Prime Numbers Greater Than 3 And Their Gaps Are Handed

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Abstract

We build a simple recursive model for the prime numbers which at its heart is the prime sieve of Eratosthenes. We also show for prime numbers greater than 3 and their gaps posses a handedness which forbids a large range of possibilities for the choice of intervals in arithmetic progressions[1].

1 The Recursive Nature Of Prime Numbers

Starting with Peano's axioms we begin to construct the whole numbers using zero and successor. We can even introduce direction and build the negative numbers. This is adequate for addition, and we can use it to build the operator of multiplication by recursive application of addition.

But we also know, the whole numbers have deeper structure, a subset called the prime numbers. We're going to start with Peano's axioms plus the concept of co-prime to establish a more rigorous definition of prime numbers.

The usual definition of co-prime is, two numbers a and b are co-prime if they share no common prime number as a factor. We need to loose the concept of prime number and hence factor, because we can't build a model of prime numbers which begins with a model of prime numbers.

We say two numbers are co-prime if the set of numbers $\{p, c\}$ such that $p * c = a$ and $p < c$ and $\{p, d\}$ such that $p * d = b$ and $p < d$, where $\{p, c, d\}$ are any whole numbers, have only the unique solution $\{p = 1, c = a, d = b\}$. It's actually the standard definition but we relax the notion of prime and say any whole number.

We begin by assigning the number one to be the zeroth prime number, $P_0 = 1$. The number one is not a prime number but by the above definition, it is co-prime to every number, including itself, so it's our initial condition for our new set. This does not effect the usual counting if we only consider the index of the prime, since it has index zero, we don't count it.

Now we build a special number, called the primorial, which is the product of all the prime numbers to some maximum index and we write it as

$$P_m\# = \prod_{j=0}^{j\leq m} P_j$$

Next, we define the next prime number, it is the smallest number co-prime to our product and greater than 1.

That is, we have the number one as our start condition and only the concept of the next prime number. The recipe to find this next number is finding a number that isn't something we know about already with the concept of multiplicative independence.

Because this is a necessarily weak construction, it is the slowest imaginable method of finding the prime numbers. Of course as we integrate our knowledge we can build faster tools, but as this also necessarily highlights the fact we will always face insurmountable odds against discovering too much structure simply because of their infinitude.

1.1 The Square Of The Next Prime Number

Given that we have found the first few prime numbers, or at least, the first real one following our model; it happens to be 2, we can also immediately deduce that what ever that number is, its square is also co-prime to our product. Moreover, every number ≥ 1 and less than P_{m+1}^2 which is also co-prime to $P_m\#$ must also be a prime number.

That is, given the first few numbers $\{1, 2, 3, 4\}$ every number less than 4 is a prime number by construction because $4 = 2^2$ and 2 is demonstrably our next prime number after 1.

So we don't need to test the number 3 for being a prime number, we can demonstrate it is prime by it being bracketed by the next prime square. And if we look at the numbers co-prime to $2\# = \{1, 3, 5, 7, 9\}$ we discover $\{5, 7\}$ for free as it were.

Further more, we've learned we only need to examine odd numbers so we can leap ahead by 2 at each step, not bothering to test the even numbers.

2 Prime Handedness

We get a further simplification once we have introduced the number 3 in our product. This comes about by what we call gap handedness. That is gaps, or the span between any numbers in our set, can now pair together in only certain configurations.

For notation, we'll say P_j^m is the j^{th} ordered number in the set of numbers co-prime to $P_m\#$.

Because of the recursive nature of the construction, well-founded substructure we find in earlier sets must apply to all future sets. At $3\#$ we learn every future co-prime and hence prime number must be a solution of either

$$P_j^m = 1 + 6n$$

or

$$P_j^m = 5 + 6n$$

This induces our gap handedness for prime numbers greater than 3, since the difference between any 2 different numbers is

$$P_{j+k}^m - P_j^m = \{1, 5\} + 6l + \{1, 5\} + 6m = \{0, 2, 4\} + 6n$$

This makes many gap pairs and hence many general arithmetic progressions[1] impossible. For example, there can never be a co-prime constellation of the form $P_{j+2}^m = 8 + P_{j+1}^m = 16 + P_j^m$, or gaps pairs of $\{8, 8\}$.

We'll say a gap is left handed if its left hand co-prime is of the form $5 + 6n$, that is, the next possibility for a co-prime is one of $2 + 6n$, or $\{2, 6, 8, 12, 14, 18, 20, \dots\}$. While right handed gaps anchor on solutions to $1 + 6n$ with the next possibilities being $\{4, 6, 10, 12, 16, 18, 22, \dots\}$.

Notice that gaps of size $6n$ can be both right handed and left handed, so we can see gap pairs and span pairs of $\{6n, 6m\}$. This is because there are two ways to make any span of any multiple of 6, $\{2, 4, 2, 4, \dots, 2, 4\}$ or $\{4, 2, 4, 2, \dots, 4, 2\}$. Such spans also preserve the handedness of the co-prime. If a left handed co-prime is followed by a left handed span of 6 and that number is co-prime, that number is also left handed.

A purely left or right handed gap looks like $\{2, 4, 2, 4, \dots, 2, 4, 2\}$ or $\{4, 2, 4, 2, \dots, 4, 2, 4\}$, which is why they can not follow each other because it requires an impossible solution to our 2 equations. So purely like handed gaps can never touch. They also change the hand of the following number. While gap pairs with each member divisible by 6 must touch with the same hands.

From here we can conclude only arithmetic progressions of intervals divisible by 6 are possible with all other possibilities banned.

2.1 Arithmetic Progressions In Prime Numbers

Since pure left hands, or right hands can never touch they can never form arithmetic progressions. That is we will never find arithmetic progressions with intervals of $2 + 6n$ or $4 + 6n$.

However, arbitrary progressions do occur, so their intervals must be divisible by 6. In following work we will show how to construct arithmetic progressions of arbitrary length in numbers co-prime to a primorial.

References

[1] Ben Green and Terence Tao. The primes contain arbitrarily long arithmetic progressions, 08 April 2004. URL `http://arxiv.org/abs/math/0404188`.