Author Dr Mark Timothy Sheldrick

Power Series Solution for Photon Trajectories In Static Stationary Spherically Symmetric Gravitational Field

Abstract

Here a simple power series solution for the path of a photon in a static, stationary spherically symmetric gravitational field in a vacuum (described by the Schwarzschild solution) is derived in the form of a power series of the angular coordinate (φ) of the photon around the axis in spherical coordinates (r,θ,φ). Standard results for photon trajectories under the influence of gravity are derived by this method. The series solution simplifies where there is a part of the trajectory with a zero first derivative (tangent) and simplifies in a different way when the trajectory crosses the r=3m radius.

Keywords

Angular coordinate, Black Hole, Geodesic, Power Series, Relativity, Schwarzschild solution,

Introduction

For the Schwarzschild solution, a static, stationary, spherically symmetric gravitational field in a vacuum, there are many ways of describing orbits and trajectories of particles and photons. Here will be described a method for determining the path of a photon based on a power series of the angular coordinate of the photon.

Previous methods used have been based on other features, like using the energy and angular momentum to determining turning points in the possible orbits or trajectories [1] [2] [3]. Another method used previously was based on using the power series of 1/r, the radial distance of the photon [4] [5]. Numeric integration, or straight computer simulations by iterative calculations of the path of photons in a gravity field have been done [6] [7] [12] [16] [19], which can be extended to the more complicated case of the Kerr solution [10] [11] [17] [8] [9]. Calculations using the weak field approximations are also in use] [20]. Various other analytic methods have also been used, like the use of elliptical functions [14] [13] [15] [18]

This paper introduces another method as an extra tool in the analysis of the behaviour of photons in the Schwarzschild solution. The power series of the angular coordinate has been used as a basis for an approximation by [21] and [22] but they used only up to second order terms to make the approximation at the solving stage.

Relativistic Equation Definition

The starting point of this analysis will be the equation:

1) $$d^2u/d\varphi^2 + u = 3mu^2$$

This is the exact relativistic equation governing the trajectories of all photons in orbits around a massive central object that is spherically symmetric. This is taken from “Relativity, Special, General, and Cosmological”. 2nd Edition by Wolfgang Rindler [Ref 1], Chapter 11 section 11 Equation 11.62. It is derived there for the static, stationary Schwarzschild solution of a spherically symmetric gravitational field in a vacuum. Note that this is for a central massive object that is not rotating and has no net charge or external magnetic field.

Here $m = GM/c^2$

Where G = Gravitational Constant

M = Mass of the central object providing the gravitational field

c = Velocity of Light

and $u = 1/r$

Where r = Radial distance of photon from the centre by the ‘circumference’ method.

and φ = angular position of photon in the orbital plane.

In general u will be a function of φ. So the appropriate form of the equation for u will be:

$u(\varphi) = 1/r(\varphi)$

and so:

1.1) $d^2u(\varphi)/d\varphi^2 + u(\varphi) = 3mu(\varphi)^2$

Power Series Definition

To solve this differential equation we will assume that the function $u(\varphi)$ can be represented as the power series:

2) $u(\varphi) = A_0 + A_1\varphi + A_2\varphi^2 + A_3\varphi^3 + A_4\varphi^4 + A_5\varphi^5 + A_6\varphi^6 + \dots$

By differentiating equation 2) twice we get:

3) $d^2u(\varphi)/d\varphi^2 = 2A_2 + 2.3.A_3\varphi + 3.4.A_4\varphi^2 + 4.5.A_5\varphi^3 + 5.6.A_6\varphi^4 + 6.7.A_6\varphi^5 + 7.8.A_6\varphi^6 \dots$

By squaring equation 2) we get:

4) $u(\varphi)^2 = A_0^2 + 2A_0 A_1\varphi + (2A_0A_2 + A_1^2)\varphi^2 + (2A_0A_3 + 2A_1A_2)\varphi^3 + (2A_0A_4 + 2A_1A_3 + A_2^2)\varphi^4$

$+ (2A_0A_5 + 2A_1A_4 + 2A_2A_3)\varphi^5 + (2A_0A_6 + 2A_1A_5 + 2A_2A_4 + A_3^2)\varphi^6 + \dots$

By substituting equations 2, 3 and 4 into equation 1.1 and rearranging, we get the following result:

5) $2.A_2 + 2.3.A_3\varphi + 3.4.A_4\varphi^2 + 4.5.A_5\varphi^3 + 5.6.A_6\varphi^4 + 6.7.A_7\varphi^5 + 7.8.A_8\varphi^6 + \dots =$

