Biquaternion EM forces with hidden momentum and an extended Lorentz Force Law, the Lorentz-Larmor Law

E.P.J. de Haas (Paul)

*Nijmegen, The Netherlands**

(Dated: August 3, 2014)

Abstract

In this paper we apply our version of biquaternion math-phys to electrodynamics, especially to moving electromagnetic dipole moments. After a terminological introduction and applying the developed math-phys to the Maxwell environment, we propose to fuse the Larmor angular velocity with the Lorentz Force Law, producing the biquaternion Lorentz-Larmor Law. This might be a more economic way to deal with cyclotron and tokamak related physics, where the Lorentz Force Law and the Larmor angular velocity actively coexist. Then we propose a biquaternion formulation for the energy-torque-hidden-momentum product related to a moving electromagnetic dipole moment. This expression is then used to get a relativistically invariant force equation on moving electromagnetic dipole moments. This equation is then used to get at the force on a hidden electromagnetic dipole moment and for a derivation of the Ahanorov-Casher force on a magnetic dipole moving in an external electric field. As a conclusion we briefly relate our findings to Mansuripurs recent critique of the Lorentz Force Law and the electrodynamic expressions for the energy-momentum tensor.

CONTENTS

1. THE BIQUATERION STRUCTURE WITH VECTORS AS 2X2 COMPLEX MATRICES

1.1. A quaternion basis for the metric

Quaternions can be represented by the basis $(\hat{\mathbf{1}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$. This basis has the properties $\hat{\mathrm{I}}\hat{\mathrm{I}} = \hat{\mathrm{J}}\hat{\mathrm{J}} = \hat{\mathrm{K}}\hat{\mathrm{K}} = -\hat{\mathbf{1}}$; $\hat{\mathbf{1}}\hat{\mathbf{1}} = \hat{\mathbf{1}}$; $\hat{\mathbf{1}}\hat{\mathrm{K}} = \hat{\mathrm{K}}\hat{\mathbf{1}} = \hat{\mathrm{K}}$ for $\hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}}$; $\hat{\mathrm{I}}\hat{\mathrm{J}} = -\hat{\mathrm{J}}\hat{\mathrm{I}} = \hat{\mathrm{K}}$; $\hat{\mathrm{J}}\hat{\mathrm{K}} = -\hat{\mathrm{K}}\hat{\mathrm{J}} = \hat{\mathrm{I}}$; $\hat{\mathrm{K}}\hat{\mathrm{I}} = -\hat{\mathrm{I}}\hat{\mathrm{K}} = \hat{\mathrm{J}}$. A quaternion number in its summation representation is given by $A = a_0\hat{\mathbf{1}} + a_1\hat{\mathrm{I}} + a_2\hat{\mathrm{J}} + a_3\hat{\mathrm{K}}$, in which the a_μ are real numbers . Bi-quaternions or complex quaternions are given by

$$
\begin{aligned}
C = A + \mathbf{i}B = c_0\hat{\mathbf{1}} + c_1\hat{\mathrm{I}} + c_2\hat{\mathrm{J}} + c_3\hat{\mathrm{K}} \\
(a_0 + \mathbf{i}b_0)\hat{\mathbf{1}} + (a_1 + \mathbf{i}b_1)\hat{\mathrm{I}} + (a_2 + \mathbf{i}b_2)\hat{\mathrm{J}} + (a_3 + \mathbf{i}b_3)\hat{\mathrm{K}} = \\
a_0\hat{\mathbf{1}} + a_1\hat{\mathrm{I}} + a_2\hat{\mathrm{J}} + a_3\hat{\mathrm{K}} + \mathbf{i}b_0\hat{\mathbf{1}} + \mathbf{i}b_1\hat{\mathrm{I}} + \mathbf{i}b_2\hat{\mathrm{J}} + \mathbf{i}b_3\hat{\mathrm{K}},
\end{aligned} \tag{1}
$$

in which the $c_\mu = a_\mu + \mathbf{i}b_\mu$ are complex numbers and the a_μ and b_μ are real numbers.

The complex conjugate of a bi-quaternion C is given by $\widetilde{C} = A - \mathbf{i}B$. The quaternion conjugate of a bi-quaternion is given by

$$
\begin{aligned}
C^\dagger = A^\dagger + \mathbf{i}B^\dagger = \\
(a_0 + \mathbf{i}b_0)\hat{\mathbf{1}} - (a_1 + \mathbf{i}b_1)\hat{\mathrm{I}} - (a_2 + \mathbf{i}b_2)\hat{\mathrm{J}} - (a_3 + \mathbf{i}b_3)\hat{\mathrm{K}}.
\end{aligned} \tag{2}
$$

A set of four numbers, real or complex, is given by

$$C_\mu = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}, \tag{3}$$

or by

$$C^\mu = [c_0, c_1, c_2, c_3] \tag{4}$$

a set of four numbers $\in \mathbb{R}$ or $\in \mathbb{C}$. The quaternion basis can be given as a set $\hat{\mathrm{K}}_\mu$ as

$$\hat{\mathrm{K}}_\mu = \begin{bmatrix} \hat{\mathtt{1}} \\ \hat{\mathrm{I}} \\ \hat{\mathrm{J}} \\ \hat{\mathrm{K}} \end{bmatrix}, \tag{5}$$

Then a quaternion C can also be written as

$$C = C^\mu \hat{\mathrm{K}}_\mu = [c_0, c_1, c_2, c_3] \begin{bmatrix} \hat{\mathtt{1}} \\ \hat{\mathrm{I}} \\ \hat{\mathrm{J}} \\ \hat{\mathrm{K}} \end{bmatrix} = c_0 \hat{\mathtt{1}} + c_1 \hat{\mathrm{I}} + c_2 \hat{\mathrm{J}} + c_3 \hat{\mathrm{K}} \tag{6}$$

We apply this to the space-time four vector of relativistic bi-quaternion 4-space R with the four numbers R_μ as

$$R_\mu = \begin{bmatrix} \mathbf{i}ct \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix}. \tag{7}$$

so with $r_1, r_2, r_3 \in \mathbb{R}$ and $r_0 \in \mathbb{C}$. Then we have $R = R^\mu \hat{\mathrm{K}}_\mu$ or

$$R = R^\mu \hat{\mathrm{K}}_\mu = r_0 \hat{\mathtt{1}} + r_1 \hat{\mathrm{I}} + r_2 \hat{\mathrm{J}} + r_3 \hat{\mathrm{K}} = r_0 \hat{\mathtt{1}} + \mathbf{r} \cdot \hat{\mathbf{K}}. \tag{8}$$

We use the three-vector analogue of $R^\mu \hat{\mathrm{K}}_\mu$ when we write $\mathbf{r} \cdot \hat{\mathbf{K}}$. In this notation we have

$$R^T = -r_0 \hat{\mathtt{1}} + r_1 \hat{\mathrm{I}} + r_2 \hat{\mathrm{J}} + r_3 \hat{\mathrm{K}} = -r_0 \hat{\mathtt{1}} + \mathbf{r} \cdot \hat{\mathbf{K}} \tag{9}$$

and

$$R^P = r_0 \hat{\mathtt{1}} - r_1 \hat{\mathrm{I}} - r_2 \hat{\mathrm{J}} - r_3 \hat{\mathrm{K}} = r_0 \hat{\mathtt{1}} - \mathbf{r} \cdot \hat{\mathbf{K}}. \tag{10}$$

1.2. Matrix representation of the quaternion basis

Quaternions can be represented by 2x2 matrices. Several representations are possible. Our choice is based upon the following operations

$$\begin{bmatrix} r_0 & r_1 \\ r_2 & r_3 \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{i} \end{bmatrix} = \begin{bmatrix} r_0 + \mathbf{i} r_1 \\ r_2 + \mathbf{i} r_3 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \tag{11}$$

and

$$R = \begin{bmatrix} p & q \\ -\widetilde{q} & \widetilde{p} \end{bmatrix} = \begin{bmatrix} r_0 + \mathbf{i} r_1 & r_2 + \mathbf{i} r_3 \\ -r_2 + \mathbf{i} r_3 & r_0 - \mathbf{i} r_1 \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{bmatrix}. \tag{12}$$

The numbers $R_{00}, R_{01}, R_{10}, R_{11} \in \mathbb{C}$. Then we can write R as

$$R = r_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{13}$$

This gives us the quaternions as the following matrices:

$$\hat{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hat{\mathrm{I}} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \hat{\mathrm{J}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \hat{\mathrm{K}} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{14}$$

We can compare these to the Pauli spin matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. If we exchange the σ_x and the σ_z, we have $\hat{\mathbf{K}} = \mathbf{i}\vec{\sigma}$ and $\hat{\mathrm{K}}^{\mu} = (\sigma_0, \mathbf{i}\vec{\sigma})$.

In our matrix representation we have doubled the metric because every coordinate appears twice in the matrix. This is directly related to the fact that SU(2) is a double representation of SO(3). We have to realize that a choice of mathematics regarding the metric is not without consequences. If we put spin in the metric instead of localizing it in particles, then particle spin number becomes a property concerning the kind of coupling of a particle to the spin aspect of the metric. So on the one hand we double the metric by adding a Möbius related double to it, but on the other hand we simplify the particle part of physics by taking spin away from it. We are however not making reality claims, we consider this as a research on the level of mathematical physics, where we are arguing on the basis of an if-then level without claiming the if-part to be true or real.

1.3. The Lorentz transformation as a twist of the metric

Usually the Lorentz transformation is given as a coordinate transformation against a Minkowski spacetime background. This spacetime background is an inert, static theatre in which the physics of special relativity takes place. Without gravity, this metric is presumed to be flat or inert. A Lorentz transformation acts upon the coordinates, not upon the metric. This is the context of Special Relativity. Quantum mechanics, from Schrödinger's to Dirac's version is defined in this environment of Special Relativity. The metric of Quantum Theory is Minkowski flat or inert. But what will happen to the Lorentz transformation and its non-effect on the metric if we take spin away from particles and put it in into the metric?

A normal Lorentz transformation between two reference frames connected by a relative velocity v in the $x-$ or Î-direction, with the usual $\gamma = 1/\sqrt{1 - v^2/c^2}$, $\beta = v/c$ and $r_0 = \mathbf{i}ct$, can be expressed as

$$\begin{bmatrix} r'_0 \\ r'_1 \end{bmatrix} = \begin{bmatrix} \gamma & -\mathbf{i}\beta\gamma \\ \mathbf{i}\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \mathbf{i}\beta\gamma r_1 \\ \gamma r_1 + \mathbf{i}\beta\gamma r_0 \end{bmatrix}. \tag{15}$$

We want to connect this to our matrix representation of R as in Eq.(12) which gives

$$R'_{00} = r'_0 + \mathbf{i}r'_1 = \gamma r_0 - \mathbf{i}\beta\gamma r_1 + \mathbf{i}\gamma r_1 - \beta\gamma r_0 \tag{16}$$

$$R'_{11} = r'_0 - \mathbf{i}r'_1 = \gamma r_0 - \mathbf{i}\beta\gamma r_1 - \mathbf{i}\gamma r_1 + \beta\gamma r_0. \tag{17}$$

If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, this can be rewritten as

$$R'_{00} = r'_0 + \mathbf{i}r'_1 = (\gamma - \beta\gamma)(r_0 + \mathbf{i}r_1) = R_{00}e^{-\psi} \tag{18}$$

$$R'_{11} = r'_0 - \mathbf{i}r'_1 = (\gamma + \beta\gamma)(r_0 - \mathbf{i}r_1) = R_{11}e^{\psi}. \tag{19}$$

As a result we have

$$R^L = \begin{bmatrix} R'_{00} & R'_{01} \\ R'_{10} & R'_{11} \end{bmatrix} = \begin{bmatrix} R_{00}e^{-\psi} & R_{01} \\ R_{10} & R_{11}e^{\psi} \end{bmatrix} = \Lambda^{-1}R\Lambda^{-1}. \tag{20}$$

In the expression $R^L = \Lambda^{-1}R\Lambda^{-1}$ we used the matrix U as

$$\Lambda = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix}. \tag{21}$$

But this means that we can write the result of a Lorentz transformation on R with a Lorentz velocity in the Î-direction between the two reference systems as

$$R^L = r_0 \begin{bmatrix} e^{-\psi} & 0 \\ 0 & e^{\psi} \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i}e^{-\psi} & 0 \\ 0 & -\mathbf{i}e^{\psi} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{22}$$

This can be written as

$$R^L = r_0 \Lambda^{-1} \hat{1} \Lambda^{-1} + r_1 \Lambda^{-1} \hat{\mathrm{I}} \Lambda^{-1} + r_2 \hat{\mathrm{J}} + r_3 \hat{\mathrm{K}} = r_0 \hat{1}^L + r_1 \hat{\mathrm{I}}^L + r_2 \hat{\mathrm{J}} + r_3 \hat{\mathrm{K}}. \tag{23}$$

But because we started with Eq.(15), we now have two equivalent options to express the result of a Lorentz transformation

$$R^L = r_0' \hat{1} + r_1' \hat{\mathrm{I}} + r_2 \hat{\mathrm{J}} + r_3 \hat{\mathrm{K}} = r_0 \hat{1}^L + r_1 \hat{\mathrm{I}}^L + r_2 \hat{\mathrm{J}} + r_3 \hat{\mathrm{K}}, \tag{24}$$

either as a coordinate transformation or as a basis transformation.

This shows that we do not need to have an inert metric any more. In our metric, a Lorentz transformation can leave the coordinates invariant and only change or rotate the basis on the level of spin-matrices. Mathematically we can formulate a Lorentz transformation as a matrix internal twist of the quaternion matrix basis, leaving the coordinates unchanged. A Lorentz transformation thus twists the metric. This implies that our metric is not the Minkowski metric of Special Relativity any more. But is it a metric that can accommodate Quantum Physics?

This result only works for Lorentz transformation between v_x-, v_1- or $\hat{\mathrm{I}}$-aligned reference systems. Reference systems which do not have their relative Lorentz velocity aligned in the $\hat{\mathrm{I}}$-direction will have to be rotated into such an alignment before the Lorentz transformation in the form $R^L = \Lambda^{-1} R \Lambda^{-1}$ is applied. With this requirement we restrict ourselves to a limited realm of applications. In other words, we are working with a simplified model.

The interesting thing about the $e^{\psi} = \gamma + \beta\gamma$ term is that it represents a relativistic Doppler-correction applied to the frequency ν of light-signals exchanged between two inertial reference systems.

$$\frac{\nu}{\nu_0} = e^{\psi}. \tag{25}$$

So if we twist the matrix basis internally as to compensate for the relativistic Doppler shift, then the coordinates can remain invariant under a Lorentz transformation. As to the $\hat{\mathrm{I}}$-alignment issue, two reference systems that exchange light signals in order to communicate might as well align their x-axes along the light signal communication direction.

