E. Hitzer, New Developments in Clifford Fourier Transforms, in N. E. Mastorakis, P. M. Pardalos, R. P. Agarwal, L. Kocinac (eds.), Advances in Applied and Pure Mathematics, Proceedings of the 2014 International Conference on Pure Mathematics, Applied Mathematics, Computational Methods (PMAMCM 2014), Santorini Island, Greece, July 17-21, 2014, Mathematics and Computers in Science and Engineering Series, Vol. 29, pp. 19-25, 2014.

New Developments in Clifford Fourier Transforms

Eckhard Hitzer

Abstract—We show how real and complex Fourier transforms are extended to W.R. Hamiltons algebra of quaternions and to W.K. Clifford's geometric algebras. This was initially motivated by applications in nuclear magnetic resonance and electric engineering. Followed by an ever wider range of applications in color image and signal processing. Cliffords geometric algebras are complete algebras, algebraically encoding a vector space and all its subspace elements. Applications include electromagnetism, and the processing of images, color images, vector field and climate data. Further developments of Clifford Fourier Transforms include operator exponential representations, and extensions to wider classes of integral transforms, like Clifford algebra versions of linear canonical transforms and wavelets.

Keywords—Fourier transform, Clifford algebra, geometric algebra, quaternions, signal processing, linear canonical transform

I. Introduction

We begin by introducing Clifford Fourier transforms, including the important class of quaternion Fourier transforms mainly along the lines of [31] and [5], adding further detail, emphasize and new developments.

There is the alternative operator exponential Clifford Fourier transform (CFT) approach, mainly pursued by the Clifford Analysis Group at the university of Ghent (Belgium) [5]. New work in this direction closely related to the roots of -1 approach explained below is in preparation [11].

We mainly provide an overview of research based on the holistic investigation [28] of real geometric square roots of -1 in Clifford algebras $Cl(p,q)$ over real vector spaces $\mathbb{R}^{p,q}$. These algebras include real and complex numbers, quaternions, Pauli- and Dirac algebra, space time algebra, spinor algebra, Lie algebras, conformal geometric algebra and many more. The resulting CFTs are therefore perfectly tailored to work on functions valued in these algebras. In general the continuous manifolds of $\sqrt{-1}$ in $Cl(p,q)$ consist of several conjugacy classes and their connected components. Simple examples are shown in Fig. 1.

A CFT analyzes scalar, vector and multivector signals in terms of sine and cosine waves with multivector coefficients. Basically, the imaginary unit $i \in \mathbb{C}$ in the transformation kernel $e^i\phi = \cos\phi + i\sin\phi$ is replaced by a $\sqrt{-1}$ in $Cl(p,q)$. This produces a host of CFTs, an incomplete brief overview is sketched in Fig. 2, see also the historical overview in [5]. Additionally the $\sqrt{-1}$ in $Cl(p,q)$ allow to construct further types of integral transformations, notably Clifford wavelets [21], [37].

E. Hitzer is with the Department of Material Science, International Christian University, Mitaka, Tokyo, 181-8585 Japan e-mail: hitzer@icu.ac.jp.
thanks
Manuscript received May 31, 2014; revised ...

II. Clifford's geometric algebra

Definition 1 (Clifford's geometric algebra [15], [36]). *Let $\{e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_n\}$, with $n = p+q$, $e_k^2 = \varepsilon_k$, $\varepsilon_k = +1$ for $k = 1,\ldots,p$, $\varepsilon_k = -1$ for $k = p+1,\ldots,n$, be an orthonormal base of the inner product vector space $\mathbb{R}^{p,q}$ with a geometric product according to the multiplication rules*

$$e_k e_l + e_l e_k = 2\varepsilon_k \delta_{k,l}, \qquad k,l = 1,\ldots n, \tag{1}$$

where $\delta_{k,l}$ is the Kronecker symbol with $\delta_{k,l} = 1$ for $k = l$, and $\delta_{k,l} = 0$ for $k \neq l$. This non-commutative product and the additional axiom of associativity generate the 2^n-dimensional Clifford geometric algebra $Cl(p,q) = Cl(\mathbb{R}^{p,q}) = Cl_{p,q} = \mathcal{G}_{p,q} = \mathbb{R}_{p,q}$ over $\mathbb{R}$. The set $\{e_A : A \subseteq \{1,\ldots,n\}\}$ with $e_A = e_{h_1}e_{h_2}\ldots e_{h_k}$, $1 \leq h_1 < \ldots < h_k \leq n$, $e_\emptyset = 1$, forms a graded (blade) basis of $Cl(p,q)$. The grades k range from 0 for scalars, 1 for vectors, 2 for bivectors, s for s-vectors, up to n for pseudoscalars. The vector space $\mathbb{R}^{p,q}$ is included in $Cl(p,q)$ as the subset of 1-vectors. The general elements of $Cl(p,q)$ are real linear combinations of basis blades e_A, called Clifford numbers, multivectors or hypercomplex numbers.

In general $\langle A\rangle_k$ denotes the grade k part of $A \in Cl(p,q)$. The parts of grade 0 and $k+s$, respectively, of the geometric product of a k-vector $A_k \in Cl(p,q)$ with an s-vector $B_s \in Cl(p,q)$

$$A_k * B_s := \langle A_k B_s\rangle_0, \qquad A_k \wedge B_s := \langle A_k B_s\rangle_{k+s}, \tag{2}$$

are called *scalar product* and *outer product*, respectively.

For Euclidean vector spaces $(n = p)$ we use $\mathbb{R}^n = \mathbb{R}^{n,0}$ and $Cl(n) = Cl(n,0)$. Every k-vector B that can be written as the outer product $B = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge \ldots \wedge \boldsymbol{b}_k$ of k vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k \in \mathbb{R}^{p,q}$ is called a *simple* k-vector or *blade*.

Multivectors $M \in Cl(p,q)$ have k-vector parts $(0 \leq k \leq n)$: scalar part $Sc(M) = \langle M\rangle = \langle M\rangle_0 = M_0 \in \mathbb{R}$, vector part $\langle M\rangle_1 \in \mathbb{R}^{p,q}$, bi-vector part $\langle M\rangle_2$, …, and pseudoscalar part $\langle M\rangle_n \in \bigwedge^n \mathbb{R}^{p,q}$

$$M = \sum_A M_A \boldsymbol{e}_A = \langle M\rangle + \langle M\rangle_1 + \langle M\rangle_2 + \ldots + \langle M\rangle_n\,. \tag{3}$$

The *principal reverse* of $M \in Cl(p,q)$ defined as

$$\widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle \overline{M}\rangle_k, \tag{4}$$

often replaces complex conjugation and quaternion conjugation. Taking the *reverse* is equivalent to reversing the order of products of basis vectors in the basis blades e_A. The operation $\overline{M}$ means to change in the basis decomposition of M the sign of every vector of negative square $\overline{e_A} = \varepsilon_{h_1}e_{h_1}\varepsilon_{h_2}e_{h_2}\ldots\varepsilon_{h_k}e_{h_k}$, $1 \leq h_1 < \ldots < h_k \leq n$. Reversion, $\overline{M}$, and principal reversion are all involutions.

