The Strong Finiteness of Double Mersenne Primes and the Infinity of Root Mersenne Primes and Near-square Primes of Mersenne Primes

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Abstract

In this paper we present the strong finiteness of double Mersenne primes to be a subset of Mersenne primes, the infinity of so-called root Mersenne primes to be also a subset of Mersenne primes and the infinity of so-called near-square primes of Mersenne primes by generalizing our previous conjecture about primality of Mersenne number. These results and our previous results about the strong finiteness of Fermat, double Fermat and Catalan-type Fermat primes[1] give an elementary but complete understanding for the infinity or the strong finiteness of some prime number sequences of the form $2^x \pm 1$, which all have a corresponding original continuous natural (prime) number sequence. It is interesting that the generalization to near-square primes of Mersenne primes $W_p = 2M_p^2 - 1$ has brought us positive result.

Keywords: Mersenne prime; double Mersenne prime; root Mersenne prime; near-square prime of Mersenne prime; strong finiteness; infinity.

2010 Mathematics Subject Classification: 11A41, 11A51, 11A07

1. The strong finiteness of double Mersenne primes

Definition 1.1 If p is a prime number then $M_p=2^p-1$ is called a Mersenne number.

Definition 1.2 If Mersenne number $M_p=2^p-1$ is prime then the number M_p is called Mersenne prime.

Definition 1.3 If exponent of a Mersenne number is a Mersenne prime M_p i.e. $MM_p=2^{M_p}-1$ then the Mersenne number MM_p is called double Mersenne number.

Definition 1.4 If a double Mersenne number $MM_p=2^{M_p}-1$ is prime then the double Mersenne number MM_p is called double Mersenne prime.

Considering all double Mersenne primes to arise from double Mersenne numbers of the form $2^{M_p}-1$, we have the following definition.

Definition 1.5 Exponents of all Mersenne primes M_p are called basic sequence of number of doube Mersenne primes.

From Definition 1.5 we see basic sequence of number of double Mersenne primes is an infinite sequence if Mersenne primes are infinite. Lenstra, Pomerance and Wagstaff have conjectured that there is an infinite number of Mersenne primes in studying the number of primes p less than x with 2^p-1 being prime[2]. Our arguments[3.4] also presented the same result i. e. Mersenne primes are infinite. Thus we have an elementary result, that is, Mersenne primes are infinite so that basic sequence of number of double Mersenne primes is an infinite sequence. Further we have the following definition.

Definition 1.6 If the first few continuous exponents p of Mersenne primes M_p make double Mersenne number $MM_p = 2^{M_p} - 1$ become double Mersenne primes in basic sequence of number of double Mersenne primes then these exponents p of Mersenne primes are called original continuous prime number sequence of double Mersenne primes.

Lemma 1.1 The original continuous prime number sequence of double Mersenne primes is p =2,3,5,7.

Proof. Since MM_p for p = 2,3,5,7 are known double Mersenne primes but MM_{13} is not double Mersenne prime because of existence of known factors for MM_{13}[5], by Definition 1.6 we can confirm there exists an original continuous prime number sequence of double Mersenne primes i.e. p =2,3,5,7.

Definition 1.7 Double Mersenne primes are strongly finite if the first few continuous terms generated from the original continuous prime number sequence are prime but all larger terms are composite.

Conjecture 1.1 Double Mersenne primes are infinite if both the sum of corresponding original continuous prime number sequence and the first such prime are Fermat primes, but such primes are strongly finite if one of them is not Fermat prime.

Conjecture 1.1 is a generalization of our previous conjecture about primality of Mersenne number[4].

Corollary 1.1 If Conjecture 1.1 is true, then double Mersenne primes are strongly finite.

Proof. Since the sum of original continuous prime number sequence of double Mersenne primes i.e. 2+3+5+7=17 is a Fermat prime i.e. F_2 but the first double Mersenne prime MM_2=7 is not a Fermat prime, we will get the result.

Corollary 1.1 means that double Mersenne primes are strongly finite, that is, every double Mersenne number MM_p is composite for p >7. We should note that the simple Mersenne conjecture also led to the same result[6].

Proposition 1.1 MM_{127} and all of the following terms are composite in Catalan-Mersenne number sequence.