$3m\,[A_0^2 + 2A_0 A_1\varphi + (2A_0A_2 + A_1^2)\varphi^2 + (2A_0A_3 + 2A_1A_2)\varphi^3 + (2A_0A_4 + 2A_1A_3 + A_2^2)\varphi^4$
$+ (2A_0A_5 + 2A_1A_4 + 2A_2A_3)\varphi^5 + (2A_0A_6 + 2A_1A_5 + 2A_2A_4 + A_3^2)\varphi^6 + \dots]$

$-\,[A_0 + A_1\varphi + A_2\varphi^2 + A_3\varphi^3 + A_4\varphi^4 + A_5\varphi^5 + A_6\varphi^6 + \dots]$

Power Series Resolution

Now by the standard technique of noting that as this equation holds for all values of φ then the factors of each power of φ must match, we get the series of equations:

6.0) $2.A_2 = 3m.A_0^2 - A_0$

6.1) $2.3.A_3 = 3m.2.A_0 A_1 - A_1$

6.2) $3.4.A_4 = 3m.(2A_0A_2+A_1^2) - A_2$

6.3) $4.5.A_5 = 3m.(2A_0A_3+2A_1A_2) - A_3$

6.4) $5.6.A_6 = 3m.(2A_0A_4+2A_1A_3+A_2^2) - A_4$

6.5) $6.7.A_7 = 3m.(2A_0A_5+2A_1A_4+2A_2A_3) - A_5$

6.6) $7.8.A_8 = 3m.(2A_0A_6+2A_1A_5+2A_2A_4+A_3^2) – A_6$

etc.

Now we will pick the initial conditions that will make the solution to these equations as easy as possible.

Simple Case Where Trajectory Does Not Cross 3m Radius

The following derivation is for the situation where the closest approach of the photon trajectory to the central mass is > 3m or for a trajectory contained within the 3m radius. This results in a simpler set of terms.

For the case of the trajectory being all outside the 3m radius, we will set the zero point of the angle φ to be at the point of closest approach of the photon trajectory to the central object. We can expect the path to be symmetric on either side of this point and we will set the value of u at this point to be $u_0 = 1/r_0$.

For the case of the trajectory being all within the 3m radius, the zero point of the angle φ is set to be at the point of greatest distance of the photon trajectory from the central object, we know this to be so from previous analysis and so take advantage of that knowledge. We can expect the path to be symmetric on either side of this point and we will set the value of u at this point to be $u_0 = 1/r_0$.

More importantly, at this point (for both cases) the rate of change of u with respect to φ will be zero as it has to be at a tangent (90^o) to the direction of r, to the central point.

From equation 2)

7) $u(0) = A_0 = u_0$

The single differential of equation 2) is:

8) $du(\varphi)/d\varphi = A_1 + 2A_2\varphi + 3A_3\varphi^2 + \ldots.$

At φ=0 equation 8) becomes:

9) $du(0)/d\varphi = A_1 = 0$

With these first two results we can now solve the set of equations in sequence:

10.1) $A_2 = (3m.u_0^2 - u_0)/2$

10.2) $A_3 = (3m.2.A_0\,A_1 - A_1\,)/2.3 = 0$

10.3) $A_4 = (3m.(2A_0A_2+A_1^2) - A_2\,)/(3.4) = (3m.(2u_0(3m.u_0^2 - u_0)/2 +0) -(3m.u_0^2 - u_0)/2\,)/(3.4)$

$= (3m.u_0^2 - u_0)(6mu_0\text{-}1)/4!$

10.4) $A_5 = (3m.(2A_0A_3+2A_1A_2) - A_3\,)/(4.5) = \quad (3m.(2A_0.0+2.0.A_2) - 0\,)/(4.5) = 0$

10.5) $A_6 = (3m.(2A_0A_4+2A_1A_3+A_2^2) - A_4\,)/(5.6) = \quad (3m.\{2u_0A_4+0+A_2^2\} - A_4\,)/(5.6)$

$= (3m.\{2u_0[\,(3m.u_0^2 - u_0)(6mu_0\text{-}1)/4!]+0+ [(3m.u_0^2 - u_0)/2]^2\} - (3m.u_0^2 - u_0)(6mu_0\text{-}1)/4!\,)/(5.6)$

$= (3m.u_0^2 - u_0).\ (3m.\{2u_0(6mu_0\text{-}1)+6.(3m.u_0^2 - u_0)\} – [6mu_0\text{-}1]\,)/6!$

$= (3m.u_0^2 - u_0).\ (6m.u_0(6mu_0\text{-}1)+18m.u_0(3m.u_0 - 1) –6mu_0+1)/6!$

$= (3m.u_0^2 - u_0).\ (90.m^2.u_0^2 – 30.m.u_0 + 1)/6!$

10.6) $A_7 = (3m.(2A_0A_5+2A_1A_4+2A_2A_3) - A_5\,)/(6.7) = \quad (3m.(2.A_0.0+2.0.A_4+2A_2.0) -0\,)/(6.7) = 0$