For $M, N \in Cl(p,q)$ we get $M * \widetilde{N} = \sum_A M_A N_A$. Two multivectors $M, N \in Cl(p,q)$ are *orthogonal* if and only if $M * \widetilde{N} = 0$. The modulus $|M|$ of a multivector $M \in Cl(p,q)$ is defined as

$$|M|^2 = M * \widetilde{M} = \sum_A M_A^2. \tag{5}$$

A. Multivector signal functions

A multivector valued function $f : \mathbb{R}^{p,q} \to Cl(p,q)$, has 2^n blade components $(f_A : \mathbb{R}^{p,q} \to \mathbb{R})$

$$f(\boldsymbol{x}) = \sum_A f_A(\boldsymbol{x}) \boldsymbol{e}_A. \tag{6}$$

We define the *inner product* of two functions $f, g : \mathbb{R}^{p,q} \to Cl(p,q)$ by

$$\begin{aligned}(f,g) &= \int_{\mathbb{R}^{p,q}} f(\boldsymbol{x}) \widetilde{g(\boldsymbol{x})} \, d^n \boldsymbol{x} \\ &= \sum_{A,B} \boldsymbol{e}_A \widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^{p,q}} f_A(\boldsymbol{x}) g_B(\boldsymbol{x}) \, d^n \boldsymbol{x},\end{aligned} \tag{7}$$

with the *symmetric scalar part*

$$\langle f, g \rangle = \int_{\mathbb{R}^{p,q}} f(\boldsymbol{x}) * \widetilde{g(\boldsymbol{x})} \, d^n \boldsymbol{x} = \sum_A \int_{\mathbb{R}^{p,q}} f_A(\boldsymbol{x}) g_A(\boldsymbol{x}) \, d^n \boldsymbol{x}, \tag{8}$$

and the $L^2(\mathbb{R}^{p,q}; Cl(p,q))$*-norm*

$$\|f\|^2 = \langle (f,f) \rangle = \int_{\mathbb{R}^{p,q}} |f(\boldsymbol{x})|^2 d^n \boldsymbol{x} = \sum_A \int_{\mathbb{R}^{p,q}} f_A^2(\boldsymbol{x}) \, d^n \boldsymbol{x}, \tag{9}$$

$$L^2(\mathbb{R}^{p,q}; Cl(p,q)) = \{ f : \mathbb{R}^{p,q} \to Cl(p,q) \mid \|f\| < \infty \}. \tag{10}$$

B. Square roots of -1 in Clifford algebras

Every Clifford algebra $Cl(p,q)$, $s_8 = (p-q) \bmod 8$, is isomorphic to one of the following (square) matrix algebras[1] $\mathcal{M}(2d, \mathbb{R})$, $\mathcal{M}(d, \mathbb{H})$, $\mathcal{M}(2d, \mathbb{R}^2)$, $\mathcal{M}(d, \mathbb{H}^2)$ or $\mathcal{M}(2d, \mathbb{C})$. The first argument of $\mathcal{M}$ is the dimension, the second the associated ring[2] $\mathbb{R}$ for $s_8 = 0, 2$, $\mathbb{R}^2$ for $s_8 = 1$, $\mathbb{C}$ for $s_8 = 3, 7$, $\mathbb{H}$ for $s_8 = 4, 6$, and $\mathbb{H}^2$ for $s_8 = 5$. For even n: $d = 2^{(n-2)/2}$, for odd n: $d = 2^{(n-3)/2}$.

It has been shown [27], [28] that $Sc(f) = 0$ for every square root of $-\mathbf{1}$ in every matrix algebra $\mathcal{A}$ isomorphic to $Cl(p,q)$. One can distinguish *ordinary* square roots of $-\mathbf{1}$, and *exceptional* ones. All square roots of $-\mathbf{1}$ in $Cl(p,q)$ can be computed using the package CLIFFORD for Maple [1], [3], [29], [38].

In all cases the *ordinary* square roots f of $-\mathbf{1}$ constitute a *unique conjugacy class* of dimension $\dim(\mathcal{A})/2$, which has *as many connected components as the group* $\mathbb{G}(\mathcal{A})$ of invertible elements in $\mathcal{A}$. Furthermore, we have $\mathrm{Spec}(f) = 0$ (zero pseudoscalar part) if the associated ring is $\mathbb{R}^2$, $\mathbb{H}^2$, or $\mathbb{C}$. The exceptional square roots of $-\mathbf{1}$ *only* exist if $\mathcal{A} \cong \mathcal{M}(2d, \mathbb{C})$.

For $\mathcal{A} = \mathcal{M}(2d, \mathbb{R})$, the centralizer (set of all elements in $Cl(p,q)$ commuting with f) and the conjugacy class of a square root f of $-\mathbf{1}$ both have $\mathbb{R}$-dimension $2d^2$ with *two connected components*. For the simplest case $d = 1$ we have the algebra $Cl(2,0)$ isomorphic to $\mathcal{M}(2, \mathbb{R})$.

For $\mathcal{A} = \mathcal{M}(2d, \mathbb{R}^2) = \mathcal{M}(2d, \mathbb{R}) \times \mathcal{M}(2d, \mathbb{R})$, the square roots of $(-\mathbf{1}, -\mathbf{1})$ are pairs of two square roots of $-\mathbf{1}$ in $\mathcal{M}(2d, \mathbb{R})$. They constitute a unique conjugacy class with *four connected components*, each of dimension $4d^2$. Regarding the four connected components, the group of inner automorphisms $\mathrm{Inn}(\mathcal{A})$ induces the permutations of the Klein group, whereas the quotient group $\mathrm{Aut}(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$ is isomorphic to the group of isometries of a Euclidean square in 2D. The simplest example with $d = 1$ is $Cl(2,1)$ isomorphic to $M(2, \mathbb{R}^2) = \mathcal{M}(2, \mathbb{R}) \times \mathcal{M}(2, \mathbb{R})$.

For $\mathcal{A} = \mathcal{M}(d, \mathbb{H})$, the submanifold of the square roots f of $-\mathbf{1}$ is a *single connected conjugacy class* of $\mathbb{R}$-dimension $2d^2$ equal to the $\mathbb{R}$-dimension of the centralizer of every f. The easiest example is $\mathbb{H}$ itself for $d = 1$.