1.4. Multiplication of vectors and the Lorentz transformation of the product

In general, multiplication of two vectors A and B follows matrix multiplication, with $A_{ij}, B_{ij} \in \mathbb{C}$.

$$AB = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} = C. \tag{26}$$

So we have

$$C = AB = \begin{bmatrix} A_{00}B_{00} + A_{01}B_{10} & A_{00}B_{01} + A_{01}B_{11} \\ A_{10}B_{00} + A_{11}B_{10} & A_{10}B_{01} + A_{11}B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}. \tag{27}$$

Of course, vectors A, B and C can be expressed with their a_μ, b_μ, c_μ coordinates and if we use them, after some elementary but elaborate calculations and rearrangements we arrive at the following result of the multiplication expressed in the a_μ, b_μ and c_μ as

$$\begin{aligned} c_0 &= a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \\ c_1 &= a_2b_3 - a_3b_2 + a_0b_1 + a_1b_0 \\ c_2 &= a_3b_1 - a_1b_3 + a_0b_2 + a_2b_0 \\ c_3 &= a_1b_2 - a_2b_1 + a_0b_3 + a_3b_0 \end{aligned} \tag{28}$$

In short, if we use the classic Euclidean dot and cross products of Euclidean threevectors, this gives for the coordinates

$$\begin{aligned} c_0 &= a_0b_0 - \mathbf{a} \cdot \mathbf{b} \\ \mathbf{c} &= \mathbf{a} \times \mathbf{b} + a_0\mathbf{b} + \mathbf{a}b_0 \end{aligned} \tag{29}$$

And in the quaternion notation we get

$$C = AB = (a_0b_0 - \mathbf{a} \cdot \mathbf{b})\hat{\imath} + (\mathbf{a} \times \mathbf{b} + a_0\mathbf{b} + \mathbf{a}b_0) \cdot \hat{\mathbf{K}} \tag{30}$$

From this it immediately follows that

$$A^TA = (-a_0a_0 - \mathbf{a} \cdot \mathbf{a})\hat{\imath} + (\mathbf{a} \times \mathbf{a} - a_0\mathbf{a} + \mathbf{a}a_0) \cdot \hat{\mathbf{K}} = (-a_0a_0 - \mathbf{a} \cdot \mathbf{a})\hat{\imath}, \tag{31}$$

and, with $a_0 = \mathbf{i}ca_t$, we get

$$A^TA = (-a_0a_0 - \mathbf{a} \cdot \mathbf{a})\hat{\imath} = (c^2a_t^2 - \mathbf{a}^2)\hat{\imath} = c^2a_\tau^2\hat{\imath}. \tag{32}$$

We will refer to the level of (27) and (28) as the analogue of the machine code level of computer languages and to the level of (29) and (30) as analogue to higher level computer languages. At the level of "machine language" we use the matrix aspect of the basis of the metric. At that level things are commutative again and from there can we rearrange and construct again towards higher levels codes-language. Another important fact: the multiplication ABC is associative, meaning that $(AB)C = A(BC)$. This is a crucial property in regard to the Lorentz transformation of complex products of biquaternions. But it is not commutative, so the differentiation rule for product functions cannot be applied directly, because $\partial(AB) \neq (\partial A)B + A(\partial B)$. In the case of $\partial(AB)$, we have to go to 'machine-code' level to be able to apply the differentiation rule for product functions, after which one tries to go back to higher language levels.

One very important matter concerns the Lorentz transformation of the product $C = AB$, so $C^L = (AB)^L$. Calculations show that we have the following rule for the Lorentz transformation of the product C of two fourvectors A and B who individually transform as $A^L = \Lambda^{-1}A\Lambda^{-1}$ and $B^L = \Lambda^{-1}B\Lambda^{-1}$:

$$C^L = (AB)^L = A^{-L}B^L = \Lambda A\Lambda\Lambda^{-1}B\Lambda^{-1} = \Lambda AB\Lambda^{-1} = \Lambda C\Lambda^{-1}. \tag{33}$$

This implies that $(AB)^L \neq A^L B^L$ if A and B are fourvectors.

As a result, it is easy to prove that the quadratic $A^T A = c^2 a_\tau^2 \hat{1}$ is Lorentz invariant. We have

$$\begin{aligned}(A^T A)^L = (A^T)^{-L}A^L = \Lambda A^T\Lambda\Lambda^{-1}A\Lambda^{-1} = \Lambda A^T A\Lambda^{-1} = \\ \Lambda(c^2 a_\tau^2)\hat{1}\Lambda^{-1} = \Lambda\Lambda^{-1}(c^2 a_\tau^2)\hat{1} = c^2 a_\tau^2 \hat{1} = A^T A.\end{aligned} \tag{34}$$

Here we touch a delicate matter, an heuristic rule so to speak, which for four-vectors is

$$(A^T)^L \equiv \Lambda A^T \Lambda = (A^T)^{-L}, \tag{35}$$

or in words; the Lorentz transformation of the time-inverse is the inverse Lorentz transformation compared to the Lorentz transformation of the non time inverse original.

The main quadratic form of the metric is

$$dR^T dR = (c^2 dt^2 - d\mathbf{r}^2)\hat{1} = c^2 d\tau^2 \hat{1} = -ds^2 \hat{1} \tag{36}$$

with $ds = \mathbf{i}cd\tau$, while the quadratic giving the distance of a point R to the origin of its reference system is given by

$$R^T R = (c^2t^2 - \mathbf{r}^2)\hat{1} = c^2\tau^2\hat{1} = -s^2\hat{1} \tag{37}$$

with $s = \mathbf{i}c\tau$. Both quadratics are Lorentz invariant scalars, as has been shown for every quadratic of fourvectors. Important to note is the fact that a pure rotation of a reference system in order to get $\hat{\mathrm{I}}$-alignment leaves the quadratics invariant.

1.5. Adding the dynamic vectors

If we want to apply the previous to relativistic electrodynamics and to quantum physics, we need to develop the mathematical language further. We start by adding the most relevant dynamic four vectors. The basic definitions we use are quite common in the formulations of relativistic dynamics.[1], We start with a particle with a given three vector velocity as $\mathbf{v}$, a rest mass as m_0 and an inertial mass $m_i = \gamma m_0$, with the usual $\gamma = (\sqrt{1 - v^2/c^2})^{-1}$. We use the Latin suffixes as abbreviations for words, not for numbers. So m_i stands for inertial mass and U_p for potential energy. The Greek suffixes are used as indicating a summation over the numbers 0, 1, 2 and 3. So P^μ stands for a momentum four-vector coordinate row with components $(p_0 = \mathbf{i}\frac{1}{c}U_i, p_1, p_2, p_3)$. The momentum three-vector is written as $\mathbf{p}$ and has components (p_1, p_2, p_3).

We define the coordinate velocity four vector as

$$V = V^\mu \hat{\mathrm{K}}_\mu = \frac{d}{dt} R^\mu \hat{\mathrm{K}}_\mu = \mathbf{i}c\hat{1} + \mathbf{v} \cdot \hat{\mathbf{K}} = v_0\hat{1} + \mathbf{v} \cdot \hat{\mathbf{K}}. \tag{38}$$

The proper velocity four vector on the other hand will be defined using the proper time $\tau = t_0$, with $t = \gamma t_0 = \gamma\tau$, as

$$U = U^\mu \hat{\mathrm{K}}_\mu = \frac{d}{d\tau} R^\mu \hat{\mathrm{K}}_\mu = \frac{d}{\frac{1}{\gamma}dt} R^\mu \hat{\mathrm{K}}_\mu = \gamma V^\mu \hat{\mathrm{K}}_\mu = u_0\hat{1} + \mathbf{u} \cdot \hat{\mathbf{K}}. \tag{39}$$

The momentum four vector will be

$$P = P^\mu \hat{\mathrm{K}}_\mu = m_i V^\mu \hat{\mathrm{K}}_\mu = m_i V = m_0 U^\mu \hat{\mathrm{K}}_\mu = m_0 U. \tag{40}$$

The four vector partial derivative $\partial = \partial^\mu \hat{\mathrm{K}}_\mu$ will be defined using the coordinate four set

$$\partial^\mu = \left[-\mathbf{i}\frac{1}{c}\partial_t, \nabla_1, \nabla_2, \nabla_3\right] = [\partial_0, \partial_1, \partial_2, \partial_3] \equiv \frac{\partial}{\partial R_\mu}. \tag{41}$$

The electrodynamic potential four vector $A = A^{\mu}\hat{\mathrm{K}}_{\mu}$ will be defined by the coordinate four set

$$A^{\mu} = \left[\mathbf{i}\frac{1}{c}\phi, A_1, A_2, A_3\right] = [A_0, A_1, A_2, A_3] \tag{42}$$

The electric four current density vector $J = J^{\mu}\hat{\mathrm{K}}_{\mu}$ will be defined by the coordinate four set

$$J^{\mu} = [\mathbf{i}c\rho_e, J_1, J_2, J_3] = [J_0, J_1, J_2, J_3]\,, \tag{43}$$

with ρ_e as the electric charge density. The electric four current with a charge q will also be written as J_{μ} and it will be indicated in the context which one is used.

Although we defined these fourvectors using the coordinate column notation, we will mostly use the matrix or summation notation, as for example with $P = P^{\mu}\hat{\mathrm{K}}_{\mu}$, written as

$$P = p_0\hat{\mathbf{1}} + p_1\hat{\mathrm{I}} + p_2\hat{\mathrm{J}} + p_3\hat{\mathrm{K}} = p_0\hat{\mathbf{1}} + \mathbf{p}\cdot\hat{\mathbf{K}} = \begin{bmatrix} p_0 + \mathbf{i}p_1 & p_2 + \mathbf{i}p_3 \\ -p_2 + \mathbf{i}p_3 & p_0 - \mathbf{i}p_1 \end{bmatrix} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}. \tag{44}$$

2. THE MAXWELL ENVIRONMENT IN OUR BIQUATERNION LANGUAGE

2.1. The EM field in our language

I we apply the multiplication rules to the electromagnetic field with four derivative ∂ and four potential A we get $B = \partial^T A$ as

$$B = \partial^T A = (-\partial_0 A_0 - \boldsymbol{\nabla}\cdot\mathbf{A})\hat{\mathbf{1}} + (\boldsymbol{\nabla}\times\mathbf{A} - \partial_0\mathbf{A} + \boldsymbol{\nabla}A_0)\cdot\hat{\mathbf{K}} \tag{45}$$

and if we insert $\partial_0 = -\mathbf{i}\frac{1}{c}\partial_t$ and $a_0 = \mathbf{i}\frac{1}{c}\phi$ we get

$$B = \partial^T A = (-\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\mathbf{A})\hat{\mathbf{1}} + (\boldsymbol{\nabla}\times\mathbf{A} - \mathbf{i}\frac{1}{c}(-\partial_t\mathbf{A} - \boldsymbol{\nabla}\phi))\cdot\hat{\mathbf{K}}. \tag{46}$$

If we apply the Lorenz gauge $\mathbb{B}_0 = -\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\mathbf{A} = 0$ and the usual EM definitions of the fields in terms of the potentials we get

$$B = \partial^T A = (\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E})\cdot\hat{\mathbf{K}} = \vec{\mathbb{B}}\cdot\hat{\mathbf{K}}. \tag{47}$$

It is important that we have a clear notation concerning B. We have

$$B = \mathbb{B}_1\hat{\mathrm{I}} + \mathbb{B}_2\hat{\mathrm{J}} + \mathbb{B}_3\hat{\mathrm{K}} = \vec{\mathbb{B}}\cdot\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{i}\mathbb{B}_1 & \mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 \\ -\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 & -\mathbf{i}\mathbb{B}_1 \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}. \tag{48}$$

For the Lorentz transformation of B we can apply the result of the previous section to get $B^L = (\partial^T A)^L = (\partial^T)^{-L} A^L = \Lambda(\partial^T)\Lambda\Lambda^{-1}A\Lambda^{-1} = \Lambda(\partial^T A)\Lambda^{-1} = \Lambda B \Lambda^{-1}$, so

$$B^L = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \begin{bmatrix} e^{-\frac{\psi}{2}} & 0 \\ 0 & e^{\frac{\psi}{2}} \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01}e^{\psi} \\ B_{10}e^{-\psi} & B_{11} \end{bmatrix} \tag{49}$$

which, when written out with **E** and **B** leads to the usual result for the Lorentz transformation of the EM field with the Lorentz velocity in the x-direction. But it can also be written as a transformation of the basis, while leaving the coordinates invariant:

$$\begin{aligned} B^L = \Lambda B \Lambda^{-1} = \mathbb{B}_1 \Lambda \hat{\mathrm{I}} \Lambda^{-1} + \mathbb{B}_2 \Lambda \hat{\mathrm{J}} \Lambda^{-1} + \mathbb{B}_3 \Lambda \hat{\mathrm{K}} \Lambda^{-1} = \\ \mathbb{B}_1 \hat{\mathrm{I}} + \mathbb{B}_2 \hat{\mathrm{J}}^L + \mathbb{B}_3 \hat{\mathrm{K}}^L = \mathbb{B}_1 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + \mathbb{B}_2 \begin{bmatrix} 0 & e^{\psi} \\ -e^{-\psi} & 0 \end{bmatrix} + \mathbb{B}_3 \begin{bmatrix} 0 & \mathbf{i}e^{\psi} \\ \mathbf{i}e^{-\psi} & 0 \end{bmatrix}. \end{aligned} \tag{50}$$

The Lorentz transformation of the EM field can be performed by internally twisting the $(\hat{\mathrm{J}}, \hat{\mathrm{K}})$-surface perpendicular to the Lorentz velocity and in the process leaving the EM-coordinates invariant.

That the above equals the usual Lorentz transformation of the EM field can be shown by going back to the 1908 paper by Minkowski, where he wrote the transformation in a form equivalent to[2]

$$\begin{bmatrix} \mathbb{B}'_1 \\ \mathbb{B}'_2 \\ \mathbb{B}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & \mathbf{i}\beta\gamma \\ 0 & -\mathbf{i}\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} \mathbb{B}_1 \\ \mathbb{B}_2 \\ \mathbb{B}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{B}_1 \\ \gamma\mathbb{B}_2 + \mathbf{i}\beta\gamma\mathbb{B}_3 \\ \gamma\mathbb{B}_3 - \mathbf{i}\beta\gamma\mathbb{B}_2 \end{bmatrix} \tag{51}$$

So we have

$$B'_{01} = \mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = \gamma\mathbb{B}_2 + \mathbf{i}\beta\gamma\mathbb{B}_3 + \mathbf{i}\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2 \tag{52}$$

and

$$B'_{10} = -\mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = -\gamma\mathbb{B}_2 - \mathbf{i}\beta\gamma\mathbb{B}_3 + \mathbf{i}\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2. \tag{53}$$

If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, this can be rewritten as

$$B'_{01} = \mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = (\gamma + \beta\gamma)(\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3) = B_{01}e^{\psi} \tag{54}$$

and

$$B'_{10} = -\mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = (\gamma - \beta\gamma)(-\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3) = B_{10}e^{-\psi}. \tag{55}$$

2.2. The biquaternion Maxwell equation

The Maxwell equations in our language can be given as, using $J = \rho V$

$$\partial B = \mu_0 J. \tag{56}$$

We will write out $\partial B = \mu_0 J$. First we remind that $B = \mathbb{B}^\mu \hat{\mathrm{K}}_\mu = \vec{\mathbb{B}} \cdot \hat{\mathbf{K}}$ with $\mathbb{B}_0 = 0$ and $\vec{\mathbb{B}} = \mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}$.

$$\partial B = (-\boldsymbol{\nabla} \cdot \vec{\mathbb{B}})\hat{1} + (\boldsymbol{\nabla} \times \vec{\mathbb{B}} - \mathbf{i}\frac{1}{c}\partial_t \vec{\mathbb{B}}) \cdot \hat{\mathbf{K}} = \mathbf{i}c\mu_0\rho\hat{1} + \mu_0 \mathbf{J} \cdot \hat{\mathbf{K}} \tag{57}$$

so we have two sub-equations, the scalar part

$$-\boldsymbol{\nabla} \cdot \vec{\mathbb{B}} = \mathbf{i}c\mu_0\rho \tag{58}$$

and the vector part

$$\boldsymbol{\nabla} \times \vec{\mathbb{B}} - \mathbf{i}\frac{1}{c}\partial_t \vec{\mathbb{B}} = \mu_0 \mathbf{J} \tag{59}$$

Because $\vec{\mathbb{B}} = \mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}$, these two sub-equations contain the set of four inhomogeneous Maxwell Equations. The scalar part contains

$$-\boldsymbol{\nabla} \cdot \mathbf{B} + \mathbf{i}\frac{1}{c}\boldsymbol{\nabla} \cdot \mathbf{E} = \mathbf{i}c\mu_0\rho \tag{60}$$

which can be split into a real part and a imaginary part

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{61}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{62}$$

and the vector part, containing

$$\boldsymbol{\nabla} \times (\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}) - \mathbf{i}\frac{1}{c}\partial_t(\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}) = \mu_0 \mathbf{J} \tag{63}$$

can be split into the real an imaginary parts

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} = \mu_0 \mathbf{J} \tag{64}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \partial_t \mathbf{B} = 0 \tag{65}$$

This demonstrates that our matrix multiplication contains the Maxwell-Lorentz structure. If we do not split the basis equation $\partial B = \mu_0 J$ but continue to write it in it's full biquaternion basis notation, we get for the left hand side

$$\partial B = (-\boldsymbol{\nabla} \cdot \vec{\mathbb{B}})\hat{1} + (\boldsymbol{\nabla} \times \vec{\mathbb{B}} - \mathbf{i}\frac{1}{c}\partial_t \vec{\mathbb{B}}) \cdot \hat{\mathbf{K}} = \tag{66}$$

$$(-\boldsymbol{\nabla} \cdot (\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}))\hat{1} + (\boldsymbol{\nabla} \times (\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}) - \mathbf{i}\frac{1}{c}\partial_t(\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E})) \cdot \hat{\mathbf{K}} = \tag{67}$$

$$-\boldsymbol{\nabla} \cdot \mathbf{B}\hat{1} + \mathbf{i}\frac{1}{c}\boldsymbol{\nabla} \cdot \mathbf{E}\hat{1} + \boldsymbol{\nabla} \times \mathbf{B} \cdot \hat{\mathbf{K}} - \mathbf{i}\frac{1}{c}\boldsymbol{\nabla} \times \mathbf{E} \cdot \hat{\mathbf{K}} - \mathbf{i}\frac{1}{c}\partial_t \mathbf{B} \cdot \hat{\mathbf{K}} - \frac{1}{c^2}\partial_t \mathbf{E} \cdot \hat{\mathbf{K}}. \tag{68}$$

This leads us to the full biquaternion version of $\partial B = \mu_0 J$ as

$$-\boldsymbol{\nabla} \cdot \mathbf{B}\hat{1} + \mathbf{i}\frac{1}{c}\boldsymbol{\nabla} \cdot \mathbf{E}\hat{1} + \boldsymbol{\nabla} \times \mathbf{B} \cdot \hat{\mathbf{K}} - \mathbf{i}\frac{1}{c}\boldsymbol{\nabla} \times \mathbf{E} \cdot \hat{\mathbf{K}} - \mathbf{i}\frac{1}{c}\partial_t \mathbf{B} \cdot \hat{\mathbf{K}} - \frac{1}{c^2}\partial_t \mathbf{E} \cdot \hat{\mathbf{K}} = \tag{69}$$

$$\mathbf{i}c\mu_0\rho\hat{1} + \mu_0 \mathbf{J} \cdot \hat{\mathbf{K}} \tag{70}$$

By splitting this first according to a time-like part $\hat{1}$ and a space-like part $\hat{\mathbf{K}}$ and then in a real part $\Re$ and an imaginary part $\Im$, we get the four Maxwell equations.

