An Application of Hardy-Littlewood Conjecture

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Abstract. In this paper, we assume that weaker Hardy-Littlewood Conjecture, we got a better upper bound of the exceptional real zero for a class of prime number module.

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Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and in all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

In 1923, Hardy and Littlewood conjectured

$$\sum_{\substack{3\leq p_1,p_2\leq N\\ p_1+p_2=N}} 1 \approx \frac{N}{\varphi(N)}\prod_{p\,\dagger N}\left(1-\frac{1}{(p-1)^2}\right)\frac{N}{\log^2 N}$$

where N is even integer and $N\geq 6$, p_1,p_2 are the prime numbers, $\varphi(n)$ is Euler function.

Under a weaker assumption, we got a better upper bound of the exceptional real zero for a class of the prime number module.

Weaker Hardy-Littlewood Conjecture. Let N is even integer and $N\geq 6$, p_1,p_2 are the prime numbers. There is an absolute constant $\delta>0$, we have

$$\sum_{\substack{3\le p_1,p_2\le N\\ p_1+p_2=N}} 1 \ge \frac{\delta N}{\log^2 N}$$

Under the above conjecture, we have the following theorem

Theorem. Let q is a prime number and $q \equiv 3\,(mod\,4)$, it has exceptional real character χ, and its Dirichlet $L(s,\chi)$ function has an exceptional real zero β. If Weaker Hardy-Littlewood Conjecture is correct, then there is a positive constant c , we have

$$\beta \le 1 - \frac{c}{\log^2 q}$$

Now, we do some preparation work.

Lemma 1.

$$\sum_{k=1}^{m} e\left(\frac{kn}{m}\right) = \begin{cases} m & if\ n \equiv 0\ (mod\ m) \\ 0 & otherwise \end{cases}$$

where $e(x) = e^{2\pi i x}$
The lemma 1 is obvious

Lemma 2. There is a constant $c_1 > 0$ such that

$$\pi(x) = Lix + O\left(x\exp(-c_1\sqrt{\log x})\right)$$

uniformly for $x \ge 2$. Where $Lix = \int_2^x \frac{du}{\log u}$, and $\exp(x) = e^x$
The lemma 2 follows from the References [2], Theorem 6.9 of the page 179.

It is easy to see that

$$Lix = \int_2^x \frac{du}{\log u} = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

Lemma 3. Let c_2 be the positive constant. if $(a,q)=1$, then

$$\pi(x;q,a)=\frac{Lix}{\varphi(q)}-\frac{\chi(a)}{\varphi(q)}\int_2^x\frac{u^{\beta-1}}{\log u}du+O\left(x\exp(-c_2\sqrt{\log x})\right)$$

when there is an exceptional character χ modulo q and β is the concomitant zero.

The lemma 3 follows from the References [2], Corollary 11.20 of the page 381

It is easy to see that

$$\int_2^x\frac{u^{\beta-1}}{\log u}du=\frac{x^\beta}{\beta\log x}+O\left(\frac{x^\beta}{\log^2 x}\right)$$

Lemma 4. if $(n,m)=1$, then

$$\sum_{\substack{k=1\\(k,m)=1}}^{m} e\left(\frac{nk}{m}\right)=\mu(m)$$

where $\mu(m)$ is Möbius function.
The lemma 4 follows from the References [1], the page 45.

Lemma 5. if χ is a primitive character modulo m, then

$$\sum_{k=1}^{m}\chi(k)e\left(\frac{nk}{m}\right)=\overline{\chi}(n)\tau(\chi)$$

where $\tau(\chi)=\sum_{k=1}^{m}\chi(k)e(\frac{k}{m})$.
The lemma 5 follows from the References [1], the page 47.

Lemma 6. if m is odd square-free and χ is a primitive real character modulo m, then

$$\tau(\chi) = \begin{cases} \sqrt{m} & if \ m \equiv 1 \ (mod \ 4) \\ i\sqrt{m} & if \ m \equiv 3 \ (mod \ 4) \end{cases}$$

The lemma 6 follows from the References [1], the theorem 3.3 of the page 49.

PROOF OF THEOREM.

The first part.

By Lemma 1, when $x \geq q^4$, we have

$$\sum_{k=1}^{q} \left(\sum_{3 \leq p \leq x} e\left(\frac{kp}{q}\right) \right)^2 = \sum_{k=1}^{q} \sum_{3 \leq p_1 \leq x} \sum_{3 \leq p_2 \leq x} e\left(\frac{k(p_1+p_2)}{q}\right)$$

$$= \sum_{3 \leq p_1 \leq x} \sum_{3 \leq p_2 \leq x} \sum_{k=1}^{q} e\left(\frac{k(p_1+p_2)}{q}\right) = q \sum_{\substack{3 \leq p_1, p_2 \leq x \\ p_1+p_2 \equiv (q)}} 1 \geq q \sum_{n=1}^{[\frac{x}{2q}]} \sum_{\substack{3 \leq p_1, p_2 \leq x \\ p_1+p_2 = 2nq}} 1$$

by Weaker Hardy-Littlewood Conjecture, the above formula

$$\geq q \sum_{n=1}^{[\frac{x}{2q}]} \frac{\delta 2nq}{\log^2 2nq} \geq q \sum_{n=1}^{[\frac{x}{2q}]} \frac{\delta 2nq}{\log^2 x} \geq \frac{2\delta q^2}{\log^2 x} \sum_{n=1}^{[\frac{x}{2q}]} n$$

$$= \frac{2\delta q^2}{\log^2 x} \cdot \frac{[\frac{x}{2q}]([\frac{x}{2q}]+1)}{2} \geq \frac{\delta x^2}{4\log^2 x} + O\left(\frac{xq}{\log^2 x}\right)$$

The second part.