10.7) $A_8 = (3m.(2A_0A_6+2A_1A_5+2A_2A_4+A_3^2) – A_6)/(7.8)$

$= (3m.(2u_0\,(3m.u_0^2 - u_0).\ (90.m^2.u_0^2 – 30.m.u_0 + 1)/6! +2\,[(3m.u_0^2 - u_0)/2].[\,(3m.u_0^2 - u_0)(6mu_0\text{-}1)/4!]) - (3m.u_0^2 - u_0).\ (90.m^2.u_0^2 – 30.m.u_0 + 1)/6!)/(7.8)$

$= ((6m.u_0.(90.m^2.u_0^2 – 30.m.u_0 + 1)+5.6.3m.(3m.u_0^2 - u_0)\ (6mu_0\text{-}1)) - (90.m^2.u_0^2 – 30.m.u_0 + 1))(3m.u_0^2 - u_0)/8!$

$= (540.m^3.u_0^3 - 180.m^2.u_0^2 +6m.u_0 +(270.m^2.u_0^2\text{-}90.m.u_0)(6mu_0\text{-}1) - 90.m^2.u_0^2 + 30.m.u_0\text{-}\ 1).(3m.u_0^2 - u_0)/8!$

$= (2160.m^3.u_0^3 - 1080.m^2.u_0^2 +126.m.u_0 - 1).u_0(3m.u_0 - 1)/8!$

$= (360m^2u_0^2 -120mu_0 + 1)(6mu_0 -1)u_0(3mu_0 - 1)/8!$

Extra terms can be calculated as required, but note that all odd indexes of A_i are 0 in value [See appendix A]. This gives us:

11.1) $1/r(\varphi) = u(\varphi) = \quad u_0 + u_0(3mu_0\text{-}\ 1)\varphi^2/2 + u_0(3mu_0 - 1)(6mu_0\text{-}1)\varphi^4/4! +$

$u_0(3mu_0 - 1).(90m^2u_0^2 – 30mu_0 + 1)\varphi^6/6! +$

$u_0(3mu_0 - 1).(360m^2u_0^2 -120mu_0 + 1)(6mu_0 -1)\varphi^8/8! + ….$

Note that there is a factor $u_0(3mu_0 - 1)$ common to all even index factors from A_2 onwards, A_0 being just u_0.

The sequence of terms when fully evaluated would give an exact solution; however there does not seem to be a simple reduction of the sequence into predictable functions of increasing powers of φ.

The differential and double differential are straight forward to calculate:

11.2) $du(\varphi)/d\varphi = u_0(3m.u_0 - 1)\varphi + u_0(3m.u_0 - 1)(6mu_0-1)\varphi^3/3! +$

$u_0(3m.u_0 - 1).(90.m^2.u_0^2 – 30.m.u_0 + 1)\varphi^5/5! +$

$u_0(3m.u_0 - 1).(360m^2u_0^2 -120mu_0 + 1)(6mu_0 -1)\varphi^7/7! + \ldots$

11.3) $d^2u(\varphi)/d\varphi^2 = u_0(3m.u_0 - 1)(6mu_0-1)\varphi^2/2! +$

$u_0(3m.u_0 - 1).(90.m^2.u_0^2 – 30.m.u_0 + 1)\varphi^4/4! +$

$u_0(3m.u_0 - 1).(360m^2u_0^2 -120mu_0 + 1)(6mu_0 -1)\varphi^6/6! + \ldots$

As for the left hand side, to convert back to r coordinates for the single differential:

$u = r^{-1}$

12.1) $du/dr = -r^{-2}$

12.2) $dr/d\varphi = dr/du . du/d\varphi = -r^2 du/d\varphi$

12.3) $du/d\varphi = -r^{-2} dr/d\varphi$

And for the double differential:

13.1) $d^2u(\varphi)/d\varphi^2 = d/d\varphi (du/d\varphi) = d/d\varphi (-r^{-2} dr/d\varphi) = d/d\varphi (-u^2 dr/d\varphi)$

13.2) $d^2u(\varphi)/d\varphi^2 = -u^2 d^2r/d\varphi^2 - dr/d\varphi .d(u^2)/d\varphi$

Now as we have:

14) $d(u^2)/d\varphi = d/d\varphi (u.u) = u.(du/d\varphi) + (du/d\varphi) .u = 2u(du/d\varphi)$

We get:

15.1) $d^2u(\varphi)/d\varphi^2 = -u^2 d^2r/d\varphi^2 - dr/d\varphi .2u(du/d\varphi)$

15.2) $d^2u(\varphi)/d\varphi^2 = -r^{-2} d^2r/d\varphi^2 - 2r^{-1} dr/d\varphi . (-r^{-2} dr/d\varphi) = -r^{-2} d^2r/d\varphi^2 + 2r^{-3}(dr/d\varphi)^2$

So the full r based single differential is:

16) $- dr/d\varphi = r^2(3m/r_0 - 1)\varphi/r_0 + r^2(3m/r_0 - 1)(6m/r_0 -1)\varphi^3/(3!.r_0) +$

$r^2(3m/r_0 - 1).(90.m^2/r_0^2 – 30m/r_0 + 1)\varphi^5/(5!.r_0) +$

$r^2(3m/r_0 - 1).(360m^2/r_0^2 -120m/r_0 + 1)(6m/r_0 -1)\varphi^7/(7!.r_0) + \ldots$

The r based double differential is:

17) $- d^2r/d\varphi^2 = - 2(dr/d\varphi)^2 /r + r^2(3m/r_0 - 1)(6m/r_0-1)\varphi^2/(2.r_0) +$

$r^2(3m/r_0 - 1).(90m^2/r_0^2 – 30m/r_0 + 1)\varphi^4/(4!.r_0) +$

$r^2(3m/r_0 - 1).(360m^2/r_0^2 -120m/r_0 + 1)(6m/r_0 -1)\varphi^6/(6!.r_0) + ….$

Circular Light Orbit About Massive Central Object

The common factor of u_0 $(3m.u_0- 1)$ allows the deduction that at the value of $u_0 =1/3m$, the equation results in all terms except for A_0 being 0.

12) $u(\varphi) = u_0 = 1/3m$ or $r(\varphi) = 3m$ for all φ. [When $u_0 =1/3m$]

This gives the r = 3m (= $3GM/c^2$) circular orbit of photons around a strong gravity field. Also both the differential and the double differential values are zero for all φ in this case.

Newtonian Approximation

In the extreme case of very large r, u_0 ->0; as terms in powers of u_0 are then small compared to 1, on substituting back in for r, we get the equation:

13) $1/r(\varphi) = 1/r_0 . [1 – (\varphi^2/2 - \varphi^4/4! + \varphi^6/6! – \varphi^8/8! +…)]$

Here the geometric sequence in the square brackets is that for Cos φ.

14) $1/r(\varphi) = [\text{Cos } \varphi]/r_0$

This is the equation for a straight line, a simple geometric result, from Cos φ = A/H in a triangle where r_0 =adjacent (A) and $r(\varphi)$ = Hypotenuse (H). So the Newtonian result for a straight line light path is obtained in the limit as r becomes large.

Deflection of Light Past A Star's Edge

The next approximation to consider is the first relativistic approximation. First we will pull the cosine series out of the sequence of terms:

15) $u(\varphi) = u_0 [1 + (3m.u_0 - 1).(\varphi^2/2 - \varphi^4/4! + \varphi^6/6! – \varphi^8/8! +…) +$

$(3m.u_0 - 1).(6mu_0)\varphi^4/4! + (3m.u_0 - 1).(90.m^2.u_0^2 – 30.m.u_0)\varphi^6/6! +$

$(3m.u_0 - 1).(2160.m^3.u_0^3 - 1080.m^2.u_0^2 +126.m.u_0)\varphi^8/8! + ….]$

15.1) $u(\varphi) = u_0 [1 + (3m.u_0 - 1).(1-\text{Cos } \varphi) +$

$(3m.u_0 - 1).(6mu_0)\varphi^4/4! + (3m.u_0 - 1).(90.m^2.u_0^2 – 30.m.u_0)\varphi^6/6! +$

$(3m.u_0 - 1).(2160.m^3.u_0^3 - 1080.m^2.u_0^2 + 126.m.u_0)\varphi^8/8! + \ldots]$

15.2) $u(\varphi) = u_0 [\text{Cos } \varphi + 3mu_0(1\text{-Cos } \varphi) + (3mu_0 - 1).mu_0\varphi^4\{6/4! + (90mu_0 - 30)\varphi^2/6! +$

$(2160.m^2.u_0^2 - 1080.m.u_0 + 126)\varphi^4/8! + \ldots\}]$

Now we make an approximation and remove powers of mu_0 that can be neglected for solar masses and radius. For example $(3mu_0 - 1) \approx -1$.