2.3. Lorentz form invariance of the Maxwell equation

As for the Lorentz invariance of the Maxwell Equations, Lorentz form invariance, this can be demonstrated quite easily through

$$\partial^L B^L = \Lambda^{-1}\partial\Lambda^{-1}\Lambda B\Lambda^{-1} = \Lambda^{-1}\partial B\Lambda^{-1} = \mu_0\Lambda^{-1}J\Lambda^{-1} = \mu_0 J^L \tag{71}$$

So if we have $\partial B = \mu_0 J$ in one inertial frame of reference S, we have $\partial' B' = \mu_0 J'$ in any other frame of reference S', the definition of Lorentz transformation form invariance of an equation.

We can also read it as a proof that the Maxwell current Lorentz transforms as a four-vector, not as a six-vector. We can even interpret the product ∂B as consisting of two four-vectors, by separating them differently as is done when obtaining the four Maxwell equations.

$$\partial B = [\mathbf{i}\frac{1}{c}\boldsymbol{\nabla} \cdot \mathbf{E}\hat{1} + (\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E}) \cdot \hat{\mathbf{K}}] - \mathbf{i}\frac{1}{c}[-\mathbf{i}c\boldsymbol{\nabla} \cdot \mathbf{B}\hat{1} + (\boldsymbol{\nabla} \times \mathbf{E} + \partial_t \mathbf{B}) \cdot \hat{\mathbf{K}}] \tag{72}$$

Then the first four-vector set equal to $\mu_0 J$ makes up the two inhomogeneous equations and the second four-vector is its homogeneous dual. It is easy to show that they transform as four-vectors. They aren't genuine four-vectors though but pseudo four-vectors because the two four-vectors are entangled. This entanglement is not without real implications as it has the em wave equation as a consequence. The above split into two four-vectors also leads to speculations concerning the possible existence of a magnetic four current as the dual for the electric four current.

2.4. Wave equations of the potentials and the fields

We can also look at ∂B from a different perspective, using $B = \partial^T A$, concentrate on the left side of the equation and write Maxwell's Equations as a wave equation of the potentials

$$\partial B = \partial\partial^T A = -(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)A \tag{73}$$

The term inside the brackets is a Lorentz invariant scalar operator, equal to $-\frac{1}{c^2}\partial_\tau^2$. The wave equation version of ∂B can be interpreted as $\partial B = \frac{1}{c^2}\partial_\tau^2 A = \mu_0 J$ and thus also as $\partial_\tau^2 A = \epsilon_0 J$. The Lorentz transformation of ∂B then only affects the fourvector A, giving $A^L = \Lambda^{-1} A \Lambda^{-1}$ and the scalar part can be inside or outside the Λ^{-1} operators. This leads to

$$\begin{aligned}(\partial B)^L = (-\boldsymbol{\nabla}^2 + \frac{1}{c^2}\partial_t^2)A^L = (-\boldsymbol{\nabla}^2 + \frac{1}{c^2}\partial_t^2)\Lambda^{-1} A \Lambda^{-1} = \\ \Lambda^{-1}(-\boldsymbol{\nabla}^2 + \frac{1}{c^2}\partial_t^2) A \Lambda^{-1} = \Lambda^{-1}\partial\partial^T A \Lambda^{-1} = \\ \Lambda^{-1}\partial\Lambda^{-1}\Lambda\partial^T\Lambda\Lambda^{-1}A\Lambda^{-1} = \partial^L(\partial^T)^L A^L = \partial^L B^L\end{aligned} \tag{74}$$

What is important here is the appearance of $\Lambda\partial^T\Lambda$ as representing $(\partial^T)^L$.

We also have an example of biquaternion multiplication being associative as $\partial(\partial^T A) = (\partial\partial^T)A$ on the level of the 'machine code'. The difference between ∂B and $-(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)A$ is created by different groupings of the elements of $\partial(\partial^T A) = (\partial\partial^T)A$ on 'machine code' level, resulting in different products at higher aggregation levels.

Maxwell's inhomogeneous wave equations can be written as

$$(-\partial^T\partial)B = -\mu_0\partial^T J \tag{75}$$

with the Lorentz invariant quadratic derivative,

$$-\partial^T\partial = (\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)\hat{1} \tag{76}$$

resulting in the homogeneous wave equations of the EM field in free space as

$$(-\partial^T\partial)B = \boldsymbol{\nabla}^2 B - \frac{1}{c^2}\partial_t^2 B = 0. \tag{77}$$

2.5. EM energy and the Lagrangian

As for the electromagnetic energy of a pure EM field, we have the two products BB and $B^T B$, with

$$B^T = (\partial^T A)^T = \partial A^T = (\mathbf{B} + \mathbf{i}\frac{1}{c}\mathbf{E}) \cdot \hat{\mathbf{K}} = \vec{\mathbb{B}}^* \cdot \hat{\mathbf{K}}. \tag{78}$$

in which we used $\vec{\mathbb{B}}^*$ instead of $\widetilde{\vec{\mathbb{B}}}$ as the complex conjugate of $\vec{\mathbb{B}}$.

We then can calculate the results, which gives

$$BB = (\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 - 2\mathbf{i}\frac{1}{c}\mathbf{B} \cdot \mathbf{E})\hat{1} = \vec{\mathbb{B}} \cdot \vec{\mathbb{B}}\hat{1} = 2\mu_0(u_B - u_E - 2\mathbf{i}\sqrt{u_B u_E})\hat{1} \tag{79}$$

for the first, so that we have the relativistic invariant EM energy formulation

$$\frac{1}{2\mu_0}BB = \frac{1}{2\mu_0}\vec{\mathbb{B}} \cdot \vec{\mathbb{B}}\hat{1} = (u_B - u_E - 2\mathbf{i}\sqrt{u_B u_E})\hat{1}. \tag{80}$$

The fact that this product is Lorentz invariant follows from $B^L = \Lambda B \Lambda^{-1}$ and the fact that BB result in a scalar value, so

$$B^L B^L = \Lambda B \Lambda^{-1} \Lambda B \Lambda^{-1} = \Lambda BB \Lambda^{-1} = \vec{\mathbb{B}} \cdot \vec{\mathbb{B}} \Lambda \hat{1} \Lambda^{-1} = \vec{\mathbb{B}} \cdot \vec{\mathbb{B}}\hat{1} = BB. \tag{81}$$

We also have the interesting product $\partial(\frac{1}{2\mu_0}BB)$, the four divergence of this Lorentz invariant EM energy related product. Using the Maxwell equations $\partial B = \mu_0 J$ and the Lorentz force density law $JB = \mathcal{F}$, we get

$$\partial(\frac{1}{2\mu_0}BB) = \frac{1}{2\mu_0}\partial BB = JB = \mathcal{F}. \tag{82}$$

If we choose for the Lagrangian density $\mathcal{L} = \Re(\frac{1}{2\mu_0}BB)$, then this matches the usual form.

2.6. EM energy and the Poynting four vector

For the second EM energy related product $B^T B$ we get

$$\begin{aligned} B^T B = -\overrightarrow{\mathbb{B}}^* \cdot \overrightarrow{\mathbb{B}}\,\hat{1} + (\overrightarrow{\mathbb{B}}^* \times \overrightarrow{\mathbb{B}}) \cdot \hat{\mathbf{K}} = \\ -(\mathbf{B}^2 + \frac{1}{c^2}\mathbf{E}^2)\hat{1} + 2\mathbf{i}\frac{1}{c}(\mathbf{E} \times \mathbf{B}) \cdot \hat{\mathbf{K}} = \\ \mathbf{i}\frac{2\mu_0}{c}(\mathbf{i}cu_{EM}\hat{1} + \mathbf{S} \cdot \hat{\mathbf{K}}) = \mathbf{i}2\mu_0 c(\mathbf{i}\frac{1}{c}u_{EM}\hat{1} + \mathbf{g} \cdot \hat{\mathbf{K}}), \end{aligned} \tag{83}$$

with the Poynting vector as $\mu_0\mathbf{S} = \mathbf{E}\times\mathbf{B}$, the EM momentum density $c^2\mathbf{g} = \mathbf{S}$ and the EM energy as $2\mu_0 u_{EM} = \mathbf{B}^2 + \frac{1}{c^2}\mathbf{E}^2$. Thus we get the usual EM four momentum density G and the four energy current density S as

$$G = \frac{1}{c^2}S = \frac{-\mathbf{i}}{2\mu_0 c}B^T B \tag{84}$$

For the Lorentz transformation of $B^T B$, as a four vector it should be

$$\begin{aligned} G^L = \Lambda^{-1}G\Lambda^{-1} = \frac{-\mathbf{i}}{2\mu_0 c}\Lambda^{-1}B^T B\Lambda^{-1} = \\ \frac{-\mathbf{i}}{2\mu_0 c}\Lambda^{-1}B^T\Lambda^{-1}\Lambda B\Lambda^{-1} = \frac{-\mathbf{i}}{2\mu_0 c}\Lambda^{-1}B^T\Lambda^{-1}B^L, \end{aligned} \tag{85}$$

but this gives the problematic $\Lambda^{-1}B^T\Lambda^{-1}$ which doesn't seem to give $(B^T)^L$. Given $B^T = (\partial^T A)^T = \partial A^T$, the Lorentz transformation of the elements results in

$$\partial^L(A^T)^L = \Lambda^{-1}\partial\Lambda^{-1}\Lambda A^T\Lambda = \Lambda^{-1}\partial A^T\Lambda = \Lambda^{-1}\partial A^T\Lambda = \Lambda^{-1}B\Lambda = (B^T)^L \tag{86}$$

as the Lorentz transformation of B^T. The Lorentz transformation of the separate elements of $B^T B$ then results in the problematic

$$(B^T)^L B^L = \Lambda^{-1}B^T\Lambda\Lambda B\Lambda^{-1} = \Lambda^{-1}B^T\Lambda^2 B\Lambda^{-1}. \tag{87}$$

As a result we conclude that

$$G^L \neq \frac{-\mathbf{i}}{2\mu_0 c}(B^T)^L B^L \tag{88}$$

imlying that in our biquaternion language the EM four momentum density equation Eq. (84) isn't form invariant under a Lorentz transformation. So here we enter the discussion, in our math-phys language and already in the vacuum context, of the genuine EM momentum density four vector, a discussion strongly related to the correct formulation of the EM energy density tensor.

Problematic also is the product $\partial(\frac{1}{\mu_0}BB^T)$, for which we get

$$\partial(\frac{1}{\mu_0}BB^T) = \frac{1}{\mu_0}\partial BB^T \neq \mathcal{F}_{Lorentz}. \tag{89}$$

We get the signs wrong, also with $\partial(\frac{1}{\mu_0}B^T B)$. The fact that in our math-phys environment things do not run smoothly and unproblematic as regards to the relativistic Poynting vector and the EM four momentum density indicates that fundamental problems in physics have a tendency to stay around, independent of the formalism. If we compare the situation regarding the EM Lagrangian energy as constructed from the Lorentz invariant BB, which seem undisputed, to the problematic situation around the EM four momentum density, where unanimous consensus still is out of view, we might conclude that there is no final solution the problem of the correct Lorentz invariant formulation of the relativistic Poynting vector and correlated EM four momentum density. The highest achievable seems a formulation that functions in every reference system but not between reference systems due to the non form invariance of it's formulation connected to $B^T B$.

3. FROM LORENTZ FORCE LAW TO LORENTZ-LARMOR LAW

3.1. From non-existent magnetic monopoles to well known Larmor angular velocity

As we have demonstrated, the biquaternion language perfectly reflects the structures of the Maxwell Equations EM Complex, not only its successes but also its dilemma's. But the Lorentz Force Law, another crucial part of the EM Complex, only involves only half this structure as $F_L = \mathbf{i}\frac{1}{c}(\mathbf{J}\cdot\mathbf{E})\hat{\imath} + q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\hat{\mathbf{K}}$.

It is easy to show that the total biquaternion version of the Lorentz force law as $JB = F$ is Lorentz form invariant. Then why should a subsection of it, a subsection that is entangled to the other parts, be Lorentz form invariant on its own. That doesn't seem logical.

But on the other hand we know that in the standard relativistic notation the inhomogeneous two Maxwell equations are written as the relativistically form invariant $\partial^\nu B_{\nu\mu} = \mu_0 J_\mu$, which then fits with the Lorentz force law as $J^\nu B_{\nu\mu} = F_\mu$. So in the standard notation we have a match between the Maxwell equation structure and the Lorentz force law structure as far as the inhomogeneous half of Maxwell's equations are concerned. But what about the other half of the Maxwell complex?

In the standard interpretation, this other half concerns the existance or non-existance of magnetic sources. If magnetic monopoles do exist, the other two of Maxwell's equations aren't homogeneous any more and then a magnetic monopole Lorentz force law starts to make sense.[3] This would result in the extended Lorentz Force Law $\mathbf{F} = e(\mathbf{E}+\mathbf{v}\times\mathbf{B}) + g(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E})$ with electric charge e and magnetic charge g. But as long as magnetic charges aren't discovered, we have to assume $g = 0$. In the usual biquaternion extension of the Lorentz Force Law we see this interpretation reflected. The full JB as a mathematical entity contains the Lorentz Force elements and $\mathbf{i}\frac{1}{c}\mathbf{J}\cdot\mathbf{B}\hat{\imath} + q(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E})\cdot\hat{\mathbf{K}}$ which two elements, scalar and vector, are then set equal to zero as $\mathbf{J}\cdot\mathbf{B} = 0$ and $\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E} = 0$, reflecting the absence of magnetic charges when $g = 0$.

However, in many practical cases, these two elements aren't zero at all. This allows for a much more down to earth interpretation. We will show that the dual analogue of the Lorentz force law can make perfect sense in a much easier, non-speculative, non-magnetic monopole way. As the standard Lorentz force law provides the acceleration $\mathbf{a}$ of charges, the dual part can provide information on the angular velocity $\boldsymbol{\omega}$ of these charges. We use the fact that the Larmor angular velocity of a particle trapped in a magnetic field is given by $\boldsymbol{\omega} = \frac{q}{m}\mathbf{B}$, a quantity independent of the velocity of this trapped particle. Thus we will be able to connect the cyclotron experience to the biquaternion extended Lorentz Force Law, the Lorentz-Larmor Law. This is a non-speculative approach in the sense that we use the biquaternion structure to (re)organize the accumulated experience regarding electromagnetism since the days of Lorentz at the beginning of the twentieth century.