When $1 \le k \le q-1$, we have

$$\sum_{3\le p\le x} e\left(\frac{pk}{q}\right) = \sum_{\substack{3\le p\le x\\(p,q)=1}} e\left(\frac{pk}{q}\right) + 1 = 1 + \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{ak}{q}\right) \sum_{\substack{3\le p\le x\\ p\equiv a(q)}} 1$$

by Lemma 3, Lemma 4, Lemma 5 and Lemma 6, the above formula

$$= 1 + \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{ak}{q}\right)\left(\frac{Lix}{\varphi(q)} - \frac{\chi(a)}{\varphi(q)}\int_2^x \frac{u^{\beta-1}}{\log u}du + O\left(x\exp(-c_2\sqrt{\log x})\right)\right)$$

$$= \frac{\mu(q)Lix}{q-1} - \frac{\tau(\chi)\chi(k)}{q-1}\int_2^x \frac{u^{\beta-1}}{\log u}du + O\left(qx\exp(-c_2\sqrt{\log x})\right)$$

$$= -\frac{i\sqrt{q}\,\chi(k)}{q-1}\int_2^x \frac{u^{\beta-1}}{\log u}du + O\left(\frac{x}{q\log x} + qx\exp(-c_2\sqrt{\log x})\right)$$

therefore

$$\left(\sum_{3\le p\le x} e\left(\frac{pk}{q}\right)\right)^2 = \left(-\frac{i\sqrt{q}\,\chi(k)}{q-1}\int_2^x \frac{u^{\beta-1}}{\log u}du\right)^2$$

$$+O\left(\frac{x^2}{q^{\frac{3}{2}}\log^2 x} + q^2x^2\exp(-c_2\sqrt{\log x}\right)$$

$$= -\frac{q}{(q-1)^2}\left(\int_2^x \frac{u^{\beta-1}}{\log u}du\right)^2 + O\left(\frac{x^2}{q^{\frac{3}{2}}\log^2 x} + q^2x^2\exp(-c_2\sqrt{\log x}\right)$$

$$= -\frac{q}{\beta^2(q-1)^2}\frac{x^{2\beta}}{\log^2 x} + O\left(\frac{x^2}{q\log^3 x} + \frac{x^2}{q^{\frac{3}{2}}\log^2 x} + q^2x^2\exp(-c_2\sqrt{\log x}\right)$$

therefore

$$\sum_{k=1}^{q}\left(\sum_{3\le p\le x} e\left(\frac{pk}{q}\right)\right)^2 = \left(\sum_{3\le p\le x} 1\right)^2 + \sum_{k=1}^{q-1}\left(\sum_{3\le p\le x} e\left(\frac{pk}{q}\right)\right)^2$$

by Lemma 2, the above formula

$$= \frac{x^2}{\log^2 x} - \frac{q\,x^{2\beta}}{\beta^2(q-1)\log^2 x} + O\left(\frac{x^2}{\log^3 x} + \frac{x^2}{\sqrt{q}\log^2 x} + q^3x^2\exp(-c_3\sqrt{\log x})\right)$$

We integrated the first part and second part

$$\frac{x^{2\beta}}{\log^2 x} \le (1-\frac{\delta}{4})\frac{x^2}{\log^2 x} + O\left(\frac{x^2}{\log^3 x} + \frac{x^2}{\sqrt{q}\log^2 x} + q^3x^2\exp(-c_3\sqrt{\log x})\right)$$

$$x^{2\beta-2} \le 1 - \frac{\delta}{4} + O\left(\frac{1}{\log x} + \frac{1}{\sqrt{q}} + q^3\log^2 x\exp(-c_3\sqrt{\log x})\right)$$

we take $\log x = (\frac{4}{c_3}\log q)^2$, then

$$x^{2\beta-2} \leq 1 - \frac{\delta}{4} + \frac{c_4}{\log^2 q}$$

we take $\log q \geq \sqrt{\frac{8c_4}{\delta}}$, then

$$x^{2\beta-2} \leq 1 - \frac{\delta}{8}$$

$$\beta - 1 \leq \frac{\log(1-\frac{\delta}{8})}{2\log x} = -\frac{\log(\frac{8}{8-\delta})}{2\log x}$$

therefore

$$\beta \leq 1 - \frac{c}{\log^2 q}$$

This completes the proof of Theorem.

REFERENCES

[1] Henryk Iwaniec, Emmanuel Kowalski, *Analytic Number Theory,* American mathematical Society, 2004.

[2] Hugh L. Montgomery, Robert C. Vaughan, *Multiplicative Number Theory I. Classical Theory,* Cambridge University Press, 2006.