For $\mathcal{A} = \mathcal{M}(d, \mathbb{H}^2) = \mathcal{M}(d, \mathbb{H}) \times \mathcal{M}(d, \mathbb{H})$, the square roots of $(-\mathbf{1}, -\mathbf{1})$ are pairs of two square roots (f, f') of $-\mathbf{1}$ in $\mathcal{M}(d, \mathbb{H})$ and constitute a *unique connected conjugacy class* of $\mathbb{R}$-dimension $4d^2$. The group $\mathrm{Aut}(\mathcal{A})$ has two connected components: the neutral component $\mathrm{Inn}(\mathcal{A})$ connected to the identity and the second component containing the swap automorphism $(f, f') \mapsto (f', f)$. The simplest case for $d = 1$ is $\mathbb{H}^2$ isomorphic to $Cl(0,3)$.

For $\mathcal{A} = \mathcal{M}(2d, \mathbb{C})$, the square roots of $-\mathbf{1}$ are in *bijection to the idempotents* [2]. First, the *ordinary* square roots of $-\mathbf{1}$ (with $k = 0$) constitute a conjugacy class of $\mathbb{R}$-dimension $4d^2$ of a *single connected component* which is invariant under $\mathrm{Aut}(\mathcal{A})$. Second, there are $2d$ *conjugacy classes* of *exceptional* square roots of $-\mathbf{1}$, each composed of a *single connected component*, characterized by the equality $\mathrm{Spec}(f) = k/d$ (the pseudoscalar coefficient) with $\pm k \in \{1, 2, \dots, d\}$, and their $\mathbb{R}$-dimensions are $4(d^2 - k^2)$. The group $\mathrm{Aut}(\mathcal{A})$ includes conjugation of the pseudoscalar $\omega \mapsto -\omega$ which maps the conjugacy class associated with k to the class associated with $-k$. The simplest case for $d = 1$ is the Pauli matrix algebra isomorphic to the geometric algebra $Cl(3,0)$ of 3D Euclidean space $\mathbb{R}^3$, and to complex biquaternions [42].

C. Quaternions

Quaternions are a special Clifford algebra, because the algebra of quaternions $\mathbb{H}$ is isomorphic to $Cl(0,2)$, and to the even grade subalgebra of the Clifford algebra of three-dimensional Euclidean space $Cl^+(3,0)$. But quaternions were initially known independently of Clifford algebras and have their own specific notation, which we briefly introduce here.

Gauss, Rodrigues and Hamilton's four-dimensional (4D) quaternion algebra $\mathbb{H}$ is defined over $\mathbb{R}$ with three imaginary units:

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \\ \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{11}$$

[1] Compare chapter 16 on *matrix representations and periodicity of 8*, as well as Table 1 on p. 217 of [36].

[2] Associated ring means, that the matrix elements are from the respective ring $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{H}^2$.

Every quaternion can be written explicitly as

$$q = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} \in \mathbb{H}, \quad q_r, q_i, q_j, q_k \in \mathbb{R}, \tag{12}$$

and has a *quaternion conjugate* (equivalent[3] to Clifford conjugation in $Cl^+(3,0)$ and $Cl(0,2)$)

$$\overline{q} = q_r - q_i\boldsymbol{i} - q_j\boldsymbol{j} - q_k\boldsymbol{k}, \quad \overline{pq} = \overline{q}\,\overline{p}, \tag{13}$$

which leaves the scalar part q_r unchanged. This leads to the *norm* of $q \in \mathbb{H}$

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}, \qquad |pq| = |p||q|. \tag{14}$$

The part $V(q) = q - q_r = \frac{1}{2}(q - \overline{q}) = q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k}$ is called a *pure* quaternion, and it squares to the negative number $-(q_i^2 + q_j^2 + q_k^2)$. Every unit quaternion (i.e. $|q| = 1$) can be written as:

$$\begin{aligned} q &= q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} = q_r + \sqrt{q_i^2 + q_j^2 + q_k^2}\,\boldsymbol{\mu}(q) \\ &= \cos\alpha + \boldsymbol{\mu}(q)\sin\alpha = e^{\alpha\,\boldsymbol{\mu}(q)}, \end{aligned} \tag{15}$$

where

$$\cos\alpha = q_r, \qquad \sin\alpha = \sqrt{q_i^2 + q_j^2 + q_k^2},$$

$$\boldsymbol{\mu}(q) = \frac{V(q)}{|q|} = \frac{q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k}}{\sqrt{q_i^2 + q_j^2 + q_k^2}}, \qquad \text{and} \qquad \boldsymbol{\mu}(q)^2 = -1. \tag{16}$$

The *inverse* of a non-zero quaternion is

$$q^{-1} = \frac{\overline{q}}{|q|^2} = \frac{\overline{q}}{q\overline{q}}. \tag{17}$$

The *scalar part* of a quaternion is defined as

$$Sc(q) = q_r = \frac{1}{2}(q + \overline{q}), \tag{18}$$

with *symmetries*

$$\begin{aligned} Sc(pq) = Sc(qp) &= p_r q_r - p_i q_i - p_j q_j - p_k q_k, \\ Sc(q) &= Sc(\overline{q}), \quad \forall p, q \in \mathbb{H}, \end{aligned} \tag{19}$$

and *linearity*

$$\begin{aligned} Sc(\alpha p + \beta q) = \alpha Sc(p) + \beta Sc(q) = \alpha p_r + \beta q_r, \\ \forall p, q \in \mathbb{H}, \ \ \alpha, \beta \in \mathbb{R}. \end{aligned} \tag{20}$$

The scalar part and the quaternion conjugate allow the definition of the $\mathbb{R}^4$ *inner product*[4] of two quaternions p, q as

$$Sc(p\overline{q}) = p_r q_r + p_i q_i + p_j q_j + p_k q_k \in \mathbb{R}. \tag{21}$$

Definition 2 (Orthogonality of quaternions)**.** *Two quaternions* $p, q \in \mathbb{H}$ *are* orthogonal $p \perp q$, *if and only if the inner product* $Sc(p\overline{q}) = 0$.

[3]This may be important in generalisations of the QFT, such as to a space-time Fourier transform in [19], or a general two-sided Clifford Fourier transform in [24].

[4]Note that we do not use the notation $p \cdot q$, which is unconventional for full quaternions.