3.2. The biquaternion version of the Lorentz Force Law: the Lorentz-Larmor Law

The underlying complete biquaternion structure not only applies to the Maxwell equations but also to Lorentz Force Law. The Lorentz force law can be written as $JB = F$, using $J = qV$. A slightly more Lorentz invariant form is given by $\gamma F = qUB$, with $U = \gamma V$. However, the product qVB contains more than the original Lorentz Force $\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$. From its structure here, it should be clear that the biquaternion qVB contains a set of four equations, analogous to the full set of Maxwell's Equations. As a four force it should also contain a time like part, a Lorentz power $P = \mathbf{J}\cdot\mathbf{E}$. From the analogy with Maxwell we expect also a $\mathbf{J}\cdot\mathbf{B}$ and a $q\mathbf{B} - q\frac{1}{c^2}\mathbf{v}\times\mathbf{E}$ part. So lets calculate it exactly. This starts as

$$F = qVB = q(\mathbf{i}c\hat{\imath} + \mathbf{v}\cdot\hat{\mathbf{K}})(\vec{\mathbb{B}}\cdot\hat{\mathbf{K}}) = (-q\mathbf{v}\cdot\vec{\mathbb{B}})\hat{\imath} + (q\mathbf{v}\times\vec{\mathbb{B}} + \mathbf{i}cq\vec{\mathbb{B}})\cdot\hat{\mathbf{K}} \tag{90}$$

and using $\vec{\mathbb{B}} = \mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}$, we get

$$F = qVB = \mathbf{i}\frac{1}{c}\mathbf{J}\cdot\mathbf{E}\hat{\imath} - \mathbf{J}\cdot\mathbf{B}\hat{\imath} + (q\mathbf{v}\times\mathbf{B} - \mathbf{i}q\frac{1}{c}\mathbf{v}\times\mathbf{E} + \mathbf{i}cq\mathbf{B} + q\mathbf{E})\cdot\hat{\mathbf{K}} \tag{91}$$

This can be split into two four vector like parts, the Lorentz part and the Larmor part, as

$$F = [\mathbf{i}\frac{1}{c}\mathbf{J}\cdot\mathbf{E}\hat{\imath} + q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\hat{\mathbf{K}}] + \mathbf{i}c[\mathbf{i}\frac{1}{c}\mathbf{J}\cdot\mathbf{B}\hat{\imath} + q(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E})\cdot\hat{\mathbf{K}}]. \tag{92}$$

So the first part of qVB clearly is the traditional Lorentz four force part. This part results in an acceleration of the charge q, both linear and centripetal. Usually the second part is either set equal to zero or connected to a hypothetical magnetic monopole four force but we decide to connect it to the Larmor or cyclotron angular velocity experimental facts.

The connection of this second part to the Larmor or cyclotron angular velocity ω_L can be made evident by writing $F = m_0 A + \mathbf{i} m_0 c\Omega$, giving

$$A+ic\Omega = [\mathbf{i}\frac{1}{c}\mathbf{v}\cdot\frac{q}{m_0}\mathbf{E}\hat{\imath}+(\frac{q}{m_0}\mathbf{E}+\mathbf{v}\times\frac{q}{m_0}\mathbf{B})\cdot\hat{\mathbf{K}}]+\mathbf{i}c[\mathbf{i}\frac{1}{c}\mathbf{v}\cdot\frac{q}{m_0}\mathbf{B}\hat{\imath}+(\frac{q}{m_0}\mathbf{B}-\frac{1}{c^2}\mathbf{v}\times\frac{q}{m_0}\mathbf{E})\cdot\hat{\mathbf{K}}] \quad (93)$$

and then using the cyclotron or Larmor formula $\boldsymbol{\omega} = \frac{q}{m_0}\mathbf{B}$ and the usual Coulomb acceleration $\mathbf{a} = \frac{q}{m_0}\mathbf{E}$ in order to get

$$\frac{1}{m_0}F = A + ic\Omega = [\mathbf{i}\frac{1}{c}\mathbf{v}\cdot\mathbf{a}\hat{\imath} + (\mathbf{a} + \mathbf{v}\times\boldsymbol{\omega})\cdot\hat{\mathbf{K}}] + \mathbf{i}c[\mathbf{i}\frac{1}{c}\mathbf{v}\cdot\boldsymbol{\omega}\hat{\imath} + (\boldsymbol{\omega} - \frac{1}{c^2}\mathbf{v}\times\mathbf{a})\cdot\hat{\mathbf{K}}]. \quad (94)$$

What we have here are two entangled semi four vectors, one for Lorentz acceleration and one for Larmor rotation. We call them semi because they aren't independent four vectors but entangled one's due to the $\mathbf{v}\times\boldsymbol{\omega}$ and $-\frac{1}{c^2}\mathbf{v}\times\mathbf{a}$ elements. The object $F = m_0 A + \mathbf{i} m_0 c\Omega$ cannot be interpreted as a genuine anti-symmetric six-vector either because then the two time like parts should have to be zero. The classical Lorentz force law is in our biquaternion representation equal to the Lorentz acceleration part times the mass.

If we split them we get the two seperate equations, with first the Lorentz four acceleration

$$A = [\mathbf{i}\frac{1}{c}\mathbf{v}\cdot\frac{q}{m_0}\mathbf{E}\hat{\imath} + (\frac{q}{m_0}\mathbf{E} + \mathbf{v}\times\frac{q}{m_0}\mathbf{B})\cdot\hat{\mathbf{K}}] \quad (95)$$

and then the Larmor four angular velocity

$$\Omega = \mathbf{i}\frac{1}{c}\mathbf{v}\cdot\frac{q}{m_0}\mathbf{B}\hat{\imath} + (\frac{q}{m_0}\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\frac{q}{m_0}\mathbf{E})\cdot\hat{\mathbf{K}}. \quad (96)$$

The first part represents the traditional Lorentz law acceleration, in four-vector notation. The second part is without official status as a law of nature but cyclotronists might recognize some of its elements. Now, for $m_0 A$ we have an interpretation, it gives the traditional Lorentz force F. But $m_0\Omega$ does not have such a traditional interpretation. We might interpret is as the cyclotron angular velocity and the cyclotron change of acceleration.

Maxwell Equations	Lorentz Force Law	Lorentz-Larmor Law
$\partial B = \mu_0 J$	$JB = m_0 A$	$JB = m_0 A + \mathbf{i} c m_0 \Omega$
$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\mathbf{J} \cdot \mathbf{E} = m_0 \mathbf{v} \cdot \mathbf{a}$	$\mathbf{J} \cdot \mathbf{E} = m_0 \mathbf{v} \cdot \mathbf{a}$
$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$	$\mathbf{J} \cdot \mathbf{B} = 0$	$\mathbf{J} \cdot \mathbf{B} = m_0 \mathbf{v} \cdot \boldsymbol{\omega}$
$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} = \mu_0 \mathbf{J}$	$\mathbf{J} \times \mathbf{B} + q\mathbf{E} = m_0 \mathbf{a}$	$\mathbf{J} \times \mathbf{B} + q\mathbf{E} = m_0 \mathbf{v} \times \boldsymbol{\omega} + m_0 \mathbf{a}$
$\boldsymbol{\nabla} \times \mathbf{E} + \partial_t \mathbf{B} = 0$	$\mathbf{J} \times \mathbf{E} - c^2 q \mathbf{B} = 0$	$\mathbf{J} \times \mathbf{E} - c^2 q \mathbf{B} = m_0 \mathbf{v} \times \mathbf{a} - m_0 c^2 \boldsymbol{\omega}$

TABLE I. Maxwell's Equations, Lorentz Force Law and Lorentz-Larmor Law

3.3. Some basic examples of the Lorentz-Larmor Law

The first example is the most obvious one. A point charge q is moving in a cyclotron with magnetic field $B = \mathbf{B} \cdot \hat{\mathbf{K}}$ perpendicular to the charge velocity $\mathbf{v}$. We then have

$$A + ic\Omega = (\mathbf{v} \times \frac{q}{m_0}\mathbf{B}) \cdot \hat{\mathbf{K}} + \mathbf{i}c(\frac{q}{m_0}\mathbf{B}) \cdot \hat{\mathbf{K}} = \mathbf{a} \cdot \hat{\mathbf{K}} + ic\boldsymbol{\omega} \cdot \hat{\mathbf{K}}. \tag{97}$$

This can be interpreted as having a centripetal acceleration $\mathbf{a} = (\mathbf{v} \times \frac{q}{m_0}\mathbf{B})$ and a cyclotron angular velocity $\boldsymbol{\omega} = \frac{q}{m_0}\mathbf{B}$.

The second example is that of a point charge in in central electric field. A point charge q is moving in a central field $E = \mathbf{E} \cdot \hat{\mathbf{K}}$, a field that is perpendicular to the charge velocity $\mathbf{v}$. We then have

$$A + ic\Omega = \frac{q}{m_0}\mathbf{E} \cdot \hat{\mathbf{K}} + \mathbf{i}c[-\frac{1}{c^2}\mathbf{v} \times \frac{q}{m_0}\mathbf{E})] \cdot \hat{\mathbf{K}} = \mathbf{a} \cdot \hat{\mathbf{K}} + ic\boldsymbol{\omega} \cdot \hat{\mathbf{K}} \tag{98}$$

so we have $\mathbf{a} = \frac{q}{m_0}\mathbf{E}$ and $\boldsymbol{\omega} = -\frac{1}{c^2}\mathbf{v} \times \frac{q}{m_0}\mathbf{E} = -\frac{1}{c^2}\mathbf{v} \times \mathbf{a}$, which can be interpreted as $\boldsymbol{\omega} = \frac{q}{m_0}(-\frac{1}{c^2}\mathbf{v} \times \mathbf{E}) = \frac{q}{m_0}\mathbf{B}' = -\frac{1}{c^2}\mathbf{v} \times \mathbf{a}$. So for the point charge thing happen as if he sees a constant magnetic field $\mathbf{B}'$ as in the example above. This is equal to the orbital angular velocity for a point charge rotating around a central opposite charge like the electron in hydrogen in a planetary Bohr-Rutherford model, the Larmor angular velocity.

Now what if a point charge enters a region with a partly perpendicular homogeneous magnetic field? Then we get a spiral trajectory with

$$A + ic\Omega = \frac{q}{m_0}\mathbf{v} \times \mathbf{B} \cdot \hat{\mathbf{K}} + \mathbf{i}c[\mathbf{i}\frac{1}{c}\mathbf{v} \cdot \frac{q}{m_0}\mathbf{B}\hat{\mathbf{i}} + (\frac{q}{m_0}\mathbf{B}) \cdot \hat{\mathbf{K}}] \tag{99}$$

A straighforward interpretation will give an angular velocity $\boldsymbol{\omega} = \frac{q}{m_0}\mathbf{B}$, the Larmor angular velocity of the spiral motion. And we also have a time-like part that, in the standard Lorentz force analogue, is related to a power or loss of energy. This complex power, in magnitude equal to $P = qc\mathbf{v} \cdot \mathbf{B}$ and four-vector like related to the cyclotron angular velocity hasn't got an obvious interpretation yet. Its magnitude can be given as $P = m_0 c(\mathbf{v} \cdot \boldsymbol{\omega}) = m_0 ac$.

These examples make it clear that the extension of the Lorentz force law the way we propose, the Lorentz-Larmor Law, doesn't constitute new physics. It is nothing more than a mathematical aggregation of the already known. That is the main character of almost all applications of biquaternion mathematics to physics. So instead of speculation about magnetic monopoles, we interpret the extended Lorentz force law as a reflection of the already familiar Larmor angular velocity or cyclotron angular velocity. The only new element in our approach is the time like part of the Larmor four angular velocity $\mathbf{J} \cdot \mathbf{B} = m_0 \mathbf{v} \cdot \boldsymbol{\omega}$, a timelike scalar force element. According to this mathematical scheme, the cyclotron helical motion is connected to a time-like power term $P = c\mathbf{J} \cdot \mathbf{B}$ or a time-like acceleration term $a = \mathbf{v} \cdot \boldsymbol{\omega}$ and a time-like force $F = m_0 a = m_0 \mathbf{v} \cdot \boldsymbol{\omega}$. Is there a known experimental fact, or conglomerate of facts, that will give this term its non-speculative interpretation? For a charge entering a magnetic field at an angle θ with the magnetic field lines, the product $\mathbf{J} \cdot \mathbf{B} = m_0 \mathbf{v} \cdot \boldsymbol{\omega}$ isn't zero, that is clear.

On of the reasons for using the Lorentz Larmor Law instead of the Lorentz Force Law might be that the first immediately produces the Larmor angular velocity, whereas with the Lorentz Force Law, the Larmor angular velocity has to be calculated using all kind of extra arguments and reasoning. The Lorentz Larmor Law is simply more economic.

3.4. The Lorentz transformation of the Lorentz-Larmor Law

In a different reference frame we use the Lorentz transformation on V and B separately with the Lorentz transformation operator Λ to get

$$qV^L B^L = q\Lambda^{-1} V \Lambda^{-1} \Lambda B \Lambda^{-1} = q\Lambda^{-1} V B \Lambda^{-1} = \Lambda^{-1} F \Lambda^{-1} = F^L \tag{100}$$

which proves that the extended Lorentz Larmor Law product F transforms like a four vector. We then also have

$$\begin{gathered} F^L = \Lambda^{-1} F \Lambda^{-1} = \Lambda^{-1}(m_0 A + \mathbf{i} m_0 c \Omega)\Lambda^{-1} = \\ m_0 \Lambda^{-1} A \Lambda^{-1} + \mathbf{i} m_0 c \Lambda^{-1} \Omega \Lambda^{-1} = m_0 A^L + \mathbf{i} m_0 c \Omega^L \end{gathered} \tag{101}$$

indicating that both A and Ω transform as ordinary four vectors, with the crucial difference that they are entangled four vectors. This involves the full Maxwell-Lorentz structure. In the case of Maxwell's Equations the full structure is already present in the classical formulation but the classical Lorentz Force lacks elements of this full structure. The case of Maxwell's Equations prove that the complete structure is relativistically invariant, implying the same for the full Lorentz Larmor Law structure.

In our interpretation of the full Lorentz Larmor Law, a Lorentz transformation of the complete $F = JB$ would be equivalent to transporting a complete working 1930's cyclotron lab setup in a spacecraft with a relativistic linear velocity. What would be seen as an acceleration in inertial reference frame S, would be seen as a combination of an acceleration and an angular velocity in reference frame S' and vice versa. In both inertial frames of reference, $F = JB$ would be valid. The Lorentz Larmor Law $F = JB$ is inertial reference frame invariant.

If we look back at the way we got to the Lorentz-Larmor Law, we started with the product JB, wrote it out in it's EM-details and then used the cyclotron or Larmor formula $\boldsymbol{\omega} = \frac{q}{m_0}\mathbf{B}$ and the usual Coulomb acceleration $\mathbf{a} = \frac{q}{m_0}\mathbf{E}$ to arrive at a mechanical dynamics parallel to the EM-dynamics in the form of $A + \mathbf{i}c\Omega$. We thus arrived at $JB = A + \mathbf{i}c\Omega$ and using $F = A + \mathbf{i}c\Omega$ we got $JB = F$. In this way, the product JB directly leads to the cyclotron or Larmor angular velocities represented by the biquaternion object Ω. That is a lot more economic than the way the cyclotron or Larmor angular velocities are usually derived. There is more to it, because of the relative ease of arriving at the Lorentz transformation of the product JB, we therefore also know how to get at the Lorentz transformation of Larmor angular velocities in all kinds of situations. Due to the fact that the Lorentz Force Law is part of the JB product, the Lorentz transformation of Larmor angular velocities automatically involves the Lorentz Force Law in an entangled way. One has to calculate the Lorentz transformation of the entire Lorentz-Larmor Law $JB = F$ in the primary reference system in order to arrive at the correct values in the secondary reference system.

3.5. Relativistic angular velocity in Thomas' 1927 paper

The relevance for physics becomes more clear if we go back to the extended paper of Thomas of 1927 regarding the electron in the atom in a Rutherford planetary model. In this paper, Thomas discusses the situation of the Lorentz force law and the correlated acceleration $\mathbf{a}$ and angular velocity $\boldsymbol{\omega}$. He works out the options for the relativistic Ω as a three vector $\boldsymbol{\omega}$, as a four-vector ω_μ and as an anti-symmetric six-vector $\omega_{\mu\nu}$. He then decides for the four-vector option, with the statement that it doesn't matter for the precession correction he needed for the electron in an atomic orbit.