16) $u(\varphi) \approx u_0 [\text{Cos } \varphi + 3mu_0(1\text{-Cos } \varphi) - mu_0\varphi^4\{6/4! – 30\varphi^2/6! + 126\varphi^4/8! + \ldots\}]$

Next we consider what happens as φ goes to ±π/2, as the light path passes by a solar mass at a solar radius and is ever so slightly deflected, so the φ values for the extremes of the path will be slightly different from ±π/2, but not by much and can be approximated in this situation to ±π/2 for the power terms.

17) $u(\pm\pi/2) \approx u_0 [\text{Cos } \varphi + 3mu_0(1\text{-Cos } \varphi) -mu_0(\pi/2)^4\{6/4! – 30(\pi/2)^2/6! + 126(\pi/2)^4/8! + \ldots\}]$

As at the extreme values of φ the r value tends to infinity, u will approach 0:

18) $0 \approx (1/r_0).[\text{Cos } \varphi + 3m(1/r_0)(1\text{-Cos } \varphi) -m(1/r_0)(\pi/2)^4\{6/4! – 30(\pi/2)^2/6! + 126(\pi/2)^4/8! + \ldots\}]$

Now a series of rearrangements:

19.1) $\text{Cos } \varphi + 3m(1/r_0)(1\text{-Cos } \varphi) = m(1/r_0)(\pi/2)^4\{6/4! – 30(\pi/2)^2/6! + 126(\pi/2)^4/8! + \ldots\}$

19.2) $(1-3m(1/r_0)).\text{Cos } \varphi + 3m(1/r_0) = m(1/r_0)(\pi/2)^4\{6/4! – 30(\pi/2)^2/6! + 126(\pi/2)^4/8! + \ldots\}$

19.3) $\text{Cos } \varphi = [-3m(1/r_0) + m(1/r_0)(\pi/2)^4\{6/4! – 30(\pi/2)^2/6! + 126(\pi/2)^4/8! + \ldots\}] / (1-3m(1/r_0))$

19.4) $\text{Cos } \varphi = m(1/r_0) [-3 + (\pi/2)^4\{6/4! – 30(\pi/2)^2/6! + 126(\pi/2)^4/8! + \ldots\}] / (1-3m(1/r_0))$

19.5) $\text{Cos } \varphi = m [-3 + (\pi/2)^4\{6/4! – 30(\pi/2)^2/6! + 126(\pi/2)^4/8! + \ldots\}] / (r_0-3m)$

On crunching the numbers:

20) $\text{Cos } \varphi = m [-3 + 6.088\{0.25 – 0.1028+ 0.0190 - \ldots\}] / (r_0-3m)$

$\approx m [-3 + 1.0118] / (r_0-3m)$

The result derived in Reference 1 is $\varphi' = 2m/r_0$ where φ’ is the difference from π/2, where here φ is the full π/2 plus the little extra. This value is then doubled up to add in the effect of the outward path, which by symmetry is identical in value to the inward path.

By standard trigonometric results:

21) $\text{Cos } (\pi/2+\varphi') = \text{Cos}(\pi/2)\text{Cos } \varphi' - \text{Sin}(\pi/2)\text{Sin } \varphi' = \text{Sin } \varphi' = \varphi' - \varphi'^3/3 + \varphi'^5/5 - \ldots \approx \varphi'$

So to a close approximation:

22) $\varphi' = m [-1.9882] / r_0$

This is a negative value as the angle is being extended beyond π/2 where the cosine is negative in value.

For the Sun, $m = GM/c^2 = 6.673x10^{-11}.(1.989x10^{30})/(2.99792x10^8)^2 = 1476.78$ m

And $r_0 = 6.96x10^8$ m.

Result is: $\varphi' = -4.219x10^{-6}$ radians = -0.8701 Arc seconds.

Or doubled up to include both paths: = -1.7403 Arc seconds.

The next term in the series would increase the magnitude of this by a few parts per thousand.

The standard result for the deflection of light passing by the edge of the sun is -1.75 Arc seconds.

[Ref 1].

General Notes on Solution

The ($3m.u_0$ - 1) term in every term for the differential and double differential means that the signs of these two equations will be opposite values for either side of the r=3m radius. This fits with previous standard analysis which has the photon orbits being unstable inside the r=3m radius and spiralling inwards [($3mu_0$ - 1) term is always positive, so increasing the value of u and hence decreasing the value of r.] and outside the r=3m radius the trajectory takes the photons away from the centre [($3m.u_0$ - 1) term is always negative, so decreasing the value of u and hence increasing r.].

There is no singularity in these equations for u = 1/2m.

The single differential $dr/d\varphi$ will tend to infinity as r increases to infinity, so at large distances from the central object, a photon of light that had a path near the central object will tend to the perpendicular in its direction of motion away from it (or in its initial approach).