III. Inventory of Clifford Fourier Transforms

A. General geometric Fourier transform

Recently a rigorous effort was made in [8] to design a *general geometric Fourier transform*, that incorporates most of the previously known CFTs with the help of very general sets of left and right kernel factor products

$$\begin{aligned} \mathcal{F}_{GFT}\{h\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^{p',q'}} L(\mathbf{x},\omega)h(\mathbf{x})R(\mathbf{x},\omega)d^{n'}\mathbf{x}, \\ L(\mathbf{x},\omega) &= \prod_{s\in F_L} e^{-s(\mathbf{x},\omega)}, \end{aligned} \tag{22}$$

with $p' + q' = n'$, $F_L = \{s_1(\mathbf{x},\omega), \dots, s_L(\mathbf{x},\omega)\}$ a set of mappings $\mathbb{R}^{p',q'} \times \mathbb{R}^{p',q'} \to \mathcal{I}^{p,q}$ into the manifold of real multiples of $\sqrt{-1}$ in $Cl(p,q)$. $R(\mathbf{x},\omega)$ is defined similarly, and $h : \mathbb{R}^{p',q'} \to Cl(p,q)$ is the multivector signal function.

B. CFT due to Sommen and Buelow

This clearly subsumes the *CFT due to Sommen and Buelow* [7]

$$\mathcal{F}_{SB}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} h(\mathbf{x}) \prod_{k=1}^{n} e^{-2\pi x_k \omega_k e_k} d^n\mathbf{x}, \tag{23}$$

where $\mathbf{x}, \omega \in \mathbb{R}^n$ with components x_k, ω_k, and $\{e_1, \dots e_k\}$ is an orthonormal basis of $\mathbb{R}^{0,n}$, $h : \mathbb{R}^n \to Cl(0,n)$.

C. Color image CFT

It is further possible [16] to only pick strictly mutually commuting sets of $\sqrt{-1}$ in $Cl(p,q)$, e.g. e_1e_2, $e_3e_4 \in Cl(4,0)$ and construct CFTs with therefore commuting kernel factors in analogy to (23). Also contained in (22) is the *color image CFT* of [40]

$$\begin{aligned} \mathcal{F}_{CI}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} & e^{\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{x}I_4B} e^{\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{x}B} h(\mathbf{x}) \\ & e^{-\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{x}B} e^{-\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{x}I_4B} d^2\mathbf{x}, \end{aligned} \tag{24}$$

where $B \in Cl(4,0)$ is a bivector and $I_4B \in Cl(4,0)$ its dual complementary bivector. It is especially useful for the introduction of efficient non-marginal generalized color image Fourier descriptors.

D. Two-sided CFT

The main type of CFT, which we will review here is the general *two sided CFT* [24] with only one kernel factor on each side

$$\mathcal{F}^{f,g}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{p',q'}} e^{-fu(\mathbf{x},\omega)} h(\mathbf{x}) e^{-gv(\mathbf{x},\omega)} d^{n'}\mathbf{x}, \tag{25}$$

with f, g two $\sqrt{-1}$ in $Cl(p,q)$, $u, v : \mathbb{R}^{p',q'} \times \mathbb{R}^{p',q'} \to \mathbb{R}$ and often $\mathbb{R}^{p',q'} = \mathbb{R}^{p,q}$. In the following we will discuss a family of transforms, which belong to this class of CFTs, see the lower half of Fig. 2.

E. Quaternion Fourier Transform (QFT)

One of the nowadays most widely applied CFTs is the *quaternion Fourier transform* (QFT) [19], [26]

$$\mathcal{F}^{f,g}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-fx_1\omega_1} h(\mathbf{x}) e^{-gx_2\omega_2} d^2\mathbf{x}, \qquad (26)$$

which also has variants were one of the left or right kernel factors is dropped, or both are placed together at the right or left side. It was first described by Ernst, et al, [14, pp. 307-308] (with $f = \boldsymbol{i}, g = \boldsymbol{j}$) for spectral analysis in two-dimensional nuclear magnetic resonance, suggesting to use the QFT as a method to independently adjust phase angles with respect to two frequency variables in two-dimensional spectroscopy. Later Ell [12] independently formulated and explored the QFT for the analysis of linear time-invariant systems of PDEs. The QFT was further applied by Buelow, et al [6] for image, video and texture analysis, by Sangwine et al [43], [5] for color image analysis and analysis of non-stationary improper complex signals, vector image processing, and quaternion polar signal representations. It is possible to split every quaternion-valued signal and its QFT into two quasi-complex components [26], which allow the application of complex discretization and fast FT methods. The split can be generalized to the general CFT (25) [24] in the form

$$x_{\pm} = \frac{1}{2}(x \pm fxg), \quad x \in Cl(p,q). \qquad (27)$$

In the case of quaternions the quaternion coefficient space $\mathbb{R}^4$ is thereby split into two steerable (by the choice of two pure quaternions f, g) orthogonal two-dimensional planes [26]. The geometry of this split appears closely related to the quaternion geometry of rotations [39]. For colors expressed by quaternions, these two planes become chrominance and luminance when $f = g =$ gray line [13].

F. Quaternion Fourier Stieltjes transform

Georgiev and Morais have modified the QFT to a *quaternion Fourier Stieltjes transform* [18].

$$\mathcal{F}_{Stj}(\sigma^1,\sigma^2) = \int_{\mathbb{R}^2} e^{-fx_1\omega_1} d\sigma^1(x_1) d\sigma^2(x_2) e^{-gx_2\omega_2}, \quad (28)$$

with $f = -\boldsymbol{i}, g = -\boldsymbol{j}$, $\sigma^k : \mathbb{R} \to \mathbb{H}$, $|\sigma^k| \leq \delta_k$ for real numbers $0 < \delta_k < \infty$, $k = 1, 2$.

G. Quaternion Fourier Mellin transform, Clifford Fourier Mellin transform

Introducing polar coordinates in $\mathbb{R}^2$ allows to establish a *quaternion Fourier Mellin transform* (QFMT) [30]

$$\begin{aligned}\mathcal{F}_{QM}\{h\}(\nu,k) =& \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} r^{-f\nu} h(r,\theta) e^{-gk\theta} d\theta dr/r, \\ & \forall (\nu,k) \in \mathbb{R}\times\mathbb{Z},\end{aligned} \qquad (29)$$

which can characterize 2D shapes rotation, translation and scale invariant, possibly including color encoded in the quaternion valued signal $h : \mathbb{R}^2 \to \mathbb{H}$ such that $|h|$ is summable over $\mathbb{R}_+^* \times \mathbb{S}^1$ under the measure $d\theta dr/r$, $\mathbb{R}^*$ the multiplicative group of positive non-zero numbers, and $f, g \in \mathbb{H}$ two $\sqrt{-1}$. The QFMT can be generalized straightforward to a *Clifford Fourier Mellin transform* applied to signals $h : \mathbb{R}^2 \to Cl(p,q), p+q = 2$ [23], with $f, g \in Cl(p,q), p+q = 2$.