We quote at length from Thomas' paper because his approach reflects the standard paradigm in the relativistic treatment in the area of electrodynamics.

> Let $\boldsymbol{\omega}_0$ and $\mathbf{a}_0$ be the angular velocity and the acceleration of an electron in the rest system of the lab and $\boldsymbol{\omega}_s$ be the spin angular frequency in the rest system of the electron, then in a system moving with velocity $\mathbf{v}$ relative to this electron rest system, the angular velocity might transform like a four vector ω_μ or it might transform as a six-vector $\omega_{\mu\nu}$.
>
> [...]
>
> If
>
> $$\begin{aligned}(\omega^1, \omega^2, \omega^3) = \boldsymbol{\omega} + (\gamma - 1)\frac{\boldsymbol{\omega}\cdot\mathbf{v}}{v^2}\mathbf{v} = \boldsymbol{\omega}_\perp + \gamma\boldsymbol{\omega}_{||} \\ (\omega^4) = \gamma\frac{\boldsymbol{\omega}\cdot\mathbf{v}}{c^2}\end{aligned} \tag{102}$$
>
> then ω^μ is a four-vector transforming like dx^μ and having zero time components and space components equal to $\boldsymbol{\omega}$ in any system in which the electron is instantaneously at rest.
>
> If
>
> $$\begin{aligned}(\omega_{23}, \omega_{31}, \omega_{12}) = \gamma\boldsymbol{\omega} + (1 - \gamma)\frac{\boldsymbol{\omega}\cdot\mathbf{v}}{v^2}\mathbf{v} = \gamma\boldsymbol{\omega}_\perp + \boldsymbol{\omega}_{||} \\ (\omega_{14}, \omega_{24}, \omega_{34}) = -\gamma\mathbf{v}\times\boldsymbol{\omega}\end{aligned} \tag{103}$$
>
> and $\omega_{\mu\nu} = -\omega_{\nu\mu}$, $\omega_{\mu\nu}$ is an anti-symmetrical tensor transformed like $F_{\mu\nu}$ and having zero components $(\omega_{14}, \omega_{24}, \omega_{34})$ and components $(\omega_{23}, \omega_{31}, \omega_{12})$ equal to $\boldsymbol{\omega}$ in any system in which the electron is instantaneously at rest.

If we compare Thomas' relativistic angular velocity options to Eq. (94), then his four-vector equals part of our Ω and his six-vector parts of the two three-vector parts of our

$A + \mathbf{i}c\Omega$. Thomas was concerned with the correct treatment of electron spin. In this paper, we aren't discussing spin angular velocity but orbital angular velocities of charged particles in electromagnetic fields. The relevance here is the difference between Thomas' options and our biquaternion options. Thomas could choose between a classical three vector, a relativistic four vector and a relativistic anti-symmetrical six-vector. Our $A + \mathbf{i}c\Omega$ combines the elements of four vector and six vector into an eight vector and it transforms as two four vectors, which are however not independent but entangled. The last implies that derived Lorentz invariant quantities should involve both four vectors. But the most important issue for this subsection is the fact that in standard U(1) relativistic electrodynamics the mathematical elements at our disposal for describing angular velocity are the the two options given by Thomas, four vectors transforming as dR_μ or six vectors, anti-symmetric tensors, transforming as the EM-field tensor $F_{\mu\nu}$. In our version of biquaternion relativistic electrodynamics, more options are available. In the standard or paradigmatic mathematical scheme, a Lorentz-Larmor Law is inconceivable.

4. THE ELECTROMAGNETIC DIPOLE MOMENT INCLUDING HIDDEN MOMENTUM

4.1. The general expression for the torque

The classical torque and energy of an electromagnetic dipole in an electromagnetic field is in our language given by the product $T = MB$ with $M = \vec{\mathbb{M}} \cdot \hat{\mathbf{K}} = (\boldsymbol{\mu} + \mathbf{i}c\boldsymbol{\pi}) \cdot \hat{\mathbf{K}}$. With $B = \vec{\mathbb{B}} \cdot \hat{\mathbf{K}} = (\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}) \cdot \hat{\mathbf{K}}$, this gives

$$T = MB = (\vec{\mathbb{M}} \cdot \hat{\mathbf{K}})(\vec{\mathbb{B}} \cdot \hat{\mathbf{K}}) = -\vec{\mathbb{M}} \cdot \vec{\mathbb{B}}\hat{1} + (\vec{\mathbb{M}} \times \vec{\mathbb{B}}) \cdot \hat{\mathbf{K}}. \tag{104}$$

Now suppose that both products, M and B, transform as six-vectors or anti-symmetric tensors. For B, this has already been established. For M this is more problematic, but we may use the assumption that M transforms as B as a starting hypothesis and see how far we get. Then the torque-energy product $T = MB$ also transforms as an antisymmetric tensor or six-vector because

$$M^L B^L = \Lambda M \Lambda^{-1} \Lambda B \Lambda^{-1} = \Lambda M B \Lambda^{-1} = \Lambda T \Lambda^{-1} = T^L. \tag{105}$$

So our torque energy equation as $T = MB$ is Lorentz form invariant, under the assumption that M Lorentz transforms as B, because in that case if one has $T = MB$ in one inertial frame of reference S one has $T' = M'B'$ in every inertial frame of reference S'.

Mathematically we get

$$MB = (-\boldsymbol{\mu}\cdot\mathbf{B} - \mathbf{i}c\boldsymbol{\pi}\cdot\mathbf{B} + \frac{\mathbf{i}}{c}\boldsymbol{\mu}\cdot\mathbf{E} - \boldsymbol{\pi}\cdot\mathbf{E})\hat{\mathbf{1}} + (\boldsymbol{\mu}\times\mathbf{B} + \mathbf{i}c\boldsymbol{\pi}\times\mathbf{B} - \frac{\mathbf{i}}{c}\boldsymbol{\mu}\times\mathbf{E} + \boldsymbol{\pi}\times\mathbf{E})\cdot\hat{\mathbf{K}} \quad (106)$$

which physically reduces to a real energy-torque part

$$\Re(MB) = (-\boldsymbol{\mu}\cdot\mathbf{B} - \boldsymbol{\pi}\cdot\mathbf{E})\hat{\mathbf{1}} + (\boldsymbol{\mu}\times\mathbf{B} + \boldsymbol{\pi}\times\mathbf{E})\cdot\hat{\mathbf{K}} \quad (107)$$

and a complex or hidden momentum part

$$\Im(MB) = \mathbf{i}c(-\boldsymbol{\pi}\cdot\mathbf{B} + \frac{1}{c^2}\boldsymbol{\mu}\cdot\mathbf{E})\hat{\mathbf{1}} + \mathbf{i}c(\boldsymbol{\pi}\times\mathbf{B} - \frac{1}{c^2}\boldsymbol{\mu}\times\mathbf{E})\cdot\hat{\mathbf{K}} \quad (108)$$

representing the energy, the torque and the hidden momentum of the static interaction between M and B. We can write is as $T = \mathbb{T}_0\hat{\mathbf{1}} + \vec{\mathbb{T}}\cdot\hat{\mathbf{K}}$. If we look at Frenkel's 1926 paper in which he criticized Thomas, his six-vector torque equals the vector part of our $T = MB$ and his related energy equals the real time like part of it. Frenkel already incorporated the vector part of the hidden momentum of the electromagnetic torque in his 1926 approach.

4.2. Finding the Lorentz form invariant expression for a moving em dipole moment

If, instead of a charge moving in an electromagnetic field, we move an electromagnetic dipole through it, we don't have qVB or qUB but $\frac{1}{c}MVB$ or $\frac{1}{c}MUB$. This can be interpreted from the dipole's point of view as $\frac{1}{c}M^T(VB)$ or from the lab EM field point of view as $\frac{1}{c}(VM)B$. First we look at the Lorentz transformation properties of $VMB = cT$, $MVB = cT$ and $M^TVB = cT$, assuming again that M and M^T both behave as B and B^T under Lorentz transformations. For the first we have

$$V^LM^LB^L = \Lambda^{-1}V\Lambda^{-1}\Lambda M\Lambda^{-1}\Lambda B\Lambda^{-1} = \Lambda^{-1}VMB\Lambda^{-1} = c\Lambda^{-1}T\Lambda^{-1} = cT^L \quad (109)$$

and for the second and third we get

$$M^LV^LB^L = \Lambda M^T\Lambda^{-1}\Lambda^{-1}V\Lambda^{-1}\Lambda B\Lambda^{-1} = \Lambda M\Lambda^{-1}\Lambda^{-1}VB\Lambda^{-1} \neq cT^L \quad (110)$$

and

$$(M^T)^LV^LB^L = \Lambda^{-1}M^T\Lambda\Lambda^{-1}V\Lambda^{-1}\Lambda B\Lambda^{-1} = \Lambda^{-1}M^TVB\Lambda^{-1} = c\Lambda^{-1}T\Lambda^{-1} = cT^L. \quad (111)$$

We conclude that the products of VMB and M^TVB transform as four-vectors and that the equations $cT = VMB$ and $cT = M^TVB$ are form invariant under a Lorentz transformation, but that $cT = MVB$ isn't. From here on we will use $cT = VMB$ in the slightly altered form of $T = -\mathbf{i}\frac{1}{c}(VM)B$ as the correct expression for the biquaternion total torque of a moving dipole in a static EM field.

4.3. Expressions for $T = -\mathbf{i}\frac{1}{c}(VM)B$, in general as well as for different conditions

In the attempt to calculate $T = -\mathbf{i}\frac{1}{c}(VM)B$ we start with the product VM as

$$VM = (\mathbf{i}c\hat{\mathbf{1}} + \mathbf{v}\cdot\hat{\mathbf{K}})(\vec{\mathbb{M}}\cdot\hat{\mathbf{K}}) = (-\mathbf{v}\cdot\vec{\mathbb{M}})\hat{\mathbf{1}} + (\mathbf{v}\times\vec{\mathbb{M}} + \mathbf{i}c\vec{\mathbb{M}})\cdot\hat{\mathbf{K}} \tag{112}$$

and using $\vec{\mathbb{M}} = \boldsymbol{\mu} + \mathbf{i}c\boldsymbol{\pi}$, we get

$$VM = -\mathbf{i}c\mathbf{v}\cdot\boldsymbol{\pi}\hat{\mathbf{1}} - \mathbf{v}\cdot\boldsymbol{\mu}\hat{\mathbf{1}} + (\mathbf{v}\times\boldsymbol{\mu} + \mathbf{i}c\mathbf{v}\times\boldsymbol{\pi} + \mathbf{i}c\boldsymbol{\mu} - c^2\boldsymbol{\pi})\cdot\hat{\mathbf{K}}. \tag{113}$$

Then we multiply it by $-\mathbf{i}/c$ to get

$$-\mathbf{i}\frac{1}{c}VM = -\mathbf{v}\cdot\boldsymbol{\pi}\hat{\mathbf{1}} + \mathbf{i}\frac{1}{c}\mathbf{v}\cdot\boldsymbol{\mu}\hat{\mathbf{1}} + (-\mathbf{i}\frac{1}{c}\mathbf{v}\times\boldsymbol{\mu} + \mathbf{v}\times\boldsymbol{\pi} + \boldsymbol{\mu} + \mathbf{i}c\boldsymbol{\pi})\cdot\hat{\mathbf{K}}. \tag{114}$$

We can rearrange this into

$$-\mathbf{i}\frac{1}{c}VM = [\mathbf{i}\frac{1}{c}\mathbf{v}\cdot\boldsymbol{\mu}\hat{\mathbf{1}} + (\boldsymbol{\mu} + \mathbf{v}\times\boldsymbol{\pi})\cdot\hat{\mathbf{K}}] + \mathbf{i}c[\mathbf{i}\frac{1}{c}\mathbf{v}\cdot\boldsymbol{\pi}\hat{\mathbf{1}} + (\boldsymbol{\pi} - \frac{1}{c^2}\mathbf{v}\times\boldsymbol{\mu})\cdot\hat{\mathbf{K}}]. \tag{115}$$

in order to be able to interpret the result as two entangled four vectors, the first as an extension of $\boldsymbol{\mu}$ and the second as an extension of $\boldsymbol{\pi}$. But it can also be also written as

$$-\mathbf{i}\frac{1}{c}VM = (\mu' + \mathbf{i}c\pi')\hat{\mathbf{1}} + (\boldsymbol{\mu}' + \mathbf{i}c\boldsymbol{\pi}')\cdot\hat{\mathbf{K}} = \mathbb{M}'_0\hat{\mathbf{1}} + \vec{\mathbb{M}}'\cdot\hat{\mathbf{K}}, \tag{116}$$

using $\mu' = -\mathbf{v}\cdot\boldsymbol{\pi}$, $\pi' = \frac{1}{c^2}\mathbf{v}\cdot\boldsymbol{\mu}$, $\boldsymbol{\mu}' = \boldsymbol{\mu} + \mathbf{v}\times\boldsymbol{\pi}$ and $\boldsymbol{\pi}' = \boldsymbol{\pi} - \frac{1}{c^2}\mathbf{v}\times\boldsymbol{\mu}$.

We use this result to calculate $T = -\mathbf{i}\frac{1}{c}(VM)B$ as

$$T = -\mathbf{i}\frac{1}{c}(VM)B = \left(\mathbb{M}'_0\hat{\mathbf{1}} + \vec{\mathbb{M}}'\cdot\hat{\mathbf{K}}\right)\left(\vec{\mathbb{B}}\cdot\hat{\mathbf{K}}\right) = -\vec{\mathbb{M}}'\cdot\vec{\mathbb{B}}\hat{\mathbf{1}} + (\vec{\mathbb{M}}'\times\vec{\mathbb{B}})\cdot\hat{\mathbf{K}} + \mathbb{M}'_0\vec{\mathbb{B}}\cdot\hat{\mathbf{K}} \tag{117}$$

We can calculate the three parts seperataly as, first

$$-\vec{\mathbb{M}}'\cdot\vec{\mathbb{B}}\hat{\mathbf{1}} = -(\boldsymbol{\mu}' + \mathbf{i}c\boldsymbol{\pi}')\cdot(\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E})\hat{\mathbf{1}} = \tag{118}$$

$$\left(-\boldsymbol{\mu}'\cdot\mathbf{B} - \boldsymbol{\pi}'\cdot\mathbf{E} - \mathbf{i}c\boldsymbol{\pi}'\cdot\mathbf{B} + \mathbf{i}\frac{1}{c}\boldsymbol{\mu}'\cdot\mathbf{E}\right)\hat{\mathbf{1}}, \tag{119}$$

second

$$(\vec{\mathbb{M}}' \times \vec{\mathbb{B}}) \cdot \hat{\mathbf{K}} = (\boldsymbol{\mu}' + \mathbf{i}c\boldsymbol{\pi}') \times (\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}) \cdot \hat{\mathbf{K}} = \tag{120}$$

$$\left(\boldsymbol{\mu}' \times \mathbf{B} + \boldsymbol{\pi}' \times \mathbf{E} + \mathbf{i}c\boldsymbol{\pi}' \times \mathbf{B} - \mathbf{i}\frac{1}{c}\boldsymbol{\mu}' \times \mathbf{E}\right) \cdot \hat{\mathbf{K}} \tag{121}$$

and third

$$\mathbb{M}_0'\vec{\mathbb{B}} \cdot \hat{\mathbf{K}} = (\mu' + \mathbf{i}c\pi')(\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}) \cdot \hat{\mathbf{K}} = \tag{122}$$

$$\left(\mu'\mathbf{B} + \pi'\mathbf{E} + \mathbf{i}c\pi'\mathbf{B} - \mathbf{i}\frac{1}{c}\mu'\mathbf{E}\right) \cdot \hat{\mathbf{K}} \tag{123}$$

If we compare this to the static T which was

$$T_S = MB = \mathbb{T}_0\hat{\imath} + \vec{\mathbb{T}} \cdot \hat{\mathbf{K}} = \left(-\boldsymbol{\mu} \cdot \mathbf{B} - \boldsymbol{\pi} \cdot \mathbf{E} - \mathbf{i}c\boldsymbol{\pi} \cdot \mathbf{B} + \frac{\mathbf{i}}{c}\boldsymbol{\mu} \cdot \mathbf{E}\right)\hat{\imath} \tag{124}$$

$$+\left(\boldsymbol{\mu} \times \mathbf{B} + \boldsymbol{\pi} \times \mathbf{E} + \mathbf{i}c\boldsymbol{\pi} \times \mathbf{B} - \mathbf{i}\frac{1}{c}\boldsymbol{\mu} \times \mathbf{E}\right) \cdot \hat{\mathbf{K}} \tag{125}$$

then we see two major differences, the first the appearance of $\boldsymbol{\pi}'$ and $\boldsymbol{\mu}'$ instead of $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ and the second the really new arrivals π' and μ'.