Proof. By Lemma 1.1, Definition 1.7 and Corollary 1.1 we see every double Mersenne number MM_p is composite for p >7. Since 127>7, MM_{127} is composite so that all of the following terms are composite in Catalan-Mersenne number sequence[7].

Proposition 1.2 There are infinitely many composite Mersenne numbers.

Proof. By Corollary 2.1 (Conjecture 2.1)[4] there are infinitely many Mersenne primes M_p. Then by Definition 1.3 there are infinitely many double Mersenne numbers MM_p. By Lemma 1.1, Definition 1.7 and Corollary 1.1 we see every double Mersenne number MM_p is composite for p >7. Thus there are infinitely many composite double Mersenne numbers MM_p. Since every double Mersenne number is also a Mersenne number by Definition 1.1, Definition 1.2 and Definition 1.3, every composite double Mersenne number is also a composite Mersenne numbers. Hence there are infinitely many composite Mersenne numbers.

2. The infinity of root Mersenne primes

Definition 2.1 Mersenne primes M_p for p=2,3,5,7 and Mersenne primes M_p to satisfy congruences $p \equiv F_0 \pmod 8$ or $p \equiv F_1 \pmod 6$ are called root Mersenne primes, where F_0 =3 and F_1 =5 are Fermat primes[8].

Although every one of p=2,3,5,7 is too small to be considered whether satisfy congruences $p \equiv F_0 \pmod 8$ or $p \equiv F_1 \pmod 6$, their sum 2+3+5+7=17 satisfies congruence $17 \equiv 5 \pmod 6$ so that M_p for p=2,3,5,7 are thought root Mersenne primes[8].

By definition 2.1 we see that among 48 known Mersenne primes, there are 31 root Mersenne primes: M_2, M_3, M_5, M_7, M_{17}, M_{19}, M_{89}, M_{107}, M_{521}, M_{2203}, M_{4253}, M_{9689},

M_{9941}, M_{11213}, M_{19937}, M_{21701}, M_{86243}, M_{216091}, M_{756839}, M_{859433}, $M_{1257787}$, $M_{1398269}$, $M_{2976221}$, $M_{3021377}$, $M_{6972593}$, $M_{20996011}$, $M_{25964951}$, $M_{32582657}$, $M_{37156667}$, $M_{43112609}$ and $M_{57885161}$. Hence we have the following propositions.

Proposition 2.1 Let F_{n+1} be Fermat primes for $n \geq 0$, the number of root Mersenne primes M_p is 2^n for $F_n - 1 < p < F_{n+1} - 1$ ($n = 0,1,2,3,\cdots$) and the number of root Mersenne primes M_p is 2^{n+1} for $p < F_{n+1} - 1$.

Proof. We can verify Proposition 2.1 as follows

For $n=0$, there exists $2^0=1$ root Mersenne prime i.e. M_3 for $F_0-1<p<F_{0+1}-1$ i.e. $2<p<4$ and there exist $2^{0+1}=2$ root Mersenne primes i.e. M_2, M_3 for $p<F_{0+1}-1$ i.e. $p<4$;

For $n=1$, there exist $2^1=2$ root Mersenne primes i.e. M_5, M_7 for $F_1-1<p<F_{1+1}-1$ i.e. $4<p<16$ and there exist $2^{1+1}=4$ root Mersenne primes i.e. M_2, M_3, M_5, M_7 for $p<F_{1+1}-1$ i.e. $p<16$;

For $n=2$, there exist $2^2=4$ root Mersenne primes i.e. M_{17}, M_{19}, M_{89}, M_{107} for $F_2-1<p<F_{2+1}-1$ i.e. $16<p<256$ and there exist $2^{2+1}=8$ root Mersenne primes i.e. M_2, M_3, M_5, M_7, M_{17}, M_{19}, M_{89}, M_{107} for $p<F_{2+1}-1$ i.e. $p<256$;

For $n=3$, there exist $2^3=8$ root Mersenne primes i.e. M_{521}, M_{2203}, M_{4253}, M_{9689}, M_{9941}, M_{11213}, M_{19937}, M_{21701} for $F_3-1<p<F_{3+1}-1$ i.e. $256<p<65536$ and there exist $2^{3+1}=16$ root Mersenne primes i.e. M_2, M_3, M_5, M_7, M_{17}, M_{19}, M_{89}, M_{107}, M_{521}, M_{2203}, M_{4253}, M_{9689}, M_{9941}, M_{11213}, M_{19937}, M_{21701} for $p<F_{3+1}-1$ i.e. $p<65536$.