H. Volume-time CFT and spacetime CFT

The spacetime algebra $Cl(3,1)$ of Minkowski space with orthonormal vector basis $\{\mathbf{e}_t, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $-\mathbf{e}_t^2 = \mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^3$, has three blades $\mathbf{e}_t, i_3, i_{st}$ of time vector, unit space volume 3-vector and unit hyperspace volume 4-vector, which are isomorphic to Hamilton's three quaternion units

$$\begin{aligned}&\mathbf{e}_t^2 = -1, \quad i_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_t^* = \mathbf{e}_t i_3^{-1}, i_3^2 = -1, \\ &i_{st} = \mathbf{e}_t i_3, i_{st}^2 = -1.\end{aligned} \qquad (30)$$

The $Cl(3,1)$ subalgebra with basis $\{1, \mathbf{e}_t, i_3, i_{st}\}$ is therefore isomorphic to quaternions and allows to generalize the two-sided QFT to a *volume-time Fourier transform*

$$\mathcal{F}_{VT}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{3,1}} e^{-\mathbf{e}_t\omega_t} h(\mathbf{x}) e^{-\vec{x}\cdot\vec{\omega}} d^4\mathbf{x}, \qquad (31)$$

with $\mathbf{x} = t\mathbf{e}_t + \vec{x} \in \mathbb{R}^{3,1}$, $\vec{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, $\boldsymbol{\omega} = \omega_t\mathbf{e}_t + \vec{\omega} \in \mathbb{R}^{3,1}$, $\vec{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3$. The split (27) with $f = \mathbf{e}_t$, $g = i_3 = \mathbf{e}_t^*$ becomes the spacetime split of special relativity

$$h_{\pm} = \frac{1}{2}(1 \pm \mathbf{e}_t h \mathbf{e}_t^*). \qquad (32)$$

It is most interesting to observe, that the volume-time Fourier transform can indeed be applied to multivector signal functions valued in the whole spacetime algebra $h : \mathbb{R}^{3,1} \to Cl(3,1)$ without changing its form [19], [22]

$$\mathcal{F}_{ST}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{3,1}} e^{-\mathbf{e}_t\omega_t} h(\mathbf{x}) e^{-i_3\vec{x}\cdot\vec{\omega}} d^4\mathbf{x}. \qquad (33)$$

The split (32) applied to *spacetime Fourier transform* (33) leads to a *multivector wavepacket analysis*

$$\begin{aligned}\mathcal{F}_{ST}\{h\}(\boldsymbol{\omega}) = &\int_{\mathbb{R}^{3,1}} h_+(\mathbf{x}) e^{-i_3(\vec{x}\cdot\vec{\omega} - t\omega_t)} d^4\mathbf{x} \\ &+ \int_{\mathbb{R}^{3,1}} h_-(\mathbf{x}) e^{-i_3(\vec{x}\cdot\vec{\omega} + t\omega_t)} d^4\mathbf{x},\end{aligned} \qquad (34)$$

in terms of right and left propagating spacetime multivector wave packets.

I. One-sided CFTs

Finally, we turn to *one-sided CFTs* [25], which are obtained by setting the phase function $u = 0$ in (25). A recent discrete *spinor CFT* used for edge and texture detection is given in [4], where the signal is represented as a spinor and the $\sqrt{-1}$ is a local tangent bivector $B \in Cl(3,0)$ to the image intensity surface ($\mathbf{e}_3$ is the intensity axis).

J. Pseudoscalar kernel CFTs

The following class of *one-sided CFTs which uses a single pseudoscalar* $\sqrt{-1}$ has been well studied and applied [20]

$$\mathcal{F}_{PS}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} h(\mathbf{x}) e^{-i_n \mathbf{x}\cdot\boldsymbol{\omega}} d^n\mathbf{x},$$
$$i_n = \mathbf{e}_1\mathbf{e}_2\ldots\mathbf{e}_n, \quad n = 2,3 (\mathrm{mod}\, 4), \tag{35}$$

where $h : \mathbb{R}^n \to Cl(n,0)$, and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the orthonormal basis of $\mathbb{R}^n$. Historically the special case of (35), $n = 3$, was already introduced in 1990 [32] for the processing of electromagnetic fields. This same transform was later applied [17] to two-dimensional images embedded in $Cl(3,0)$ to yield a two-dimensional analytic signal, and in image structure processing. Moreover, the *pseudoscalar CFT* (35), $n = 3$, was successfully applied to three-dimensional vector field processing in [10], [9] with vector signal convolution based on Clifford's full geometric product of vectors. The theory of the transform has been thoroughly studied in [20].

For embedding one-dimensional signals in $\mathbb{R}^2$, [17] considered in (35) the special case of $n = 2$, and in [10], [9] this was also applied to the processing of two-dimensional vector fields.

Recent applications of (35) with $n = 2,3$, to geographic information systems and climate data can be found in [47], [46], [35].

K. Quaternion and Clifford linear canonical transforms

Real and complex linear canonical transforms parametrize a continuum of transforms, which include the Fourier, fractional Fourier, Laplace, fractional Laplace, Gauss-Weierstrass, Bargmann, Fresnel, and Lorentz transforms, as well as scaling operations. A Fourier transform transforms multiplication with the space argument x into differentiation with respect to the frequency argument ω. In Schroedinger quantum mechancis this constitutes a rotation in position-momentum phase space. A linear canonical transform transforms the position and momentum operators into linear combinations (with a two-by-two real or complex parameter matrix), preserving the fundamental position-momentum commutator relationship, at the core of the uncertainty principle. The transform operator can be made to act on appropriate spaces of functions, and can be realized in the form of integral transforms, parametrized in terms of the four real (or complex) matrix parameters [44].

KitIan Kou et al [34] introduce the quaternionic linear canonical transform (QLCT). They consider a pair of unit determinant two-by-two matrices

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \tag{36}$$

with entries $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$, $a_1 d_1 - c_1 b_1 = 1$, $a_2 d_2 - c_2 b_2 = 1$, where they disregard the cases $b_1 = 0$, $b_2 = 0$, for which the LCT is essentially a chirp multiplication.