4.4. Specific cases for $T = -\mathbf{i}\frac{1}{c}(VM)B$

Let's assume a situation in which $\boldsymbol{\pi} = 0$ and $\mathbf{E} = 0$. Then we have

$$T = -\mathbf{i}\frac{1}{c}(VM)B = \left(\mathbf{i}\frac{1}{c}\mathbf{v} \cdot \boldsymbol{\mu}\hat{\imath} + \boldsymbol{\mu} \cdot \hat{\mathbf{K}} - \mathbf{i}\frac{1}{c}(\mathbf{v} \times \boldsymbol{\mu}) \cdot \hat{\mathbf{K}}\right)\left(\mathbf{B} \cdot \hat{\mathbf{K}}\right) = \tag{126}$$

$$-\boldsymbol{\mu} \cdot \mathbf{B}\hat{\imath} + \mathbf{i}\frac{1}{c}((\mathbf{v} \times \boldsymbol{\mu}) \cdot \mathbf{B})\hat{\imath} + \tag{127}$$

$$\mathbf{i}\frac{1}{c}(\mathbf{v} \cdot \boldsymbol{\mu})\mathbf{B} \cdot \hat{\mathbf{K}} + (\boldsymbol{\mu} \times \mathbf{B}) \cdot \hat{\mathbf{K}} - \mathbf{i}\frac{1}{c}((\mathbf{v} \times \boldsymbol{\mu}) \times \mathbf{B}) \cdot \hat{\mathbf{K}}. \tag{128}$$

If on the other hand we assume a situation in which $\boldsymbol{\mu} = 0$ and $\mathbf{B} = 0$, then we get

$$T = -\mathbf{i}\frac{1}{c}(VM)B = \left((\mathbf{v} \times \boldsymbol{\pi}) \cdot \hat{\mathbf{K}} - \mathbf{v} \cdot \boldsymbol{\pi}\hat{\imath} + \mathbf{i}c\boldsymbol{\pi} \cdot \hat{\mathbf{K}}\right)(-\mathbf{i}\frac{1}{c}\mathbf{E} \cdot \hat{\mathbf{K}}) = \tag{129}$$

$$-\boldsymbol{\pi} \cdot \mathbf{E}\hat{\imath} + \mathbf{i}\frac{1}{c}(\mathbf{v} \times \boldsymbol{\pi}) \cdot \mathbf{E}\hat{\imath} + \tag{130}$$

$$\mathbf{i}\frac{1}{c}(\mathbf{v} \cdot \boldsymbol{\pi})\mathbf{E} \cdot \hat{\mathbf{K}} + (\boldsymbol{\pi} \times \mathbf{E}) \cdot \hat{\mathbf{K}} - \mathbf{i}\frac{1}{c}((\mathbf{v} \times \boldsymbol{\pi}) \times \mathbf{E}) \cdot \hat{\mathbf{K}}. \tag{131}$$

Then of course we can study the mixed situation or mixed elements of $T = -\mathbf{i}\frac{1}{c}(VM)B$. So let's assume $\boldsymbol{\pi} = 0$ and $\mathbf{B} = 0$, then we get

$$T = -\mathbf{i}\frac{1}{c}(VM)B = \left(\mathbf{i}\frac{1}{c}\mathbf{v}\cdot\boldsymbol{\mu}\hat{1} + \boldsymbol{\mu}\cdot\hat{\mathbf{K}} - \mathbf{i}\frac{1}{c}(\mathbf{v}\times\boldsymbol{\mu})\cdot\hat{\mathbf{K}}\right)\left(-\mathbf{i}\frac{1}{c}\mathbf{E}\cdot\hat{\mathbf{K}}\right) = \tag{132}$$

$$\mathbf{i}\frac{1}{c}\boldsymbol{\mu}\cdot\mathbf{E}\hat{1} + \frac{1}{c^2}(\mathbf{v}\times\boldsymbol{\mu})\cdot\mathbf{E}\hat{1} + \tag{133}$$

$$\frac{1}{c^2}(\mathbf{v}\cdot\boldsymbol{\mu})\mathbf{E}\cdot\hat{\mathbf{K}} - \mathbf{i}\frac{1}{c}(\boldsymbol{\mu}\times\mathbf{E})\cdot\hat{\mathbf{K}} - \frac{1}{c^2}((\mathbf{v}\times\boldsymbol{\mu})\times\mathbf{E})\cdot\hat{\mathbf{K}} \tag{134}$$

Finally, if we have $\boldsymbol{\mu} = 0$ and $\mathbf{E} = 0$, then we get

$$T = -\mathbf{i}\frac{1}{c}(VM)B = \left((\mathbf{v}\times\boldsymbol{\pi})\cdot\hat{\mathbf{K}} - \mathbf{v}\cdot\boldsymbol{\pi}\hat{1} + \mathbf{i}c\boldsymbol{\pi}\cdot\hat{\mathbf{K}}\right)(\mathbf{B}\cdot\hat{\mathbf{K}}) = \tag{135}$$

$$-\mathbf{i}c\boldsymbol{\pi}\cdot\mathbf{B}\hat{1} - (\mathbf{v}\times\boldsymbol{\pi})\cdot\mathbf{B}\hat{1} + \tag{136}$$

$$-(\mathbf{v}\cdot\boldsymbol{\pi})\mathbf{B}\cdot\hat{\mathbf{K}} + \mathbf{i}c(\boldsymbol{\pi}\times\mathbf{B})\cdot\hat{\mathbf{K}} + ((\mathbf{v}\times\boldsymbol{\pi})\times\mathbf{B})\cdot\hat{\mathbf{K}}. \tag{137}$$

4.5. The total torque-energy-hidden-momentum product

If we want to have an overview of the total $T = -\mathbf{i}\frac{1}{c}(VM)B$, a split in vector momentum, torque, energy and scalar momentum is still practical. We will split the total product T into a time-like real energy part U, a time-like complex scalar hidden momentum part p, a complex vector hidden momentum part $\mathbf{p}$ and a real vector torque part $\boldsymbol{\tau}$. Let's start with the torque

$$\left((\boldsymbol{\mu} + (\mathbf{v}\times\boldsymbol{\pi}))\times\mathbf{B} + (\boldsymbol{\pi} - \frac{1}{c^2}(\mathbf{v}\times\boldsymbol{\mu}))\times\mathbf{E} + \frac{1}{c^2}(\mathbf{v}\cdot\boldsymbol{\mu})\mathbf{E} - (\mathbf{v}\cdot\boldsymbol{\pi})\mathbf{B}\right)\cdot\hat{\mathbf{K}} \tag{138}$$

$$= \boldsymbol{\tau}\cdot\hat{\mathbf{K}} = (\boldsymbol{\mu}'\times\mathbf{B} + \boldsymbol{\pi}'\times\mathbf{E} + \pi'\mathbf{E} + \mu'\mathbf{B})\cdot\hat{\mathbf{K}}, \tag{139}$$

then give the hidden momentum as

$$\left(\mathbf{i}c(\boldsymbol{\pi} - \frac{1}{c^2}\mathbf{v}\times\boldsymbol{\mu})\times\mathbf{B} - \mathbf{i}\frac{1}{c}(\boldsymbol{\mu} + \mathbf{v}\times\boldsymbol{\pi})\times\mathbf{E} + \mathbf{i}\frac{1}{c}(\mathbf{v}\cdot\boldsymbol{\mu})\mathbf{B} + \mathbf{i}\frac{1}{c}(\mathbf{v}\cdot\boldsymbol{\pi})\mathbf{E}\right)\cdot\hat{\mathbf{K}} \tag{140}$$

$$= \mathbf{i}c\mathbf{p}\cdot\hat{\mathbf{K}} = \left(\mathbf{i}c\boldsymbol{\pi}'\times\mathbf{B} - \mathbf{i}\frac{1}{c}\boldsymbol{\mu}'\times\mathbf{E} + \mathbf{i}\frac{1}{c}\pi'\mathbf{B} - \mathbf{i}\frac{1}{c}\mu'\mathbf{E}\right)\cdot\hat{\mathbf{K}}, \tag{141}$$

move on to the energy

$$U\hat{1} = -(\boldsymbol{\mu} + \mathbf{v}\times\boldsymbol{\pi})\cdot\mathbf{B}\hat{1} - (\boldsymbol{\pi} - \frac{1}{c^2}\mathbf{v}\times\boldsymbol{\mu})\cdot\mathbf{E}\hat{1} = -\boldsymbol{\mu}'\cdot\mathbf{B}\hat{1} - \boldsymbol{\pi}'\cdot\mathbf{E}\hat{1} \tag{142}$$

and finally write down the complex time-like part of T or the scalar hidden momentum

$$\mathbf{i}cp\hat{1} = \mathbf{i}\frac{1}{c}(\boldsymbol{\mu} + \mathbf{v}\times\boldsymbol{\pi})\cdot\mathbf{E}\hat{1} - \mathbf{i}c(\boldsymbol{\pi} - \frac{1}{c^2}\mathbf{v}\times\boldsymbol{\mu})\cdot\mathbf{B})\hat{1} = \mathbf{i}\frac{1}{c}\boldsymbol{\mu}'\cdot\mathbf{E}\hat{1} - \mathbf{i}c(\boldsymbol{\pi}'\cdot\mathbf{B})\hat{1}. \tag{143}$$

This results in a total $T = U\hat{\imath} + \mathbf{i}cp\hat{\imath} + \boldsymbol{\tau}\cdot\hat{\mathbf{K}} + \mathbf{i}c\mathbf{p}\cdot\hat{\mathbf{K}}$ as

$$T = \left(-\boldsymbol{\mu}'\cdot\mathbf{B} - \boldsymbol{\pi}'\cdot\mathbf{E} - \mathbf{i}c\boldsymbol{\pi}'\cdot\mathbf{B} + \mathbf{i}\frac{1}{c}\boldsymbol{\mu}'\cdot\mathbf{E}\right)\hat{\imath} \tag{144}$$

$$+ \left(\boldsymbol{\mu}'\times\mathbf{B} + \boldsymbol{\pi}'\times\mathbf{E} + \pi'\mathbf{E} + \mu'\mathbf{B}\right)\cdot\hat{\mathbf{K}} \tag{145}$$

$$+ \left(\mathbf{i}c\boldsymbol{\pi}'\times\mathbf{B} - \mathbf{i}\frac{1}{c}\boldsymbol{\mu}'\times\mathbf{E} + \mathbf{i}\frac{1}{c}\pi'\mathbf{B} - \mathbf{i}\frac{1}{c}\mu'\mathbf{E}\right)\cdot\hat{\mathbf{K}} \tag{146}$$

If we compare this to the static T which was

$$T_S = MB = \mathbb{T}_0\hat{\imath} + \vec{\mathbb{T}}\cdot\hat{\mathbf{K}} = \left(-\boldsymbol{\mu}\cdot\mathbf{B} - \boldsymbol{\pi}\cdot\mathbf{E} - \mathbf{i}c\boldsymbol{\pi}\cdot\mathbf{B} + \frac{\mathbf{i}}{c}\boldsymbol{\mu}\cdot\mathbf{E}\right)\hat{\imath} \tag{147}$$

$$+ \left(\boldsymbol{\mu}\times\mathbf{B} + \boldsymbol{\pi}\times\mathbf{E} + \mathbf{i}c\boldsymbol{\pi}\times\mathbf{B} - \mathbf{i}\frac{1}{c}\boldsymbol{\mu}\times\mathbf{E}\right)\cdot\hat{\mathbf{K}}, \tag{148}$$

then we see two major differences, the first the appearance of $\boldsymbol{\pi}'$ and $\boldsymbol{\mu}'$ instead of $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ and the second the really new arrivals π' and μ'. We already defined them as

$$\boldsymbol{\pi}' = \boldsymbol{\pi} - \frac{1}{c^2}\mathbf{v}\times\boldsymbol{\mu} \tag{149}$$

$$\boldsymbol{\mu}' = \boldsymbol{\mu} + \mathbf{v}\times\boldsymbol{\pi} \tag{150}$$

$$\pi' = \frac{1}{c^2}\mathbf{v}\cdot\boldsymbol{\mu} \tag{151}$$

$$\mu' = -\mathbf{v}\cdot\boldsymbol{\pi}. \tag{152}$$

In most practical cases, a dipole will move in such a way that we have $\pi' = 0$ and $\mu' = 0$ and then the difference between the static T and the dynamic T will be the replacement of $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ by $\boldsymbol{\pi}'$ and $\boldsymbol{\mu}'$. In general it will be better to use the dynamic T at all times and realize that it reduces to the static T when $\mathbf{v} = 0$. This becomes important for the next section, where we study the electromagnetic forces on moving EM dipole moments due to a divergence of T, when $\partial T \neq 0$.

5. FORCES DUE TO THE DIVERGENCE OF THE ENERGY-TORQUE-HIDDEN-MOMENTUM PRODUCT

5.1. General perspective on ∂T in relativistic mechanics

As we have seen, $T = -\mathbf{i}\frac{1}{c}(VM)B$ represents an energy-momentum-torque product that is presented as a symmetric tensor $T_{\nu\mu}$ in relativistic electrodynamics, so without the torque. In SR and GR, Laue's condition for a conserved energy-momentum is $\partial^\nu T_{\nu\mu} = 0$ and for non-closed systems it gives $\partial^\nu T_{\nu\mu} = F_\mu$ with the relativistic force as a simple four vector. In our language this would be represented by $\partial T = 0$ and $\partial T = F$ or, with $T = V^T P$, by $\partial V^T P = 0$ and $\partial V^T P = F$. But we start with the comparable but not identical $\partial^T P = 0$ condition as a starting point of our alternative relativistic mechanics, a study which eventually should lead to a relativistically invariant force equation for the moving em-dipole in em-field situation. In the case of electrodynamics, when we have the canonical $P = qA$, we have $\partial^T A = B \neq 0$. So in the circumstances analogous to a nonzero anti-symmetric EM field, the condition $\partial^T P = q\partial^T A = 0$ is not fulfilled for the pure EM-field-momentum.

The condition $\partial^T P = 0$ or $(\mathbf{i}\frac{1}{c}\partial_t \hat{\mathrm{I}} + \boldsymbol{\nabla} \cdot \hat{\mathbf{K}})(\mathbf{i}\frac{1}{c}U_i \hat{\mathrm{I}} + \mathbf{p} \cdot \hat{\mathbf{K}})$ leads to

$$\partial^T P = (-\frac{1}{c^2}\partial_t U_i - \boldsymbol{\nabla} \cdot \mathbf{p})\hat{\mathrm{I}} + (\boldsymbol{\nabla} \times \mathbf{p} + \mathbf{i}\frac{1}{c}(\partial_t \mathbf{p} + \boldsymbol{\nabla} U_i)) \cdot \hat{\mathbf{K}} = 0. \tag{153}$$

so to three subconditions

$$\frac{1}{c^2}\partial_t U_i + \boldsymbol{\nabla} \cdot \mathbf{p} = 0 \tag{154}$$

$$\boldsymbol{\nabla} \times \mathbf{p} = 0 \tag{155}$$

$$\partial_t \mathbf{p} = -\boldsymbol{\nabla} U_i. \tag{156}$$

The first one is the continuity equation, the second means that we have zero vorticity and the third that the related force field can be connected to a potential energy. We can take the time derivative of the second condition, giving the conserved force field condition

$$\boldsymbol{\nabla} \times \mathbf{F} = 0. \tag{157}$$

The first condition can also be written as

$$\partial_t m_i + \boldsymbol{\nabla} \cdot (m_i \mathbf{v}) = 0, \tag{158}$$

the continuity equation for inertial mass.