Case Where Trajectory Crosses 3m Radius

The following derivation is for the situation where the photon trajectory to the central mass is crosses the 3m radii. This results in a more complex set of terms as we do not have a convenient place where the trajectory has a zero value for $du(\varphi)/do$, so we need to calculate the constants with odd indexes.

We will set the zero point of the angle φ to be at the point where the photon trajectory to the central object crosses the r=3m radius. We will set the value of u at this point to be $u_0 = 1/(3m)$. We use this as this value is guaranteed to exist in this situation, defined by having the light path cross r=3m. It will also lead to some useful simplification later on.

The analysis here should not go further in than the r=2m radius as time and space coordinates change completely at that radius. The Schwarzschild coordinate solution becomes invalid at that radius and so this solution system cannot be safely applied at that point.

From equation 2)

23) $u(0) = A_0 = u_0 < 1/(2m)$

The single differential of equation 2) is:

8) $du(\varphi)/d\varphi \;\; = A_1 + 2A_2\varphi + 3A_3\varphi^2 + \ldots.$

At $\varphi=0$ equation 8) becomes

24) $du(0)/d\varphi \;\; = A_1$ (a non zero value)

With these first two results we can now solve the set of equations in sequence:

25.1) $A_2 = (3m.u_0^2 - u_0)/2 = u_0(3mu_0 - 1)/2$

25.1) $A_3 = (3m.2.u_0 A_1 - A_1)/(2.3) = A_1.(6mu_0 - 1)/3!$

25.2) $A_4 = (3m.(2A_0A_2+A_1^2) - A_2)/(3.4) = (3m(u_0^2(3mu_0 - 1) +A_1^2) - u_0(3mu_0 - 1)/2)/(3.4)$

$= u_0(6mu_0- 1)(3mu_0 - 1)/4! + 6mA_1^2 /4!$

25.3) $A_5 = (3m.(2A_0A_3+2A_1A_2) - A_3)/(4.5)$

$= (3m.(2u_0A_1.(6mu_0 - 1)/3! + 2u_0A_1.(3mu_0 - 1)/2) - A_1.(6mu_0 - 1)/3!)/(4.5)$

$= (6mu_0A_1.(6mu_0 - 1) + 18mu_0A_1.(3mu_0 - 1) - A_1.(6mu_0 - 1))/5!$

$= A_1(90m^2u_0^2 - 30mu_0 + 1)/5!$

25.4) $A_6 \;\; = (3m.(2A_0A_4+2A_1A_3+A_2^2) - A_4)/5.6$

$= ([6mu_0.(mu_0^2(3mu_0 - 1) +mA_1^2) - mu_0^2(3mu_0 - 1)]/4 +$

$+ mA_1^2.(6mu_0 - 1) + 3mu_0^2(3mu_0 - 1)^2/4)/5.6$

$- u_0(6mu_0- 1)(3mu_0 - 1)/6! \;\; - 6mA_1^2/6!$

$= 36m^2u_0^3(3mu_0 - 1)/6! +36m^2u_0A_1^2 /6! - 6mu_0^2(3mu_0 - 1)/6!$

$+ 24mA_1^2.(6mu_0 - 1)/6! + 18mu_0^2(3mu_0 - 1)^2/6!$

$- u_0(6mu_0- 1).(3mu_0 - 1)/6! - 6mA_1^2/6!$

$= 30(6mu_0-1).mA_1^2/6! + (90m^2u_0^2-30mu_0 +1)u_0(3mu_0 - 1)/6!$

25.5) $A_7 \;\; = (3m.(2A_0A_5+2A_1A_4+2A_2A_3) - A_5)/6.7$

$= (6m.[u_0.A_1(90m^2u_0^2 - 30mu_0 + 1)/5! + A_1(u_0(6mu_0- 1)(3mu_0 - 1)/4! \;\; +6mA_1^2 /4!)$

$+ (u_0(3mu_0 - 1)/2)(A_1.(6mu_0 - 1)/3!)] - A_1(90m^2u_0^2 - 30mu_0 + 1)/5!)/6.7$

$= A_1[(6mu_0-1).(90m^2u_0^2 - 30mu_0 + 1)/5! + 6mu_0(6mu_0- 1)(3mu_0 - 1)/4! \;\; + 36m^2A_1^2 /4!$

$+ 3mu_0(3mu_0 - 1).(6mu_0 - 1)/3!]/6.7$

$= A_1[(6mu_0\text{-}1).\{90m^2u_0^2 - 30mu_0 + 1 + 90m^2u_0^2 - 30mu_0 + 180m^2u_0^2 - 60mu_0)\ \}/5!$