Since it is known that there exist no any undiscovered Mersenne primes M_p for $p \leq 30402457$[9] and all Fermat numbers F_n are composite for $5 \leq n \leq 32$ and there is no any found new Fermat prime for $n>4$, if suppose every Fermat number F_n is composite for $n>4$ then Proposition 2.1 holds.

Proposition 2.2 Let F_n be Fermat primes for $n \geq 1$, the number of root Mersenne primes M_p is 2^n for $p<F_n-1$.

Proof. Proposition 2.2 is a part of Proposition 2.1 and we have the following verification.

For $n=1$, there exist $2^1=2$ root Mersenne primes i.e. M_2, M_3 for $p<F_1-1$ i.e. $p<4$;

For $n=2$, there exist $2^2=4$ root Mersenne primes i.e. M_2, M_3, M_5, M_7 for $p<F_2-1$ i.e. $p<16$;

For $n=3$, there exist $2^3=8$ root Mersenne primes i.e. M_2, M_3, M_5, M_7, M_{17}, M_{19}, M_{89}, M_{107} for $p<F_3-1$ i.e. $p<256$;

For $n=4$, there exist $2^4=16$ root Mersenne primes i.e. M_2, M_3, M_5, M_7, M_{17}, M_{19}, M_{89}, M_{107}, M_{521}, M_{2203}, M_{4253}, M_{9689}, M_{9941}, M_{11213}, M_{19937}, M_{21701} for $p<F_4-1$ i.e. $p<65536$.

Since it is known that there exist no any undiscovered Mersenne primes M_p for $p \leq 30402457$[9] and all Fermat numbers F_n are composite for $5 \leq n \leq 32$ and there is no any found new Fermat prime for $n>4$, if suppose every Fermat number F_n is composite for $n>4$ then Proposition 2.2 holds.

It is such simple and clear distribution law of root Mersenne primes that makes us fell it is necessary to introduce root Mersenne primes as a subset of Mersenne primes. However, we should note that the statement of Proposition 2.1 in this paper is different from our previous statement about distribution law of root Mersenne primes[3,8,10], because F_{n+1} is restricted to Fermat prime in this paper.

Considering all root Mersenne primes to arise from Mersenne primes, we have the following definition.

Definition 2.2 Exponents p of all Mersenne primes M_p are called basic sequence of number of root Mersenne primes.

From Definition 2.2 we see basic sequence of number of root Mersenne primes is an infinite sequence if Mersenne primes are infinite. Further we have the following definition.

Definition 2.3 If the first few continuous exponents of Mersenne primes p make $M_p=2^p-1$ become root Mersenne primes in basic sequence of number of root Mersenne primes then these exponents are called original continuous prime number sequence of root Mersenne primes.

Lemma 2.1 The original continuous prime number sequence of root Mersenne primes is p =2,3,5,7.

Proof. Since M_p for p = 2,3,5,7 are root Mersenne primes but Mersenne prime M_{13} is not root Mersenne prime, by Definition 2.3 we can confirm there exists an original continuous prime number sequence of root Mersenne primes i.e. p =2,3,5,7.

Definition 2.4 Root Mersenne primes are strongly finite if the first few continuous terms generated from the original continuous prime number sequence are prime but all larger terms are composite.

Conjecture 2.1 Root Mersenne primes are infinite if both the sum of corresponding original continuous prime number sequence and the first such prime are Fermat primes, but such primes are strongly finite if one of them is not Fermat prime.

Corollary 2.1 If Conjecture 2.1 is true, then root Mersenne primes are infinite.

Proof. Since the sum of original continuous prime number sequence of root Mersenne primes i.e. 2+3+5+7=17 is a Fermat prime i.e. F_2 and the first root Mersenne prime M_2=3 is also a Fermat prime i.e. F_0, we will get the result.

3. The infinity of near-square primes of Mersenne primes

Definition 3.1 If M_p is a Mersenne prime then $W_p = 2M_p^2 - 1$ is called near-square number of Mersenne prime M_p.