If we have $\partial^T P = 0$ in one system of reference, then in another system of reference we have $(\partial^T P)^L = 0$ or $U(\partial^T P)U^{-1} = 0$. As we have seen, if in this other system we change in the basis $\hat{\mathrm{J}}$ to $\hat{\mathrm{J}}^L$ and $\hat{\mathrm{K}}$ to $\hat{\mathrm{K}}^L$, then we have invariant coordinates and our three subconditions should hold. Of course, we have to remind that we in all cases are restricted to reference systems that have aligned themselves in such a way that the Lorentz velocity between them is in the $\hat{\mathrm{I}}$-direction. In all other situations further study regarding the form invariance under Lorentz transformations is necessary. But it is important to notice that we have a relativistic formulation of a mechanical system representing a central force. This condition $\partial^T P = 0$ does not represent an anti-symmetric Lorentz force but a much simpler central force like the Coulomb force, because it is possible to write U_i as $U_i \approx U_0 + U_k$ and then $\boldsymbol{\nabla} U_i = \boldsymbol{\nabla} U_k$ so we get $\partial_t \mathbf{p} = -\boldsymbol{\nabla} U_k = -\boldsymbol{\nabla}|U_p|$ in the Newtonian limit of a central field.

In the Laue condition $\partial^\nu T_{\nu\mu} = 0$ the stress-energy density tensor is $T_{\nu\mu} = V_\nu G_\mu$. In our math-phys language we would get the not exact analog $T = V^T G$ and $\partial T = 0$, but that would imply a full homogeneous Maxwell-Lorentz structure with the product $\partial V^T G = 0$. Our stress energy density 'tensor' T gives

$$T = V^T G = (u_i - \mathbf{v} \cdot \mathbf{g})\hat{1} + (\mathbf{v} \times \mathbf{g} + \mathbf{i}c(-\mathbf{g} + \frac{1}{c^2} u_i \mathbf{v})) \cdot \hat{\mathbf{K}}. \tag{159}$$

This tensor analog contains all the elements of $T_{\nu\mu} = V_\nu G_\mu$, with the difference that the cross product $\mathbf{v} \times \mathbf{g}$ appears directly in our $T = V^T G$ whereas ony half of it is in the usual tensor and the anti-symmetric tensor product is needed to get the full cross product. We have to look closer to $\mathbf{v} \times \mathbf{g}$. If we start by $\mathbf{L} = \mathbf{r} \times \mathbf{g}$ then $\frac{d}{dt}\mathbf{L} = \frac{d}{dt}(\mathbf{r} \times \mathbf{g}) = \mathbf{v} \times \mathbf{g} + \mathbf{r} \times \mathbf{f}$. So $\mathbf{v} \times \mathbf{g}$ represents a torque due to internal stresses in a body and in the biquaternion environment T contains energy, momentum and torque.

In the case of a symmetric situation $\mathbf{v}$ has the same direction as $\mathbf{g}$, resulting in

$$T = (u_i - \mathbf{v} \cdot \mathbf{g})\hat{1} = u_0 \hat{1} \tag{160}$$

$$\mathbf{v} \times \mathbf{g} = 0 \tag{161}$$

$$\mathbf{g} = \frac{1}{c^2} u_i \mathbf{v}. \tag{162}$$

The third equation contains the mass-energy density equivalence $u_i = \rho_i c^2$, but it also implies the absence of linear stresses. The second equation implies the absence of angular stresses or torque. The first equation equals the scalar Lagrangian density, the trace of the Laue mechanical stress-energy density tensor. Then the divergence of the symmetric T gives the four force density $\mathcal{F} = -\partial T$ as $\mathcal{F} = -\partial u_0$. The direct parallel in electromagnetics would be that $\overrightarrow{\mathbb{B}} = 0$ with a nonzero Lorenz gauge $\mathbb{B}_0$ for the field and that $T_{EM} = J^T A = \rho_0 \phi_0 \hat{1}$, with $\mathcal{F} = -\partial \rho_0 \phi_0$. In the rest system this would produce a Coulomb force density and a Coulomb force power, which, for a static potential, would be zero. Thus in our relativistic dynamics, in the symmetric case the electromagnetic parallel would only produce a Coulomb force situation.

Only if $\mathbf{v}$ doesn't have the same direction as $\mathbf{g}$ will there be an anti-symmetric component present that is analog to the structure of the Maxwell-Lorentz electromagnetic field. The Lorentz force is given as $JB = F$, which can be written as $qV\partial^T A = F$ which, by using $P = qA$, results in the mechanical analog $V\partial^T P = F$. This still isn't the full $\partial V^T P = -F$. The Lorentz force law analog in our relativistic dynamics implies that $\partial^T P \neq 0$, so that $m_0\partial^T U \neq 0$. If we look closer at $V\partial^T$, we see that it contains the three parts

$$(-\frac{\partial}{\partial t} - \mathbf{v} \cdot \boldsymbol{\nabla})\hat{1} = -\frac{d}{dt}\hat{1} \tag{163}$$

$$(\mathbf{v} \times \boldsymbol{\nabla}) \cdot \hat{\mathbf{K}} \tag{164}$$

$$\mathbf{i}c(\boldsymbol{\nabla} + \frac{1}{c^2}\mathbf{v}\partial_t) \cdot \hat{\mathbf{K}}. \tag{165}$$

So the product $-V\partial^T$ is our variant of the absolute derivative, with $\frac{d}{dt}\hat{1}$ as the scalar part of it. Thus if we go from $\partial^T P = 0$ to $V\partial^T P = F$, we move in our relativistic mechanics from a pure Coulomb force structure or environment to a Lorentz force one, related to a move in our biquaternion environment from a partial derivative to an absolute derivative.

What becomes apparent is that we have a lot of highly relevant relativistic mechanics before we arrive at analog of the traditional Laue product in its full, non-symmetric realisation $\partial V^T G = -\mathcal{F}$. It seems that the divergence of the stress energy density tensor looses its central role, and thereby also its function as a role-model. Especially when we realize that the Minkowski-Laue principle example, $\partial^\nu T^{EM}_{\nu\mu} = -\mathcal{F}_\mu$, is a highly compact, complex expression containing in our translation the energy products BB, BB^T and the equations $\partial B = \mu_0 J$ and $JB = \mathcal{F}$, not all of them in simple Lorentz invariance combinations. In our math-phys language, the compactified Minkowski-Laue equation's content is spread out over several products and equations existing at different layers of complexity.

One important difference between the Minkowski-Laue stress-energy tensor and connected force equations and the biquaternion version is that in the biquaternion stress-energy tensor, the stresses are directly given as a torsion or a torque. In the resulting force equation, the torque is directly implied. Another difference that concerns us here is that the torque-energy-hidden-momentum product $T = -\mathbf{i}\frac{1}{c}(VM)B$ results in a four-vector like T and not a tensor or six-vector like T. That has implications for the form of the force equation. For a six-vector T, the adequate force equation is $F = -\partial T$, resulting in a four-vector force. But for a four-vector T, the adequate force equation is $F = -\partial^T T$, resulting in a six-vector force. Of course, the product of the static EM torque-energy product $T = MB$ is a six-vector like T with the appropriate force equation $F = -\partial T$. In the following subsection we start by showing that we cannot use both because that would lead to paradoxes if we compare a static case to a moving case brought to a standstill and thus becoming static. From a physics point of view, these two cases should be identical.

5.2. The force equation due to a divergent torque-energy-hidden-momentum

The total torque-energy-hidden-momentum product can be represented as $T = \mathbb{T}_0\hat{\mathbf{1}} + \vec{\mathbb{T}}\cdot\hat{\mathbf{K}}$. In the static case $T = MB$ transforms like a six-vector or a tensor and the logical, Lorentz form invariant force equation would seem

$$F = -\partial T = (\mathbf{i}\frac{1}{c}\partial_t\mathbb{T}_0 + \boldsymbol{\nabla}\cdot\vec{\mathbb{T}})\hat{\mathbf{1}} + (-\boldsymbol{\nabla}\times\vec{\mathbb{T}} - \boldsymbol{\nabla}\mathbb{T}_0 + \mathbf{i}\frac{1}{c}\partial_t\vec{\mathbb{T}})\cdot\hat{\mathbf{K}} \tag{166}$$

But in the dynamic case $T = -\mathbf{i}\frac{1}{c}(VM)B$ transforms as a four vector and we have the Lorentz form invariant force equation $F = -\partial^T T$ and we would arrive at a force equation

$$F = -\partial^T T = -(\mathbf{i}\frac{1}{c}\partial_t\hat{\mathbf{1}} + \boldsymbol{\nabla}\cdot\hat{\mathbf{K}})(\mathbb{T}_0\hat{\mathbf{1}} + \vec{\mathbb{T}}\cdot\hat{\mathbf{K}}) = \tag{167}$$

$$\left(-\mathbf{i}\frac{1}{c}\partial_t\mathbb{T}_0 + \boldsymbol{\nabla}\cdot\vec{\mathbb{T}}\right)\hat{\mathbf{1}} + \left(-\boldsymbol{\nabla}\times\vec{\mathbb{T}} - \boldsymbol{\nabla}\mathbb{T}_0 - \mathbf{i}\frac{1}{c}\partial_t\vec{\mathbb{T}}\right)\cdot\hat{\mathbf{K}} \tag{168}$$

in which the element ∂_t changes sign compared to the static case. Because the dynamic case is more general than the static situation, and due to the fact that the dynamic case can always be reduced to the static case by applying $\mathbf{v} \to 0$, in the following we choose the expression $F = -\partial^T T$ with $T = -\mathbf{i}\frac{1}{c}(VM)B$. We cannot use both this T and $T = MB$ because then we would produce contradictions by comparing the static case to dynamic cases where we slowly reduce $\mathbf{v}$ to zero.

We calculated the expression for $F = -\partial^T T = -\partial^T(-\mathbf{i}\frac{1}{c}(VM)B)$ without the elements with the scalar π' and μ' in them to be

$$F = \left(\partial_t(\boldsymbol{\pi}' \cdot \mathbf{B}) + \frac{1}{c^2}\partial_t(\boldsymbol{\mu}' \cdot \mathbf{E}) + \boldsymbol{\nabla} \cdot (\boldsymbol{\mu}' \times \mathbf{B}) + \boldsymbol{\nabla} \cdot (\boldsymbol{\pi}' \times \mathbf{E})\right) \hat{\imath} + \tag{169}$$

$$\mathbf{i}\left(\frac{1}{c}\partial_t(\boldsymbol{\mu}' \cdot \mathbf{B}) + \frac{1}{c}\partial_t(\boldsymbol{\pi}' \cdot \mathbf{E}) + c\boldsymbol{\nabla} \cdot (\boldsymbol{\pi}' \times \mathbf{B}) - \frac{1}{c}\boldsymbol{\nabla} \cdot (\boldsymbol{\mu}' \times \mathbf{E})\right) \hat{\imath} + \tag{170}$$

$$\left(\boldsymbol{\nabla}(\boldsymbol{\mu}' \cdot \mathbf{B}) - \boldsymbol{\nabla} \times (\boldsymbol{\mu}' \times \mathbf{B}) - \frac{1}{c^2}\partial_t(\boldsymbol{\mu}' \times \mathbf{E})\right) \cdot \hat{\mathbf{K}} +$$

$$(\boldsymbol{\nabla}(\boldsymbol{\pi}' \cdot \mathbf{E}) - \boldsymbol{\nabla} \times (\boldsymbol{\pi}' \times \mathbf{E}) + \partial_t(\boldsymbol{\pi}' \times \mathbf{B})) \cdot \hat{\mathbf{K}} + \tag{171}$$

$$\mathbf{i}\left(c\boldsymbol{\nabla}(\boldsymbol{\pi}' \cdot \mathbf{B}) - c\boldsymbol{\nabla} \times (\boldsymbol{\pi}' \times \mathbf{B}) - \frac{1}{c}\partial_t(\boldsymbol{\pi}' \times \mathbf{E})\right) \cdot \hat{\mathbf{K}} +$$

$$\mathbf{i}\left(-\frac{1}{c}\boldsymbol{\nabla}(\boldsymbol{\mu}' \cdot \mathbf{E}) + \frac{1}{c}\boldsymbol{\nabla} \times (\boldsymbol{\mu}' \times \mathbf{E}) - \frac{1}{c}\partial_t(\boldsymbol{\mu}' \times \mathbf{B})\right) \cdot \hat{\mathbf{K}}, \tag{172}$$

in which the real vector part (171) represents the force and the complex time like part (170) represents the connected power. But in the total expression of F we also get parts due to π' and μ'. The extra T, not yet given in (169)-(172), is

$$T_{extra} = \vec{\mathbb{T}} \cdot \hat{\mathbf{K}} = (-\mu'\mathbf{B} - \pi'\mathbf{E}) \cdot \hat{\mathbf{K}} + \left(-\mathbf{i}\frac{1}{c}\pi'\mathbf{B} + \mathbf{i}\frac{1}{c}\mu'\mathbf{E}\right) \cdot \hat{\mathbf{K}} \tag{173}$$

which results in extra terms

$$F = \left(\boldsymbol{\nabla} \cdot \vec{\mathbb{T}}\right) \hat{\imath} + \left(-\boldsymbol{\nabla} \times \vec{\mathbb{T}} - \mathbf{i}\frac{1}{c}\partial_t\vec{\mathbb{T}}\right) \cdot \hat{\mathbf{K}} \tag{174}$$

so with $\pi' = \frac{1}{c^2}\mathbf{v} \cdot \boldsymbol{\mu}$ and $\mu' = -\mathbf{v} \cdot \boldsymbol{\pi}$

$$F = (\boldsymbol{\nabla} \cdot (+\mu'\mathbf{B} + \pi'\mathbf{E}))\, \hat{\imath} \tag{175}$$

$$+ \left(\boldsymbol{\nabla} \cdot \left(+\mathbf{i}\frac{1}{c}\pi'\mathbf{B} - \mathbf{i}\frac{1}{c}\mu'\mathbf{E}\right)\right) \hat{\imath} \tag{176}$$

$$\left(-\boldsymbol{\nabla} \times (\mu'\mathbf{B} + \pi'\mathbf{E}) - \mathbf{i}\frac{1}{c}\partial_t\left(\mu'\mathbf{B} + \pi'\mathbf{E}\right)\right) \cdot \hat{\mathbf{K}} \tag{177}$$

$$\left(-\boldsymbol{\nabla} \times \left(\mathbf{i}\frac{1}{c}\pi'\mathbf{B} + \mathbf{i}\frac{1}{c}\mu'\mathbf{E}\right) + \frac{1}{c}\partial_t\left(\frac{1}{c}\pi'\mathbf{B} - \frac{1}{c}\mu'\mathbf{E}\right)\right) \cdot \hat{\mathbf{K}} \tag{178}$$

The real vector part of this represents the extra three-vector force, giving

$$\mathbf{F}_{extra} \cdot \hat{\mathbf{K}} = \left(\boldsymbol{\nabla} \times (-\mu'\mathbf{B} - \pi'\mathbf{E}) + \frac{1}{c}\partial_t\left(\frac{1}{c}\pi'\mathbf{B} - \frac{1}{c}\mu'\mathbf{E}\right)\right) \cdot \hat{\mathbf{K}} \tag{179}$$

5.3. The force on a moving magnetic dipole including the hidden momentum

If we from here apply some heuristics from practical situations and assume that $\pi' = 0$ and $\mu' = 0$, so with the velocity of the moving dipole moment perpendicular $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$. Then the real, or $\Re$, three-vector force in the moving case reduces to Eqn. (171), so

$$\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\mu}' \cdot \mathbf{B}) - \boldsymbol{\nabla} \times (\boldsymbol{\mu}' \times \mathbf{B}) - \frac{1}{c^2}\partial_t(\boldsymbol{\mu}' \times \mathbf{E}) + \\ \boldsymbol{\nabla}(\boldsymbol{\pi}' \cdot \mathbf{E}) - \boldsymbol{\nabla} \times (\boldsymbol{\pi}' \times \mathbf{E}) + \partial_t(\boldsymbol{\pi}' \times \mathbf{B}).$$

Then assume $\boldsymbol{\pi}' = 0$ to get

$$\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\mu}' \cdot \mathbf{B}) - \boldsymbol{\nabla} \times (\boldsymbol{\mu}' \times \mathbf{B}) - \frac{1}{c^2}\partial_t(\boldsymbol{\mu}' \times \mathbf{E}). \tag{180}$$

This result is very close to the force in the Vekstein, Kholmetskii-Missevitch-Yarman and Hnizdo discussion.[7],[8],[9],[10],[11],[12],[13],[14] But if the velocity and the dipole moments have parallel elements, then the extra force terms cannot be neglected. We arrived at this result by reduction of the Lorentz form invariant equation $F = -\partial^T T = -\partial^T(-\mathbf{i}\frac{1}{c}(VM)B)$, with as simplifying assumptions that $\pi' = 0$, $\mu' = 0$ and $\boldsymbol{\pi}' = 0$ and then focussing on the $\Re$ part of the resulting expression.