$+ 180m^2A_1^2 /5!\ ]/6.7$

$= A_1(6mu_0\text{-}1).\{360m^2u_0^2 - 120mu_0 + 1\ \}/7! + 180m^2A_1^2/7!$

25.6) $A_8 = (3m.(2A_0A_6+2A_1A_5+2A_2A_4+A_3^2) – A_6\ )/7.8$

$= 6m.u_0.[30(6mu_0\text{-}1).mA_1^2/8! + (90m^2u_0^2\text{-}30mu_0 +1)u_0(3mu_0 - 1)/8!\ ]$

$+6m.(A_1^2(90m^2u_0^2 - 30mu_0 + 1)/5!)/7.8$

$+6m.(u_0(3mu_0 - 1)/2).(u_0(6mu_0\text{-} 1)(3mu_0 - 1)/4! \quad +6mA_1^2 /4!)/7.8$

$+3m.(A_1.(6mu_0 - 1)/3!\ )^2 /7.8 - 30(6mu_0\text{-}1).mA_1^2/8! - (90m^2u_0^2\text{-}30mu_0 +1)u_0(3mu_0 - 1)/8!$

$= (6mu_0 - 1).30(6mu_0\text{-}1).mA_1^2/8! + (6mu_0 -1).(90m^2u_0^2\text{-}30mu_0 +1)u_0(3mu_0 - 1)/8!$

$+6m.A_1^2[540m^2u_0^2\text{-} 180mu_0 + 6 + 90mu_0(3mu_0 - 1) + 360m^2u_0^2\text{-}120mu_0 +10]/8!$

$+90mu_0(3mu_0 - 1).u_0(6mu_0\text{-} 1)(3mu_0 - 1) /8!$

$= 18m.A_1^2.(450m^2u_0^2\text{-} 150mu_0 + 7)/8! + (6mu_0 -1).[360m^2u_0^2 - 120mu_0 + 1]u_0(3mu_0 - 1)/8!$

Here the odd index terms all have a common factor of A_1, if the gradient value for A_1 was zero at the reference point then all these terms would go to zero and we would be left with the even index terms as in the first analysis, also the extra terms in the even indexes terms are multiples of A_1 as well.

26) $u(\varphi) = u_0 + A_1\varphi + u_0(3mu_0 - 1)\varphi^2 /2 + A_1.(6mu_0 - 1)\varphi^3 /3! + [u_0(6mu_0\text{-}1)(3mu_0\text{-}1)+6mA_1^2\ ]\varphi^4 /4!$

$+A_1[mu_0(3mu_0 - 1)/4 + 1]\varphi^5/5! + [2.(30mu_0\text{-}1).mA_1^2 +(90m^2u_0^2\text{-}30mu_0\text{-}1)u_0(3mu_0\text{-}1)]\varphi^6/6!$

$+A_1[60mu_0.(3mu_0 - 1).\{11mu_0 - 1\} + 6mu_0 + 180m^2A_1^2 - 1]\varphi^7/7!$

$+18m.A_1^2.(450m^2u_0^2\text{-} 150mu_0 + 7)\varphi^8/8! + (6mu_0 -1).[360m^2u_0^2 - 120mu_0 + 1]u_0(3mu_0 - 1)\varphi^8/8!$

+…

We will now re-introduce the specified value for $u_0 = 1/3m$. This could have been used earlier to simplify things but chose to develop the full equations up to here.

27) $u(\varphi) = 1/3m + A_1\varphi + u_0(0)\varphi^2 /2 + A_1.(1)\varphi^3 /3! + [u_0(1)(0)+6mA_1^2\ ]\varphi^4 /4!$

$+A_1[mu_0(0)/4 + 1]\varphi^5/5! + [2.(9).mA_1^2 +(-1)u_0(0)]\varphi^6/6!$

$+A_1[20.(0).\{11mu_0 - 1\} + 2 + 180m^2A_1^2 - 1]\varphi^7/7!$

$+18m.A_1^2.(50\text{-} 50 + 7)\varphi^8/8! + (2 -1).[40 - 40 + 1]u_0(0)\varphi^8/8!$ …

Which simplifies to:

28) $1/r(\varphi) = 1/3m + A_1\varphi + A_1\varphi^3/3! + 6mA_1^2\varphi^4/4! + A_1\varphi^5/5! + 18mA_1^2\varphi^6/6!$

$+A_1[1 + 180m^2A_1^2]\varphi^7/7! + 126mA_1^2\varphi^8/8! + \dots$

The differential and double differential of $u(\varphi)$ are again straight forward to calculate:

28.1) $du(\varphi)/d\varphi = A_1 + A_1\varphi^2/2! + 6mA_1^2\varphi^3/3! + A_1\varphi^4/4! + 18mA_1^2\varphi^5/5!$

$+A_1[1 + 180m^2A_1^2]\varphi^6/6! + 126mA_1^2\varphi^7/7! + \dots$

28.2) $d^2u(\varphi)/d\varphi^2 = A_1\varphi + 3mA_1^2\varphi^2 + A_1\varphi^3/3! + 18mA_1^2\varphi^4/4!$

$+A_1[1 + 180m^2A_1^2]\varphi^5/5! + 126mA_1^2\varphi^6/6! + \dots$

We can note that at $\varphi=0$ (and $r=3m$) that $d^2u(\varphi)/d\varphi^2 = 0$.

The r based single differential is:

29) $- dr/d\varphi = r(\varphi)^2A_1 \{1 + \varphi^2/2! + 6mA_1\varphi^3/3! + \varphi^4/4! + 18mA_1\varphi^5/5!$

$+[1 + 180m^2A_1^2]\varphi^6/6! + 126mA_1\varphi^7/7! + \dots\}$

The r based double differential is:

30) $-d^2r/d\varphi^2 = -2(dr/d\varphi)^2/r(\varphi) + r(\varphi)^2A_1\{\varphi + 3mA_1\varphi^2 + \varphi^3/3! + 18mA_1\varphi^4/4!$

$+ [1 + 180m^2A_1^2]\varphi^5/5! + 126mA_1\varphi^6/6! + \dots \}$

Note that $d^2r/d\varphi^2$ is not necessarily zero at $\varphi=0$ and $r=3m$.

Summary

It is possible to produce a power series approximation for the photon orbit around a strong gravitational field based on the angular position of the photon. This series allows for an arbitrarily accurate set of calculations to be made. Only terms up to the eighth power of φ are derived here but more can be calculated by this method. The equations derived here are tested against standard results and successfully reproduce the known results.

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Appendix A

Proof that all odd powers of the expansion terms for φ have a constant with an odd index in every term when the first odd index term is zero, and hence have a 0 value constant in every term.

A reprise of the relevant equations:

1.1) $d^2u(\varphi)/d\varphi^2 + u(\varphi) = 3mu(\varphi)^2$

2) $u(\varphi) = A_0 + A_1\varphi + A_2\varphi^2 + A_3\varphi^3 + A_4\varphi^4 + A_5\varphi^5 + A_6\varphi^6 + \ldots$

3) $d^2u(\varphi)/d\varphi^2 = 2A_2 + 2.3.A_3\varphi + 3.4.A_4\varphi^2 + 4.5.A_5\varphi^3 + 5.6.A_6\varphi^4 + 6.7.A_6\varphi^5 + 7.8.A_6\varphi^6 \ldots$

4) $u(\varphi)^2 = A_0^2 + 2A_0 A_1\varphi + (2A_0A_2 + A_1^2)\varphi^2 + (2A_0A_3 + 2A_1A_2)\varphi^3 + (2A_0A_4 + 2A_1A_3 + A_2^2)\varphi^4$

$+ (2A_0A_5 + 2A_1A_4 + 2A_2A_3)\varphi^5 + (2A_0A_6 + 2A_1A_5 + 2A_2A_4 + A_3^2)\varphi^6 + \ldots$

The u(φ) terms have the constants with odd indexes associated with odd powers of φ.

The $d^2u(\varphi)/d\varphi^2$ will have the odd powers of φ associated with multiples of the odd index constants due to the double differentiation.

The $u(\varphi)^2$ terms for odd powers of φ will have pairs of constants whose index terms sum to an odd number and so must include an odd index constant.

Hence in the case where the first odd index is zero value, all further odd index constants will be multiples of this by simple iteration and hence will all be zero in value.

For powers that are multiples of 4, noting that $(6mu_0-1)$ is a factor in the fourth and eighth powers, the higher powers of multiples of 4 will follow the pattern:

$A_{4n} = [3m.(2A_0A_{4n-2}+2A_2A_{4n-4}+2A_4A_{4n-6}+2A_6A_{4n-8}+\ldots) – A_{4n-2}]/((4n-1).4n)$

Which rearranges to:

$A_{4n} =[(6mu_0-1).A_{4n-2}+3m.(2A_2A_{4n-4}+2A_4A_{4n-6}+2A_6A_{4n-8}+\ldots)]/((4n-1).4n)$

Where all the terms in the square brackets (after the first) have a factor that has an index that is a multiple of 4.

By iteration again then, as the first fourth power of φ constant (A_4) has $(6mu_0-1)$ as a factor, all subsequent fourth powers of φ will have the common factor of $(6mu_0-1)$.