If Mersenne primes M_p are infinite then near-square number sequence $W_p = 2M_p^2 - 1$ generated from all Mersenne primes M_p is an infinite sequence.

Definition 3.2 If $W_p = 2M_p^2 - 1$ is a prime number then the number W_p is called near-square prime of Mersenne prime M_p.

Definition 3.3 Exponents p of all Mersenne primes M_p are called basic sequence of number of near-square primes of Mersenne primes.

If Mersenne primes are infinite then the basic sequence of number of near-square primes of Mersenne primes is infinite.

Definition 3.4 If the first few continuous exponents of Mersenne primes p make $W_p = 2M_p^2 - 1$ become near-square primes of Mersenne primes in basic sequence of number of near-square primes of Mersenne primes then these exponents are called original continuous prime number sequence of near-square primes of Mersenne primes.

Lemma 3.1 The original continuous prime number sequence of near-square primes of Mersenne primes is p =2,3.

Proof. Since the first two near-square numbers of Mersenne primes $W_p = 2M_p^2 - 1$ for p = 2,3 i.e. W_2 =17 and W_3 = 97 all are near-square primes of Mersenne primes but the third near-square number of Mersenne prime W_5 =1921 is not prime but composite, by Definition 3.4 we can confirm there exists an original continuous prime number sequence of near-square primes of Mersenne primes i.e. p =2,3.

Definition 3.5 Near-square primes of Mersenne primes are strongly finite if the first few continuous terms generated from the original continuous prime number sequence are prime but all larger terms are composite.

Conjecture 3.1 Near-square primes of Mersenne primes are infinite if both the sum of corresponding original continuous prime number sequence and the first such prime are Fermat primes, but such primes are strongly finite if one of them is not Fermat prime.

Corollary 3.1 If Conjecture 3.1 is true, then near-square primes of Mersenne primes are infinite.

Proof. Since the sum of original continuous prime number sequence of near-square primes of Mersenne primes i.e. 2+3=5 is a Fermat prime i.e. F_1 and the first near-square prime of Mersenne prime W_2 =17 is also a Fermat prime i.e. F_2, we will get the result.

Obviously this is an interesting result, because it implies our previous conjecture about primality of Mersenne number[4] not only can be generalized to subsets of Mersenne primes, which have corresponding original continuous prime number sequence, such as double Mersenne primes and root Mersenne primes, but also can be generalized to some near-square forms of Mersenne primes such as near-square primes of Mersenne primes $W_p = 2M_p^2 - 1$ to have corresponding original continuous prime number sequence i.e. p =2,3.

It is important that the infinity of near-square primes of Mersenne primes $W_p = 2M_p^2 - 1$ may lead us to find larger primes than the largest known Mersenne prime $M_{57885161}$ by some known Mersenne primes with large p-values themselves[11].

4. Conclusion

Our previous two papers[1,4] and this paper have completely presented a new idea that the infinity or the strong finiteness of prime number sequences of the form $2^x \pm 1$ can be confirmed by conjectures about primality of Mersenne number and Fermat number and their generalizations if such prime number sequences have corresponding original continuous natural (prime) number sequences. In this mathematical frame, we have considered the set of Mersenne primes and its two subsets i.e. the set of double Mersenne primes and the set of root Mersenne primes[3,8,10] as well as the set of Fermat primes and its two subsets i.e. the set of double Fermat primes[3,12] and the set of Catalan-type Fermat primes[12], and discovered it is an acceptable argument to be able to lead to many elementary but reasonable results, that is, Mersenne primes and root Mersenne primes are all infinite but double Mersenne primes, Fermat primes, double Fermat primes and Catalan-type Fermat primes are all strongly finite, because the infinity and the finiteness of Mersenne primes and Fermat primes have been very difficult and open problems for many years. It seems to be somehow similar to a well-known method that existence of Fermat primes as distrinct prime factors of n can confirm whether a regular n-sided polygon can be constructed with compass and straightedge[13]. It is interesting that the generalization to a near-square form of Mersenne primes i.e. near-square primes of Mersenne primes $W_p = 2M_p^2 - 1$ to have corresponding original continuous prime number sequence i.e. $p = 2,3$ has brought us some reasonable results, which means we may find larger primes than the largest known Mersenne primes by some known

Mersenne primes with large p-values themselves.

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