In our opinion, trying to find a Lorentz invariant formulation of Eqn. (180) by starting with Eqn.(180) and then puzzle your way towards a Lorentz invariant formulation seems a sheer impossible task.[15],[16] The only approach that seems to make sense is to go back to the Lorentz form invariant equation $F = -\partial^T T = -\partial^T(-\mathbf{i}\frac{1}{c}(VM)B)$, apply the Lorentz transformation and then start again but now with the analysis with the restrictions in that specific frame of reference.

Another important remark, relating back to the force equation $F = -\partial MB$, is that this force equation would lead to a static

$$\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\mu} \cdot \mathbf{B}) - \boldsymbol{\nabla} \times (\boldsymbol{\mu} \times \mathbf{B}) + \frac{1}{c^2}\partial_t(\boldsymbol{\mu} \times \mathbf{E}) \tag{181}$$

so to an inverse force due to the hidden momentum as compared to the moving case Eqn. (180) with the additional assumption of $\mathbf{v} \to 0$.

5.4. The appropriate force equation and the Ahanorov Casher force as an application

The Aharonov Casher (AC) effect deals with the force on a magnetic dipole moment moving in an electric field, with $\boldsymbol{\pi} = 0$ and $\mathbf{B} = 0$. In the AC setup, $\boldsymbol{\mu} \cdot \mathbf{E} = 0$ and $\mathbf{v} \cdot \boldsymbol{\mu} = 0$

and then also $(\mathbf{v}\times\boldsymbol{\mu})\times\mathbf{E}=0$. A direct calculus starts with Eqn. (171)

$$\mathbf{F}=\boldsymbol{\nabla}(\boldsymbol{\mu}'\cdot\mathbf{B})-\boldsymbol{\nabla}\times(\boldsymbol{\mu}'\times\mathbf{B})-\frac{1}{c^2}\partial_t(\boldsymbol{\mu}'\times\mathbf{E})+$$
$$\boldsymbol{\nabla}(\boldsymbol{\pi}'\cdot\mathbf{E})-\boldsymbol{\nabla}\times(\boldsymbol{\pi}'\times\mathbf{E})+\partial_t(\boldsymbol{\pi}'\times\mathbf{B}).$$

and then applying $\mathbf{B}=0$ leads to

$$\mathbf{F}=-\frac{1}{c^2}\partial_t(\boldsymbol{\mu}'\times\mathbf{E})+\boldsymbol{\nabla}(\boldsymbol{\pi}'\cdot\mathbf{E})-\boldsymbol{\nabla}\times(\boldsymbol{\pi}'\times\mathbf{E}) \tag{182}$$

after which we can use $\boldsymbol{\mu}'=\boldsymbol{\mu}+\mathbf{v}\times\boldsymbol{\pi}$ with $\boldsymbol{\pi}=0$ so $\boldsymbol{\mu}'=\boldsymbol{\mu}$ and use $\boldsymbol{\pi}'=\boldsymbol{\pi}-\frac{1}{c^2}\mathbf{v}\times\boldsymbol{\mu}$ with $\boldsymbol{\pi}=0$, so $\boldsymbol{\pi}'=-\frac{1}{c^2}\mathbf{v}\times\boldsymbol{\mu}$ in order to get at

$$\mathbf{F}=-\frac{1}{c^2}\partial_t(\boldsymbol{\mu}\times\mathbf{E})-\frac{1}{c^2}\boldsymbol{\nabla}((\mathbf{v}\times\boldsymbol{\mu})\cdot\mathbf{E})+\frac{1}{c^2}\boldsymbol{\nabla}\times((\mathbf{v}\times\boldsymbol{\mu})\times\mathbf{E}). \tag{183}$$

In the AC setup $(\mathbf{v}\times\boldsymbol{\mu})\times\mathbf{E}=0$. We apply this to get

$$\mathbf{F}_{ac}=-\frac{1}{c^2}\partial_t(\boldsymbol{\mu}\times\mathbf{E})-\frac{1}{c^2}\boldsymbol{\nabla}((\mathbf{v}\times\boldsymbol{\mu})\cdot\mathbf{E}). \tag{184}$$

Because in the AC case the E-field is time independent, we get

$$\mathbf{F}_{ac}=\frac{1}{c^2}\mathbf{E}\times\partial_t\boldsymbol{\mu}-\frac{1}{c^2}\boldsymbol{\nabla}((\mathbf{v}\times\boldsymbol{\mu})\cdot\mathbf{E}). \tag{185}$$

Using $\boldsymbol{\pi}'\cdot\mathbf{E}=\boldsymbol{\mu}\cdot\mathbf{B}'$ this can be rewritten as

$$\mathbf{F}_{ac}=\frac{1}{c^2}\mathbf{E}\times\partial_t\boldsymbol{\mu}+\boldsymbol{\nabla}(\boldsymbol{\mu}\cdot(-\frac{1}{c^2}\mathbf{v}\times\mathbf{E})) \tag{186}$$

and then using $\boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\mathbf{B}')=(\boldsymbol{\mu}\cdot\boldsymbol{\nabla})\mathbf{B}'$, it can be written as

$$\mathbf{F}_{ac}=\frac{1}{c^2}\mathbf{E}\times\partial_t\boldsymbol{\mu}-(\boldsymbol{\mu}\cdot\boldsymbol{\nabla})(\frac{1}{c^2}\mathbf{v}\times\mathbf{E}) \tag{187}$$

which is the Ahanorov-Casher force.[18],[19]

6. ON MANSURIPURS PARADOX, PLASMA PHYSICS AND THE BIQUATERNION ENVIRONMENT

Resently, the correctness of the Lorentz Force Law and especially its invariance under a Lorentz transformation was questioned by Mansuripur.[20],[21],[22] We quote from Griffeths "Mansuripur's Paradox",[23]

> On May 7, 2012, a remarkable article appeared in *Physical Review Letters*.[24] The author, Masud Mansuripur, claimed to offer "incontrovertible theoretical evidence of the incompatibility of the Lorentz [force] law with the fundamental tenets of special relativity," and concluded that "the Lorentz law must be abandoned." The Lorentz law,
>
> $$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \tag{188}$$
>
> tells us the force $\mathbf{F}$ on a charge q moving with velocity $\mathbf{v}$ through electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Together with Maxwell's equations, it is the foundation on which all of classical electrodynamics rests. If it is incorrect, 150 years of theoretical physics is in serious jeopardy.
>
> Such a provocative proposal was bound to attract attention. *Science*[25] published a full-page commentary, and within days several rebuttals were posted.[26] Critics pointed out that since the Lorentz force law can be embedded in a manifestly covariant formulation of electrodynamics, it is guaranteed to be consistent with special relativity,[27] and and some of them identified the specific source of Mansuripur's error: neglect of "hidden momentum." Nearly a year later *Physical Review Letters* published four rebuttals,[28] and *Science* printed a follow-up article declaring the "purported relativity paradox resolved."[29]
>
> Mansuripur's argument is based on a "paradox" that was explored in this journal by Victor Namias and others[30] many years ago: a magnetic dipole moving through an electric field can experience a torque, with no accompanying rotation.

In our analysis of the Lorentz Force Law, it is not an incorrect law of physics, but an incomplete one. Our expansion is the Lorentz-Larmor Law $F = JB$, which includes a four-vector Larmor angular velocity. It is related to the energy-momentum product $T = -\frac{1}{2\mu_0}BB$, where the usual $U(1)$ EM Lagrangian only contains the real vector part of this T. The Lorentz-Larmor Law is then given by $F = -\partial T = \frac{1}{2\mu_0}\partial(BB) = JB = qVB$

From there we can show the mathematical connection to forces on hidden momenta. Standard EM-relation between M and B gives in vacuum $\boldsymbol{\pi} = -\epsilon_0\mathbf{E}$ and $\boldsymbol{\mu} = \frac{1}{\mu_0}\mathbf{B}$. So $M = \boldsymbol{\mu} + \mathbf{i}c\boldsymbol{\pi} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{i}c\epsilon_0\mathbf{E} = \frac{1}{\mu_0}(\mathbf{B} - \mathbf{i}\frac{1}{c}\mathbf{E}) = \frac{1}{\mu_0}B$ or $M = \frac{1}{\mu_0}B_m$.

Apart from a sign problem and a factor half problem, we have a perfectly matching situation because $T = -\frac{1}{\mu_0} B_m B = -MB$ and $F = -\partial T = \partial(MB) = \partial(\frac{1}{\mu_0} B_m B) = J_m B$. We already know that $T = -MB$ with $F = -\partial T = \partial(MB)$.

For dynamic or moving electromagnetic dipoles we have the torque-energy-hidden-momentum T as $T = -\mathbf{i}\frac{1}{c}(VM)B$ with a related force as $F = -\partial^T T = -\partial^T(-\mathbf{i}\frac{1}{c}(VM)B)$. We want to relate this to the Lorentz-Larmor Law using $M = \frac{1}{\mu_0} B_m$ so $F = -\partial^T T = -\partial^T(-\mathbf{i}\frac{1}{c}(\frac{1}{\mu_0} V B_m)B) = \mathbf{i}\frac{1}{c\mu_0}\partial^T((VB_m)B)$. What we have here is a moving EM-field relative to a static EM-field, showing the continuity of the situation.

In the above, we skipped through all the details important for a correct physical analysis and only looked at the basis mathematical structures involved. This very fast survey shows the basis continuity of our biquaternion structure from the Lorentz-Larmor Law to the forces on moving electromagnetic dipoles. The different equations are Lorentz form invariant.

We believe that this biquaternion environment can be very useful for all physics contexts involving Larmor frequencies and not yet controlled forces on charges in complex plasma-like environments. We think of Tokamak plasma's in fusion physics, cyclotron particle accelerators and related contexts. What we have done in this paper was a further reconnaissance of the environment produced by our version of biquaternion physics, as applied to the domain of electrodynamics.

---

* haas2u@gmail.com

1 Pauli, W.: *Theory of Relativity*. Pergamon, New York (1958).

2 Minkowski, H.: "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern". Math. Ann. **6**8, 472-525 (1910); Nachr. Ges. Wiss. Göttingen, 53 (1908).

3 Dzhordzhadze, V., et al., "Accelerator Based Magnetic Monopole Search Experiments", Brookhaven National Laboratory, 2007,

4 Thomas, L.H.: The Motion of the Spinning Electron. Nature (London) 117, 524 (1926)

5 Thomas, L.H.: The Kinematics of an Electron with an Axis. Philos. Mag. 3, 1-22 (1927)

6 Frenkel, J.: Die Elektrodynamik des rotierenden Elektrons. Zeit. f. Phys. 37, 243-262 (1926)

7 Shockley, W. and James, R.P., " 'Try simplest cases' discovery of 'hidden momentum' forces on 'magnetic currents'," Phys. Rev. Lett. **18**, 876-879 (1967)

[8] Vaidman, L., “Torque and force on a magnetic dipole,” Am. J. Phys. **58**, 978-983 (1990).

[9] Vekstein, G. E., ”On the electromagnetic force on a moving dipole,” Eur. J. Phys., Vol. 18, 113-117, 1997.

[10] Kholmetskii, A. L., ”On ’hidden momentum’ of magnetic dipoles,” Apeiron, Vol. 12, 309–329 (2005).

[11] Kholmetskii, A. L., Missevitch, O. V. and Yarman, T., On the classical analysis of spin-orbit coupling in hydrogen-like atoms, Am. J. Phys. 78, 428432 (2010).

[12] Kholmetskii, A. L., O. V. Missevitch, and T. Yarman, ”Electromagnetic force on a moving dipole,” Eur. J. Phys., Vol. 32, 873, 2011.

[13] Kholmetskii, A. L., Missevitch, O. V. and Yarman, T., “Torque on a Moving Electric/Magnetic Dipole,” Prog. Electromagn. Res. B **45**, 83-99 (2012).

[14] Hnizdo, V., ”Comment on ‘Electromagnetic force on a moving dipole’, ” Eur. J. Phys., 33, L3–L6 (2012)

[15] Hnizdo, V., Hidden momentum and the electromagnetic mass of a charge and current carrying body, Am. J. Phys. 65, 5565 (1997).

[16] Comay, E., Lorentz transformation of a system carrying Hidden Momentum, Am. J. Phys. 68, 1007–1013 (2000).

[17] Hnizdo, V. Response to Lorentz transformation of a system carrying Hidden Momentum, Am. J. Phys. 68, 1014–1015 (2000).

[18] Aharonov, Y., Pearle, P., and Vaidman, L., ”Comment on ”Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag, ” Phys. Rev. A 37, 4052-4055 (1988)

[19] Al-Jaber S. M., Zhu, X. and Hennenberger, W. C. ”Interaction of a moving magnetic dipole with a static electric field”, Eur. J. Phys. 12 266-270, 1991

[20] Mansuripur, M., Trouble with the Lorentz law of force: Incompatibility with special relativity and momentum conservation, Phys. Rev. Lett. 108, 193901, 1–4 (2014).

[21] Mansuripur, M., ”The Lorentz force law and its connections to hidden momentum, the Einstein-Laub force, and the Aharonov-Casher effect,” IEEE Transactions on Magnetics, Vol. 50, No. 4, 1300110, 1–10 (2014).

[22] McDonald, K. T., ”Mansuripur’s Paradox,” www.physics.princeton.edu/mcdonald/examples/mansuripur.pdf (14 pp).

[23] Griffiths, D. J. and Hnizdo, V., "Mansuripur's paradox," Am. J. Phys. 81, 570–576 (2013)

[24] M. Mansuripur, "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation," Phys. Rev. Lett. **108**, 193901(4) (2012).

[25] A. Cho, "Textbook Electrodynamics May Contradict Relativity," Science **336**, 404 (2012).

[26] K. T. McDonald, "Mansuripur's Paradox," ⟨ www.physics.princeton.edu/∼mcdonald/examples/mansuripur.pdf⟩ (14 pp); D. A. T. Vanzella, "Comment on 'Trouble with the Lorentz law of force,'" e-print arXiv:1205.1502 (2 pp); D. J. Cross, "Resolution of the Mansuripur Paradox," e-print arXiv:1205:5451 (3 pp); P. L. Saldanha, "Comment on 'Trouble with the Lorentz law of force,'" e-print arXiv:1205:6858 (2 pp); D. J. Griffiths and V. Hnizdo, "Comment on 'Trouble with the Lorentz law of force,'" e-print arXiv:1205.4646 (3 pp). Later critiques include K. A. Milton and G. Meille, "Electromagnetic Angular Momentum and Relativity," e-print arXiv:1208:4826 (4 pp); F. De Zela, "Comment on 'Trouble with the Lorentz law of force,'" e-print arXiv:1210.7344 (2 pp); T. M. Boyer, "Examples and comments related to relativity controversies," Am. J. Phys. **80**, 962-971 (2012); A. L. Kholmetskii, O. V. Missevitch, and T. Yarman, "Torque on a Moving Electric/Magnetic Dipole," Prog. Elecromagnetics Research B, **45**, 83-99 (2012). See also M. Mansuripur, "Trouble with the Lorentz Law of Force: Response to Critics," Proc. SPIE, **8455**, 845512 (2012).

[27] The details are worked out in Cross and Vanzella (ref. 3).

[28] D. A. T. Vanzella,"Comment on 'Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation,'" Phys. Rev. Lett. **110**, 089401-1 (2013); and, under the same title, S. M. Barnett, Phys. Rev. Lett. **110**, 089402-1 (2013); P. L. Saldanha, Phys. Rev. Lett. **110**, 089403-1-2 (2013); M. Khorrami, Phys. Rev. Lett. **110**, 089404-1 (2013).

[29] "Paradox Lost," Science **339**, 496 (2013); A. Cho, "Purported Relativity Paradox Resolved," ⟨http://scim.ag/Lorpara⟩ (2013). Mansuripur himself does not accept this verdict, though he does appear to have softened his assertions somewhat: M. Mansuripur, "Mansuripur replies," Phys. Rev. Lett. **110**, 089405-1 (2013).

[30] V. Namias, "Electrodynamics of moving dipoles: The case of the missing torque," Am. J. Phys. **57**, 171-177 (1989); D. Bedford and P. Krumm, "On the origin of magnetic dynamics," Am. J. Phys. **54**, 1036-1039 (1986).