frac{1}{2} \sum_{\mathbb{P}_+}^{n} 1 = \frac{\pi_{m+}(2^N - 1)}{2}$$

,so,if we apply this to (3.3) then

$$\pi_m(2^N - 1) = 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B + \frac{\pi_{m-}(2^N - 1)}{2} + \frac{\pi_{m+}(2^N - 1)}{2} \right\}$$

$$= 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B + \frac{\pi_m(2^N - 1) - 2}{2} \right\} \text{------------- (3.4)}$$

If we arrange (3.4) then

$$\pi_m(2^N - 1) + \frac{\pi_m(2^N - 1) - 2}{2} = 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \rightarrow$$

$$\frac{3\pi_m(2^N - 1) - 2}{2} = 2n + 2 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \rightarrow$$

$$\frac{3\pi_m(2^N - 1)}{2} = 2n + 3 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \rightarrow \pi_m(2^N - 1) = \frac{2}{3} \left\{ 2n + 3 - \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \right\} \rightarrow$$

$$\pi_m(2^N - 1) = 2 + \frac{4n}{3} - \frac{2}{3} \left\{ \sum_{k=1}^{n} A + \sum_{k=1}^{n} B \right\} \text{------------- (3.5)}$$

If we substitute A,B to (3.5) then

$$\pi_m(2^N-1) = 2+\frac{4n}{3}$$

$$-\frac{2}{3}\left\{\sum_{k=1}^{n}\left(\frac{\beta\left(2^{6k-1}-1\right)}{\beta(2^{6k-1}-1)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta\left(2^{6k-1}-1\right)}{\beta(2^{6k-1}-1)-w}\right)}{m}\right)\right.$$

$$\left.+\sum_{k=1}^{n}\left(\frac{\beta\left(2^{6k+1}-1\right)}{\beta(2^{6k+1}-1)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta\left(2^{6k+1}-1\right)}{\beta(2^{6k+1}-1)-w}\right)}{m}\right)\right\}$$

$$=2+\frac{4n}{3}+\frac{2n}{3}$$

$$-\frac{2}{3}\left\{\sum_{k=1}^{n}\left(\frac{\beta\left(2^{6k-1}-1\right)}{\beta(2^{6k-1}-1)-w}+\frac{\beta\left(2^{6k+1}-1\right)}{\beta(2^{6k+1}-1)-w}\right.\right.$$

$$\left.\left.+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta\left(2^{6k-1}-1\right)}{\beta(2^{6k-1}-1)-w}\right)}{m}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta\left(2^{6k+1}-1\right)}{\beta(2^{6k+1}-1)-w}\right)}{m}\right)\right\}$$

$$=2n+2-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{\beta\left(2^{6k-1}-1\right)}{\beta(2^{6k-1}-1)-w}+\frac{\beta\left(2^{6k+1}-1\right)}{\beta(2^{6k+1}-1)-w}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi\beta\left(2^{6k-1}-1\right)}{\beta(2^{6k-1}-1)-w}\right)+sin\left(\frac{2m\pi\beta\left(2^{6k+1}-1\right)}{\beta(2^{6k+1}-1)-w}\right)}{m}\right) \text{------------- (3.6)}$$

If we substitute $w = \frac{1}{\pi}$ to (3.6) especially, then

$$\pi_m(2^N-1) = 2n+2-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{\beta(2^{6k-1}-1)}{\beta(2^{6k-1}-1)-\frac{1}{\pi}}+\frac{\beta(2^{6k+1}-1)}{\beta(2^{6k+1}-1)-\frac{1}{\pi}}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi\beta(2^{6k-1}-1)}{\beta(2^{6k-1}-1)-\frac{1}{\pi}}\right)+sin\left(\frac{2m\pi\beta(2^{6k+1}-1)}{\beta(2^{6k+1}-1)-\frac{1}{\pi}}\right)}{m}\right)$$

$$= 2n+2-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(2^{6k-1}-1)}{\pi\beta(2^{6k-1}-1)-1}+\frac{\pi\beta(2^{6k+1}-1)}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(2^{6k-1}-1)}{\pi\beta(2^{6k-1}-1)-1}\right)+sin\left(\frac{2m\pi^2\beta(2^{6k+1}-1)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right) \text{------------- } (3.7)$$