We now *generalize* the definitions of [34] using the following two kernel functions with two pure unit quaternions $f, g \in \mathbb{H}$, $f^2 = g^2 = -1$, including the cases $f = \pm g$,

$$K^f_{A_1}(x_1, \omega_1) = \frac{1}{\sqrt{f 2\pi b_1}} e^{f(a_1 x_1^2 - 2x_1\omega_1 + d_1\omega_1^2)/(2b_1)},$$
$$K^g_{A_2}(x_2, \omega_2) = \frac{1}{\sqrt{g 2\pi b_2}} e^{g(a_2 x_2^2 - 2x_2\omega_1 + d_2\omega_2^2)/(2b_2)}. \tag{37}$$

The *two-sided QLCT* of signals $h \in L^1(\mathbb{R}^2, \mathbb{H})$ can now generally be defined as

$$\mathcal{L}^{f,g}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} K^f_{A_1}(x_1, \omega_1) h(\mathbf{x}) K^g_{A_2}(x_2, \omega_2) d^2\mathbf{x}. \tag{38}$$

The *left-sided* and *right-sided QLCTs* can be defined correspondingly by placing the two kernel factors both on the left or on the right[5], respectively. For $a_1 = d_1 = a_2 = d_2 = 0$, $b_1 = b_2 = 1$, the conventional two-sided (left-sided, right-sided) QFT is recovered. We note that it will be of interest to "complexify" the matrices A_1 and A_2, by including replacing $a_1 \to a_{1r} + f a_{1f}$, $a_2 \to a_{2r} + g a_{2g}$, etc. In [34] for $f = \boldsymbol{i}$ and $g = \boldsymbol{j}$ the right-sided QLCT and its properties, including an uncertainty principle are studied in some detail.

In [45] a complex Clifford linear canonical transform is defined and studied for signals $f \in L^1(\mathbb{R}^m, C^{m+1})$, where $C^{m+1} = \mathrm{span}\{1, e_1, \ldots, e_m\} \subset Cl(0,m)$ is the subspace of paravectors in $Cl(0,m)$. This includes uncertainty principles. Motivated by Remark 2.2 in [45], we now modify this definition to generalize the one-sided CFT of [25] for real Clifford algebras $Cl(n,0)$ to a *general real Clifford linear canonical transform (CLNT)*. We define the parameter matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}, \quad ad - cb = 1. \tag{39}$$

We again omit the case $b = 0$ and define the kernel

$$K^f(\mathbf{x}, \boldsymbol{\omega}) = \frac{1}{\sqrt{f(2\pi)^n b}} e^{f(a\mathbf{x}^2 - 2\mathbf{x}\cdot\boldsymbol{\omega} + d\boldsymbol{\omega}^2)/(2b)}, \tag{40}$$

with the general square root of -1: $f \in Cl(n,0)$, $f^2 = -1$. Then the general real CLNT can be defined for signals $h \in L^1(\mathbb{R}^n; Cl(n,0))$ as

$$\mathcal{L}^f\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} h(\mathbf{x}) K^f(\mathbf{x}, \boldsymbol{\omega}) d^n\mathbf{x}. \tag{41}$$

For $a = d = 0$, $b = 1$, the conventional one-sided CFT of [25] in $Cl(n,0)$ is recovered. It is again of interest to modify the entries of the parameter matrix to $a \to a_0 + f a_f$, $b \to b_0 + f b_f$, etc.

Similarly in [33] a *Clifford version of a linear canonical transform (CLCT)* for signals $h \in L^1(\mathbb{R}^m; \mathbb{R}^{m+1})$ is formulated using two-by-two parameter matrices $A_1, \ldots, A_m$, which maps $\mathbb{R}^m \to Cl(0,m)$. The Sommen Bülow CFT (23) is recovered for parameter matrix entries $a_k = d_k = 0$, $b_k = 1$, $1 \le k \le m$.

[5] In [34] the possibility of a more general pair of unit quaternions $f, g \in \mathbb{H}$, $f^2 = g^2 = -1$, is only indicated for the case of the right-sided QLCT, but with the restriction that f, g should be an *orthonormal pair* of pure quaternions, i.e. $Sc(f\overline{g}) = 0$. Otherwise [34] always strictly sets $f = \boldsymbol{i}$ and $g = \boldsymbol{j}$.

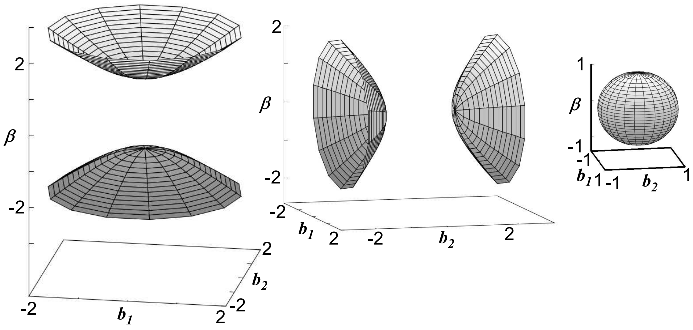

Fig. 1. Manifolds [28] of square roots f of -1 in $Cl(2,0)$ (left), $Cl(1,1)$ (center), and $Cl(0,2) \cong \mathbb{H}$ (right). The square roots are $f = \alpha + b_1 e_1 + b_2 e_2 + \beta e_{12}$, with $\alpha, b_1, b_2, \beta \in \mathbb{R}$, $\alpha = 0$, and $\beta^2 = b_1^2 e_2^2 + b_2^2 e_1^2 + e_1^2 e_2^2$.

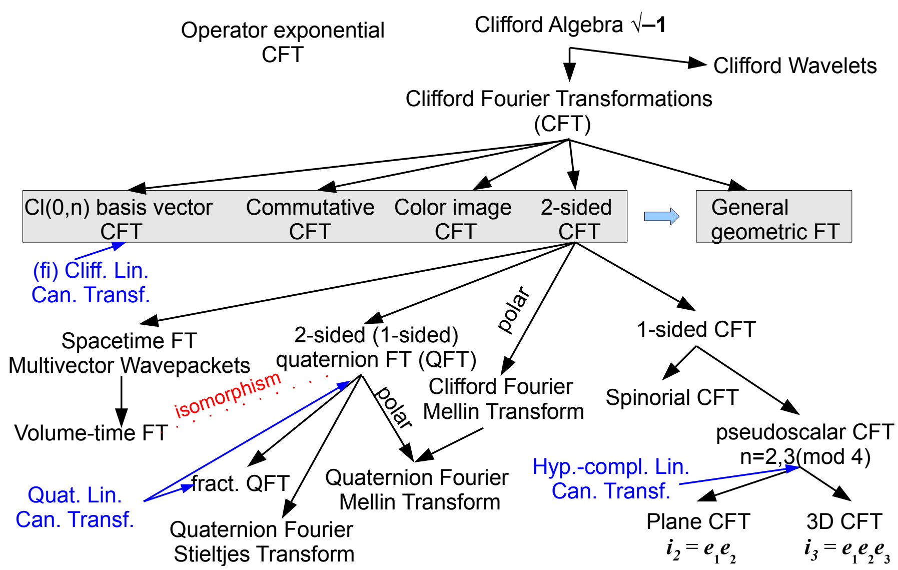

Fig. 2. Family tree of Clifford Fourier transformations.

IV. Conclusion

We have reviewed Clifford Fourier transforms which apply the manifolds of $\sqrt{-1} \in Cl(p,q)$ in order to create a rich variety of new Clifford valued Fourier transformations. The history of these transforms spans just over 30 years. Major steps in the development were: $Cl(0,n)$ CFTs, then pseudoscalar CFTs, and Quaternion FTs. In the 1990ies especially applications to electromagnetic fields/electronics and in signal/image processing dominated. This was followed by by color image processing and most recently applications in Geographic Information Systems (GIS). This paper could only feature a part of the approaches in CFT research, and only a part of the applications. Omitted were details on operator exponential CFT approach [5], and CFT for conformal geometric algebra. Regarding applications, e.g. CFT Fourier descriptor representations of shape [41] of B. Rosenhahn, et al was omitted. Note that there are further types of Clifford algebra/analysis related integral transforms: Clifford wavelets, Clifford radon transforms, Clifford Hilbert transforms, ... which we did not discuss.

Acknowledgment

Soli deo gloria. I do thank my dear family, and the organizers of the PMAMCM 2014 conference in Santorini, Greece.

References

[1] R. Abłamowicz, *Computations with Clifford and Grassmann Algebras*, Adv. Appl. Clifford Algebras **19**, No. 3–4 (2009), 499–545.

[2] R. Abłamowicz, B. Fauser, K. Podlaski, J. Rembieliński, *Idempotents of Clifford Algebras*. Czechoslovak Journal of Physics, **53** (11) (2003), 949–954.

[3] R. Abłamowicz and B. Fauser, CLIFFORD with Bigebra – A Maple Package for Computations with Clifford and Grassmann Algebras, http://math.tntech.edu/rafal/ (©1996-2012).

[4] T. Batard, M. Berthier, *CliffordFourier Transf. and Spinor Repr. of Images*, in: E. Hitzer, S.J. Sangwine (eds.), "Quaternion and Clifford Fourier Transf. and Wavelets", TIM **27**, Birkhauser, Basel, 2013, 177–195.

[5] F. Brackx, et al, *History of Quaternion and Clifford-Fourier Transf.*, in: E. Hitzer, S.J. Sangwine (eds.), "Quaternion and Clifford Fourier Transf. and Wavelets", TIM **27**, Birkhauser, Basel, 2013, xi–xxvii.

[6] T. Buelow, *Hypercomplex Spectral Signal Repr. for the Proc. and Analysis of Images*, PhD thesis, Univ. of Kiel, Germany, Inst. fuer Informatik und Prakt. Math., Aug. 1999.

[7] T. Buelow, et al, *Non-comm. Hypercomplex Fourier Transf. of Multidim. Signals*, in G. Sommer (ed.), "Geom. Comp. with Cliff. Algebras", Springer 2001, 187–207.

[8] R. Bujack, et al, *A General Geom. Fourier Transf.*, in: E. Hitzer, S.J. Sangwine (eds.), "Quaternion and Clifford Fourier Transf. and Wavelets", TIM **27**, Birkhauser, Basel, 2013, 155–176.

[9] J. Ebling, G. Scheuermann, *Clifford convolution and pattern matching on vector fields*, In Proc. IEEE Vis., **3**, IEEE Computer Society, Los Alamitos, 2003. 193–200,

[10] J. Ebling, G. Scheuermann, *Cliff. Four. transf. on vector fields*, IEEE Trans. on Vis. and Comp. Graph., **11**(4), (2005), 469–479.

[11] D. Eelbode, E. Hitzer, *Operator Exponentials for the Clifford Fourier Transform on Multivector Fields*, in preparation, 18 pages. Preprint: http://vixra.org/abs/1403.0310.

[12] T. A. Ell, *Quaternionic Fourier Transform for Analysis of Two-dimensional Linear Time-Invariant Partial Differential Systems.* in Proceedings of the 32nd IEEE Conference on Decision and Control, December 15-17, **2** (1993), 1830–1841.

[13] T.A. Ell, S.J. Sangwine, *Hypercomplex Fourier transforms of color images*, IEEE Trans. Image Process., **16**(1) (2007), 22–35.

[14] R. R. Ernst, et al, *Princ. of NMR in One and Two Dim.*, Int. Ser. of Monogr. on Chem., Oxford Univ. Press, 1987.

[15] M.I. Falcao, H.R. Malonek, *Generalized Exponentials through Appell sets in* $\mathbb{R}^{n+1}$ *and Bessel functions*, AIP Conference Proceedings, Vol. 936, pp. 738–741 (2007).

[16] M. Felsberg, et al, *Comm. Hypercomplex Fourier Transf. of Multidim. Signals*, in G. Sommer (ed.), "Geom. Comp. with Cliff. Algebras", Springer 2001, 209–229.

[17] M. Felsberg, *Low-Level Img. Proc. with the Struct. Multivec.*, PhD thesis, Univ. of Kiel, Inst. fuer Inf. & Prakt. Math., 2002.

[18] S. Georgiev, J. Morais, *Bochner's Theorems in the Framework of Quaternion Analysis* in: E. Hitzer, S.J. Sangwine (eds.), "Quaternion and Clifford Fourier Transf. and Wavelets", TIM **27**, Birkhauser, Basel, 2013, 85–104.

[19] E. Hitzer, *Quaternion Fourier Transform on Quaternion Fields and Generalizations*, AACA, **17** (2007), 497–517.

[20] E. Hitzer, B. Mawardi, *Clifford Fourier Transf. on Multivector Fields and Unc. Princ. for Dim.* $n = 2$ *(mod 4) and* $n = 3$ *(mod 4)*, P. Angles (ed.), AACA, **18**(S3,4), (2008), 715–736.

[21] E. Hitzer, *Cliff. (Geom.) Alg. Wavel. Transf.*, in V. Skala, D. Hildenbrand (eds.), Proc. GraVisMa 2009, Plzen, 2009, 94–101.

[22] E. Hitzer, *Directional Uncertainty Principle for Quaternion Fourier Transforms*, AACA, **20**(2) (2010), 271–284.

[23] E. Hitzer, *Clifford Fourier-Mellin transform with two real square roots of* -1 *in* $Cl(p, q)$, $p+q = 2$, 9th ICNPAA 2012, AIP Conf. Proc., **1493**, (2012), 480–485.