And, if we modify (3.7) then

$$\pi_m(2^N-1) = 2n+2-\frac{4n}{3}+\frac{4n}{3}-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(2^{6k-1}-1)}{\pi\beta(2^{6k-1}-1)-1}+\frac{\pi\beta(2^{6k+1}-1)}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(2^{6k-1}-1)}{\pi\beta(2^{6k-1}-1)-1}\right)+sin\left(\frac{2m\pi^2\beta(2^{6k+1}-1)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right)$$

$$= 2+\frac{2n}{3}+\frac{2}{3}\sum_{k=1}^{n}2-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta(2^{6k-1}-1)}{\pi\beta(2^{6k-1}-1)-1}+\frac{\pi\beta(2^{6k+1}-1)}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta(2^{6k-1}-1)}{\pi\beta(2^{6k-1}-1)-1}\right)+sin\left(\frac{2m\pi^2\beta(2^{6k+1}-1)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right)$$

$$= 2 + \frac{2n}{3} + \frac{2}{3}\sum_{k=1}^{n}\left(1 - \frac{\pi\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1} + 1 - \frac{\pi\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^{2}\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1}\right) + sin\left(\frac{2m\pi^{2}\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right)$$

$$= 2 + \frac{2n}{3}$$

$$+\frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta\left(2^{6k-1}-1\right)-1-\pi\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1} + \frac{\pi\beta\left(2^{6k+1}-1\right)-1-\pi\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^{2}\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1}\right) + sin\left(\frac{2m\pi^{2}\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right)$$

$$= 2 + \frac{2n}{3} - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{1}{\pi\beta(2^{6k-1}-1)-1} + \frac{1}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^{2}\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1}\right) + sin\left(\frac{2m\pi^{2}\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right) \text{----- (3.8)}$$

And, if we modify (3.8) then

$$\pi_m(2^N-1)=2+\frac{2n}{3}+\frac{2n}{3}-\frac{2n}{3}-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{1}{\pi\beta(2^{6k-1}-1)-1}+\frac{1}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1}\right)+sin\left(\frac{2m\pi^2\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right)$$

$$=2+\frac{4n}{3}-\frac{1}{3}\sum_{k=1}^{n}2-\frac{1}{3}\sum_{k=1}^{n}\left(\frac{2}{\pi\beta(2^{6k-1}-1)-1}+\frac{2}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1}\right)+sin\left(\frac{2m\pi^2\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right)$$

$$=2+\frac{4n}{3}-\frac{1}{3}\sum_{k=1}^{n}\left(1+\frac{2}{\pi\beta(2^{6k-1}-1)-1}+1+\frac{2}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1}\right)+sin\left(\frac{2m\pi^2\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right)$$

$$=2+\frac{4n}{3}-\frac{1}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta\left(2^{6k-1}-1\right)-1+2}{\pi\beta(2^{6k-1}-1)-1}+\frac{\pi\beta\left(2^{6k+1}-1\right)-1+2}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1}\right)+sin\left(\frac{2m\pi^2\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right)$$

$$=2+\frac{4n}{3}-\frac{1}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta\left(2^{6k-1}-1\right)+1}{\pi\beta(2^{6k-1}-1)-1}+\frac{\pi\beta\left(2^{6k+1}-1\right)+1}{\pi\beta(2^{6k+1}-1)-1}\right)$$

$$-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\left(\frac{sin\left(\frac{2m\pi^2\beta\left(2^{6k-1}-1\right)}{\pi\beta(2^{6k-1}-1)-1}\right)+sin\left(\frac{2m\pi^2\beta\left(2^{6k+1}-1\right)}{\pi\beta(2^{6k+1}-1)-1}\right)}{m}\right)\blacksquare$$

References

[1] Oh Jung Uk, “The formula of $\pi(N)$”, http://vixra.org/pdf/1408.0041v1.pdf

[2] wikipedia, *Mersenne prime*,
http://ko.wikipedia.org/wiki/%EB%A9%94%EB%A5%B4%EC%84%BC_%EC%86%8C%EC%88%98

[3] wikipedia, *floor functions*,
http://ko.wikipedia.org/wiki/%EB%B0%94%EB%8B%A5_%ED%95%A8%EC%88%98

Oh Jung Uk, South Korea (I am not in any institutions of mathematics)

E-mail address: ojumath@gmail.com