[24] E. Hitzer, *Two-sided Clifford Fourier transf. with two square roots of* -1 *in* $Cl(p, q)$, Advances in Applied Clifford Algebras, 2014, 20 pages, DOI: 10.1007/s00006-014-0441-9. First published in M. Berthier, L. Fuchs, C. Saint-Jean (eds.) electronic Proceedings of AGACSE 2012, La Rochelle, France, 2-4 July 2012. Preprint: http://arxiv.org/abs/1306.2092 .

[25] E. Hitzer, *The Clifford Fourier transform in real Clifford algebras*, in E. H., K. Tachibana (eds.), "Session on Geometric Algebra and Applications, IKM 2012", Special Issue of Clifford Analysis, Clifford Algebras and their Applications, Vol. 2, No. 3, pp. 227-240, (2013). First published in K. Guerlebeck, T. Lahmer and F. Werner (eds.), electronic Proc. of 19th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering, IKM 2012, Weimar, Germany, 0406 July 2012. Preprint: http://vixra.org/abs/1306.0130 .

[26] E. Hitzer, S. J. Sangwine, *The Orthogonal 2D Planes Split of Quaternions and Steerable Quaternion Fourier Transf.*, in: E. Hitzer, S.J. Sangwine (eds.), "Quaternion and Clifford Fourier Transf. and Wavelets", TIM **27**, Birkhauser, Basel, 2013, 15–39.

[27] E. Hitzer, R. Abłamowicz, *Geometric Roots of* -1 *in Clifford Algebras* $Cl(p, q)$ *with* $p + q \leq 4$. Adv. Appl. Clifford Algebras, **21**(1), (2011) 121–144, DOI: 10.1007/s00006-010-0240-x.

[28] E. Hitzer, J. Helmstetter, and R. Abłamowicz, *Square roots of -1 in real Clifford algebras*, in: E. Hitzer, S.J. Sangwine (eds.), "Quaternion and Clifford Fourier Transf. and Wavelets", TIM **27**, Birkhauser, Basel, 2013, 123–153.

[29] E. Hitzer, J. Helmstetter, and R. Abłamowicz, Maple worksheets created with `CLIFFORD` for a verification of results in [28], http://math.tntech.edu/rafal/publications.html (©2012).

[30] E. Hitzer, *Quaternionic Fourier-Mellin Transf.*, in T. Sugawa (ed.), Proc. of ICFIDCAA 2011, Hiroshima, Japan, Tohoku Univ. Press, Sendai (2013), ii, 123–131.

[31] E. Hitzer, *Extending Fourier transformations to Hamiltons quaternions and Cliffords geometric algebras*, In T. Simos, G. Psihoyios and C. Tsitouras (eds.), Numerical Analysis and Applied Mathematics ICNAAM 2013, AIP Conf. Proc. 1558, pp. 529–532 (2013). DOI: 10.1063/1.4825544, Preprint: http://viXra.org/abs/1310.0249

[32] B. Jancewicz. *Trivector Fourier transf. and electromag. Field*, J. of Math. Phys., **31**(8), (1990), 1847–1852.

[33] K. Kou, J. Morais, Y. Zhang, *Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford Analysis*, Math. Meth. Appl. Sci. 2013, **36** pp. 1028-1041. DOI: 10.1002/mma.2657.

[34] K. Kou, J-Y. Ou, J. Morais, *On Uncertainty Principle for Quaternionic Linear Canonical Transform*, Abs. and App. Anal., Vol. 2013, IC 72592, 14 pp.

[35] Linwang Yuan, et al, *Geom. Alg. for Multidim.-Unified Geogr. Inf. System*, AACA, **23** (2013), 497–518.

[36] P. Lounesto, *Clifford Algebras and Spinors*, CUP, Cambridge (UK), 2001.

[37] B. Mawardi, E. Hitzer, *Clifford Algebra Cl(3,0)-valued Wavelet Transf., Clifford Wavelet Uncertainty Inequality and Clifford Gabor Wavelets*, Int. J. of Wavelets, Multiresolution and Inf. Proc., **5**(6) (2007), 997–1019.

[38] Waterloo Maple Incorporated, *Maple, a general purpose computer algebra system*. Waterloo, `http://www.maplesoft.com` (©2012).

[39] L. Meister, H. Schaeben, *A concise quaternion geometry of rotations*, Mathematical Methods in the Applied Sciences 2005; **28**(1): pp. 101–126.

[40] J. Mennesson, et al, *Color Obj. Recogn. Based on a Clifford Fourier Transf.*, in L. Dorst, J. Lasenby, "Guide to Geom. Algebra in Pract.", Springer 2011, 175–191.

[41] B. Rosenhahn, G. Sommer *Pose estimation of free-form objects*, European Conference on Computer Vision, Springer-Verlag, Berlin, **127**, pp. 414–427, Prague, 2004, edited by Pajdla, T.; Matas, J.

[42] S. J. Sangwine, *Biquaternion (Complexified Quaternion) Roots of* -1. Adv. Appl. Clifford Algebras **16**(1) (2006), 63–68.

[43] S. J. Sangwine, *Fourier transforms of colour images using quaternion, or hypercomplex, numbers*, Electronics Letters, **32**(21) (1996), 1979–1980.

[44] K.B. Wolf, *Integral Transforms in Science and Engineering*, Chapters 9&10, New York, Plenum Press, 1979. http://www.fis.unam.mx/~bwolf/integral.html

[45] Y. Yang, K. Kou, *Uncertainty principles for hypercomplex signals in the linear canoncial transform domains*, Signal Proc., Vol. 95 (2014), pp. 67–75.

[46] Yuan Linwang, et al, *Pattern Forced Geophys. Vec. Field Segm. based on Clifford FFT*, To appear in Computer & Geoscience.

[47] Yu Zhaoyuan, et al, *Clifford Algebra for Geophys. Vector Fields*, To appear in Nonlinear Process in Geophysics.

Eckhard Hitzer Eckhard Hitzer is Senior Associate Professor at the Department of Material Science at the International Christian University in Mitaka/Tokyo, Japan. His special interests are theoretical and applied Clifford geometric algebras and Clifford analysis, including applications to crystal symmetry visualization, neural networks, signal and image processing. Additionally he is interested in environmental radiation measurements, renewable energy and energy efficient buildings.

E. Hitzer, New Developments in Clifford Fourier Transforms, in N. E. Mastorakis, P. M. Pardalos, R. P. Agarwal, L. Kocinac (eds.), Advances in Applied and Pure Mathematics, Proceedings of the 2014 International Conference on Pure Mathematics, Applied Mathematics, Computational Methods (PMAMCM 2014), Santorini Island, Greece, July 17-21, 2014, Mathematics and Computers in Science and Engineering Series, Vol. 29, pp. 19-25, 2014.